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Some results on uniformly convex linear topological spaces. (**)

1. - Introduction.

Clarkson ([1], § 26,6, p. 353) gave the definition of uniform convexity in normed linear spaces as follows:

Definition 1.1. A normed linear space X is said to be *uniformly convex* if for any ε $(0<\varepsilon\leqslant 2)$ there exists a δ $(0<\delta<1)$ such that for any x, y in X

$$\|x\|\leqslant 1\,,\quad \|y\|\leqslant 1 \quad \text{and} \quad \|x-y\|\geqslant\varepsilon \quad \text{imply} \quad \|\tfrac{1}{2}(x+y)\|\leqslant 1-\delta\,.$$

It is obvious that this definition of uniform convexity depends on the norm of the space.

If X is an inner product space and if $\|\cdot\|$ be the norm induced by the inner product, then for any two points x, y in X

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2\,.$$

This gives that the space $(X, \|\cdot\|)$ is uniformly convex. It is known that the spaces l^p and L^p are uniformly convex for all p with $1<p<\infty$ ([1], ch. V, § 26, 7 (12), p. 358). It has been shown by Milman that a uniformly convex B-space is reflexive ([1], ch. V, § 26, 6 (4), p. 354; [2], ch. V, Th. 2, p. 127).

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In the present paper we have extended the definition of uniform convexity to linear topological spaces. We have shown that a uniformly convex linear topological space is locally convex (2, Th. 2.1) and we have established that a uniformly convex linear topological space possessing the property (P) and satisfying first axiom of countability is reflexive (2, Th. 2.3).

2. - Results on uniformly convex linear topological spaces.

Definition 2.1. A balanced neighbourhood V of 0 in a linear topological space X is said to possess *the property* (I) if for any $x(\neq 0)$ in X there is a positive number ε such that $x \notin \varepsilon V$.

Definition 2.2. A balanced neighbourhood V of 0 in a linear topological space X is said to possess *the property* (II) if for every $\varepsilon > 0$ there is a δ $(0 < \delta < 1)$ such that $x, y \in \overline{V}$ and $x - y \notin \varepsilon V$ imply $(1/2)(x + y) \in (1 - \delta)\,\overline{V}$, where $\overline{V}$ denotes the closure of V.

Definition 2.3. A linear topological space X is said to be *uniformly convex* if the family $\mathscr{V}$ of all balanced neighbourhoods V of 0 in X possessing the properties (I) and (II) is a local base at 0.

Let X be a normed linear space. Suppose that X is uniformly convex according to the definition 1.1. Let $V = \{x;\, x \in X \text{ and } \|x\| < 1\}$ and let $V_k = kV$ for any $k > 0$. It is easy to see that each V_k possesses the properties (I) and (II). Clearly the family $\{V_k;\, k > 0\}$ is a local base at 0. Hence X is uniformly convex according to the definition 2.3.

Lemma 2.1. *Let A be a non-void closed subset of a linear topological space. If for any two points x, y in A, $(1/2)(x + y) \in A$, then A is convex.*

Proof. Let x_0, y_0 be any two distinct points of A. For any t $(0 \leqslant t \leqslant 1)$ write $x(t) = (1 - t)x_0 + ty_0$. Denote by E the set consisting of points

$$\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \frac{11}{16}, \frac{13}{16}, \frac{15}{16}, \ldots .$$

Then clearly $x(t) \in A$ for all t in E. Take any α with $0 < \alpha < 1$. Since E is dense in $[0, 1]$ we can choose a sequence $\{t_n\}$ from E such that $t_n \to \alpha$ as $n \to \infty$. We have $x(t_n) - x(\alpha) = (t_n - \alpha)(y_0 - x_0)$. This gives that $x(t_n) \to x(\alpha)$ as $n \to \infty$. Since A is closed and $x(t_n) \in A$ for all n, it follows that $x(\alpha) \in A$. Hence A is convex.

Lemma 2.2. *Let X be a linear topological space. If there is a balanced neighbourhood V of 0 in X possessing the property* (I), *then X is a Hausdorff space.*

Proof. Let x and y be any two distinct points of X. Let V be a balanced neighbourhood of 0 in X possessing the property (I). Since $x-y\neq 0$ there is a positive number ε such that

$$x-y\notin \varepsilon V\,. \tag{1}$$

Choose a balanced neighbourhood U of 0 in X such that $U+U\subset \varepsilon V$. Then $x+U$ and $y+U$ are neighbourhoods of x and y respectively. If $z\in(x+U)\cap\cap(y+U)$, then $z=x+x'=y+y'$, where $x', y'\in U$. So, $x-y=y'-x'\in\in U+U\subset\varepsilon V$ which contradicts (1). Thus $(x+U)\cap(y+U)=\varphi$. Hence X is a Hausdorff space.

Note 2.1. Every uniformly convex linear topological space is a Hausdorff space.

Lemma 2.3. *Let X be a linear topological space and let V be a balanced neighbourhood of* 0 *in X possessing the properties* (I) *and* (II). *Then $\overline{V}$ is convex.*

Proof. Take any two points x, y in $\overline{V}$. If $x=y$, then $(1/2)(x+y)==x\in\overline{V}$. Suppose that $x-y\neq 0$. Since V possesses the property (I), there is an $\varepsilon>0$ such that $x-y\notin\varepsilon V$. By the property (II), there is a δ $(0<\delta<1)$ such that $(1/2)(x+y)\in(1-\delta)\overline{V}\subset\overline{V}$. Thus for any two points x, y in $\overline{V}$, $(1/2)(x+y)\in\overline{V}$. So by Lemma 2.1, the set $\overline{V}$ is convex.

Theorem 2.1. *A uniformly convex linear topological space is locally convex.*

Proof. Let X be a uniformly convex linear topological space. Denote by $\mathscr{V}$ the family of all balanced neighbourhoods V of 0 in X possessing the properties (I) and (II). Then $\mathscr{V}$ is a local base at 0. By Lemma 2.3, $\overline{V}$ is convex for each V in $\mathscr{V}$. Since the family $\{\overline{V};\ V\in\mathscr{V}\}$ is also a local base at 0, it follows that the space X is locally convex.

We recall now the definition of an inner-product on a linear space.

Definition 2.4. Let X be a linear space over the field $\mathscr{C}$ of complex numbers. A mapping $f\colon X\times X\to\mathscr{C}$ is an inner-product on X if for x, y, z in

X and α, β in $\mathscr{C}$

(i) $f(y, x) = \overline{f(x, y)}$,

(ii) $f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta f(y, z)$,

(iii) $f(x, x) > 0$ for $x \neq 0$ and $f(0, 0) = 0$.

The function p_f defined by $p_f(x) = \sqrt{f(x, x)}$ is a norm on X.

Let $\mathscr{F}$ be a family of inner-products on X. The family $\mathscr{P} = \{p_f; f \in \mathscr{F}\}$ of norms on X generates a topology τ on X relative to which X is a linear topological space. We say that the topology τ is generated by the family $\mathscr{F}$ of inner-products on X. For any f in $\mathscr{F}$ write: $V_f = \{x; x \in X \text{ and } p_f(x) < 1\}$ and $\mathscr{V}_{(\mathscr{F})} = \{V_f; f \in \mathscr{F}\}$.

Since for any x, y in X and any f in $\mathscr{F}$

$$[p_f(x+y)]^2 + [p_f(x-y)]^2 = 2[p_f(x)]^2 + 2[p_f(y)]^2$$

and $p_f(x) > 0$ for $x \neq 0$ it follows that each V_f possesses the properties (I) and (II). If for any f_1, f_2 in $\mathscr{F}$, there is an f in $\mathscr{F}$ such that $f_i(x, x) \leqq f(x, x)$ for all x in X $(i = 1, 2)$ then $\mathscr{V}_{(\mathscr{F})}$ is clearly a local base at 0 in X. So we obtain:

Theorem 2.2. *Let (X, τ) be a linear topological space whose topology τ is generated by a family $\mathscr{F}$ of inner-products on X. If for any two members f_1, f_2 in $\mathscr{F}$ there is an f in $\mathscr{F}$ such that $f_i(x, x) \leqq f(x, x)$ for all x in X $(i = 1, 2)$, then the space (X, τ) is uniformly convex.*

Example 2.1. Let X denote the set of all complex-valued functions $x(t)$ continuous on $[0, 1]$. Then X is a linear space over the field of complex numbers under the usual definitions of addition and multiplication by scalars. Denote by $\mathscr{V}$ the set of all continuous strictly increasing functions $\omega(t)$ on $[0, 1]$. For x, y in X and ω in $\mathscr{V}$ define

$$f_\omega(x, y) = \int_0^1 x(t)\overline{y(t)}\, d\omega .$$

Then f_ω is an inner-product on X and so the family $\mathscr{F} = \{f_\omega; \omega \in \mathscr{V}\}$ generates a topology τ on X so that (X, τ) is a linear topological space. We can verify that $\mathscr{F}$ satisfies the condition of the Theorem 2.2. Hence the space (X, τ) is uniformly convex.

Example 2.2. Let X denote the set of all sequences $x = (x_1, x_2, x_3, \ldots)$ of complex numbers with $\sum_{n=1}^{\infty} |x_n|^2 < +\infty$.

Then X is a linear space. Denote by $\mathscr{C}$ the set of all bounded sequences $a = (a_1, a_2, a_3, \ldots)$ of positive terms. For x, y in X and a in $\mathscr{C}$ define $f_a(x, y) = \sum_{n=1}^{\infty} a_n x_n \bar{y}_n$.

Clearly f_a is an inner-product on X. The family $\mathscr{F} = \{f_a; a \in \mathscr{C}\}$ generates a topology τ on X such that (X, τ) is a linear topological space. Clearly $\mathscr{F}$ satisfies the condition of Theorem 2.2. So (X, τ) is a uniformly convex space.

Definition 2.5. A uniformly convex linear topological space (X, τ) is said to possess *the property* (P) if there is a local base $\mathscr{V}$ at 0 of balanced neighbourhoods V possessing the properties (I) and (II) such that the space (X, p_V) is a B-space for each $V \in \mathscr{V}$, where p_V is the Minkowski functional of $\bar{V}$.

Suppose that (X, τ) is a uniformly convex linear topological space satisfying the first axiom of countability. Then there is a local base $\mathscr{U} = \{U_n\}_{n=1}^{\infty}$ at 0 with $U_1 \supset U_2 \supset U_3 \supset U_4 \supset \ldots$. Denote by $\mathscr{V}$ the family of all balanced neighbourhoods of 0 possessing the properties (I) and (II). Then $\mathscr{V}$ is a local base at 0. Chose a member V_1 in $\mathscr{V}$ with $V_1 \subset U_1$. Choose a member U_{n_2} $(n_2 \geqslant 2)$ in $\mathscr{U}$ with $U_{n_2} \subset V_1$. Now take a member V_2 in $\mathscr{V}$ with $V_2 \subset U_{n_2}$. Proceeding in this way we can choose a sequence $\{V_\nu\}_{\nu=1}^{\infty}$ of members in $\mathscr{V}$ and a sequence $\{n_\nu\}_{\nu=1}^{\infty}$ of positive integers with $1 = n_1 < n_2 < n_3 < \ldots$ such that $V_\nu \subset U_{n_\nu}$ $(\nu = 1, 2, 3, \ldots)$. Then $\mathscr{V}_0 = \{V_\nu\}_{\nu=1}^{\infty}$ is a base at 0 and $V_1 \supset V_2 \supset V_3 \supset \ldots$.

Theorem 2.3. *A uniformly convex linear topological space possessing the property* (P) *and satisfying the first axiom of countability is reflexive.*

Proof. Let (X, τ) be a uniformly convex linear topological space which possesses the property (P) and satisfies the first axiom of countability. Then there is a local base $\mathscr{V}_0 = \{V_n\}_{n=1}^{\infty}$ at 0 of balanced neighbourhoods possessing the properties (I) and (II) with $V_1 \supset V_2 \supset V_3 \supset \ldots$. Denote by p_n the Minkowski functional of $\bar{V}_n$.

We prove the theorem by the following steps.

Step I. Denote by X' the dual space of (X, τ) and by X'_n the dual space of (X, p_n). We show that $X' = U_{n=1}^{\infty} X'_n$. Let $f \in X'$. Then $f^{-1}(A) \in \tau$, where $A = (-1, 1)$. Since $0 \in f^{-1}(A)$, there is a member $V_n \in \mathscr{V}_0$ such that $V_n \subset f^{-1}(A)$. Choose $\varepsilon > 0$ arbitrarily. Write $U = \varepsilon V_n$. Since $U \subset f^{-1}(\varepsilon A)$ we have $f(U) \subset \varepsilon A$. This gives $|f(x)| < \varepsilon$ for all $x \in U$. So f is continuous at 0 in the topology of (X, p_n). Hence $f \in X'_n$. It is easy to see that $X'_n \subset X'$ for $n = 1, 2, 3, \ldots$. Therefore $X' = U_{n=1}^{\infty} X'_n$.

Step II. Since (X, τ) possesses the property (P), the space (X, p_n) is a B-space for each n. Due to the properties (I) and (II), we see that (X, p_n) is uniformly convex. Hence by Milman's Theorem ([2], ch. V, §22, Th. 2. p. 127) the space (X, p_n) is reflexive. The set $\overline{V}_n$ is closed, convex, balanced and bounded in the space (X, p_n). Hence by Theorem 2 ([2], p. 140) the set $\overline{V}_n$ is compact relative to the weak topology τ_{ω_n} of X induced by X'_n.

Step III. Let B be a closed, convex, balanced and bounded set in (X, τ). Let $x_0 \in X \sim B$. By Mazur Theorem ([2], ch. IV, § 6, Th. 3, p. 108) there is a continuous linear functional f on (X, τ) such that $f(x_0) > 1$ and $|f(x)| \leqslant 1$ for all $x \in B$.

Write $\varrho = f(x_0)$ and $U = \{x;\, x \in X \text{ and } |f(x)| \leqslant 1\}$. Then U is a weak neighbounrhood of 0. Take an α with $0 < \alpha < (\varrho - 1)$. Clearly $x_0 + \alpha U$ is a weak neighbourhood of x_0. If $z \in x_0 + \alpha U$, then $z = x_0 + \alpha x$, where $x \in U$. We have $f(z) = f(x_0) + \alpha f(x) = \varrho + \alpha f(x) \geqslant \varrho - \alpha > 1$. This shows that $z \notin U$. Since $B \subset U$ we see $x_0 + \alpha U$ does not contain any point of B. So x_0 is not a τ_ω-accumulation point of B, where τ_ω is the weak topology on X induced by X'. Hence B is closed in the weak topology of τ_ω of X.

Step IV. Let B be any closed, convex, balanced and bounded set in (X, τ). Then by step III, B is closed in the weak topology τ_ω. Let $\{x_n\}$ be any sequence in B.

For each V_n there is a positive number α_n such that $B \subset \alpha_n \overline{V}_n$.

Since $B \subset \alpha_1 \overline{V}_1$ and $\alpha_1 \overline{V}_1$ is compact relative to the weak topology $\tau_{\omega 1}$ there is a subsequence $\{x_n^{(1)}\}$ of $\{x_n\}$ such that $\{x_n^{(1)}\}$ converges to a point $x_0^{(1)} \in \alpha_1 \overline{V}_1$ in the topology τ_{ω_1}.

Again, since $\{x_n^{(1)}\} \subset \alpha_2 \overline{V}_2$ and $\alpha_2 \overline{V}_2$ is compact in (X, τ_{ω_2}) there is a subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$ such that $\{x_n^{(2)}\}$ converges to a point $x_0^{(2)} \in \alpha_2 \overline{V}_2$ in the topology τ_{ω_2}.

Proceeding in this way we obtain sequences $\{x_n^{(m)}\}$ $(m = 1, 2, 3, \ldots)$ such that $\{x_n^{(m)}\}$ is subsequence of $\{x_n^{(m-1)}\}$ and $\{x_n^{(m)}\}$ converges to a point $x_0^{(m)} \in \alpha_m \overline{V}_m$ in the topology τ_{ω_m}.

Write $z_n = x_n^{(n)}$ $(n = 1, 2, 3, \ldots)$. For each positive integer m, $\{z_n\}_{n=m}^{\infty}$ is a subsequence of $\{x_n^{(m)}\}$ which gives that $\{z_n\}_{n=1}^{\infty}$ converges to $x_0^{(m)}$ in the topology τ_{ω_m}. Let $f \in X'$. Then by step I $f \in X'_m$ for some positive integer m; so $\{f(z_n)\}$ converges for all f in X'.

Let $f \in X'_1$. Take any positive integer m. Since $X'_1 \subset X'_m$, $f \in X'_m$. So $f(z_n) \to f(x_0^{(1)})$ and $f(z_n) \to f(x_0^{(m)})$ as $n \to \infty$. This gives that $f(x_0^{(1)}) = f(x_0^{(m)})$ for all $f \in X_1'$. So $x_0^{(1)} = x_0^{(m)}$ for $m = 2, 3, \ldots$. Hence $z_n \to x_0^{(1)}$ in the weak topology τ_ω. Since $\{z_n\} \subset B$ and B is τ_ω-closed, $x_0^{(1)} \in B$. Therefore B is sequentially τ_ω-compact.

Step V. The topology τ of X can be generated by the quasi-norm $\|\cdot\|$ on X defined by

$$\|x\| = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} \cdot \frac{p_\nu(x)}{1 + p_\nu(x)} .$$

Since $p_1(x) \leqslant p_2(x) \leqslant p_3(x) \leqslant \ldots$ for all $x \in X$ and (X, p_ν) is a B-space for $\nu = 1, 2, 3, \ldots$ we can verify that (X, τ) is an F-space.

Now let M be any bounded set in (X, τ). Denote by B the convex hull of the set IM, where $I = \{\lambda;\ |\lambda| \leqslant 1\}$. Then $\bar{B}$ is a closed, convex, balanced and bounded set in (X, τ). By step IV, $\bar{B}$ is sequentially τ_ω-compact. By step II $\bar{B}_\omega = \bar{B}$, where $\bar{B}_\omega$ is the τ_ω-closure of B. Since $\bar{M}_\omega \subset \bar{B}_\omega$, $\bar{M}_\omega$ is sequentially τ_ω-compact. Hence $\bar{M}_\omega$ is countably τ_ω-compact. Now by Theorem (7) ([1], § 24, 2(7), p. 315) the space (X, τ) is reflexive.

Theorem 2.4. *Let (X, τ) be a linear topological space whose topology τ is generated by a sequence $\{p_n\}_{n=1}^{\infty}$ of norms on X satisfying the following conditions*

(i) $p_1(x) \leqslant p_2(x) \leqslant p_3(x) \leqslant \ldots$ *for all* $x \in X$,

(ii) (X, p_n) *is a reflexive B-space for each n.*

Then (X, τ) is reflexive.

Proof. Let $V_n = \{x;\ x \in X \text{ and } p_n(x) < 1\}$ $(n = 1, 2, 3, \ldots)$. Then $V_1 \supset V_2 \supset V_3 \supset \ldots$. Clearly $\{V_n\}_{n=1}^{\infty}$ is a local base at 0. As in step I of Theorem 2.3 we can show that $X' = U_{n=1}^{\infty} X_n'$, where X' and X_n' are dual spaces of (X, τ) and (X, p_n) respectively. The set $\bar{V}_n$ is closed, convex, balanced and bounded in the space (X, p_n). Since (X, p_n) is reflexive by Theorem 2 ([2], p. 140) the set $\bar{V}_n$ is compact relative to the weak topology τ_{ω_n} of X induced by X_n'. Now we can complete the proof proceeding as in Theorem 2.3.

Example 2.3. Let X denote the set of all sequences $x = (x_1, x_2, x_3, \ldots)$ of complex numbers such that $\sum_{i=1}^{\infty} |x_i|^a < +\infty$ for every $a > 1$.

Then X is a linear space over the field of complex numbers. For any $a > 1$ and $x \in X$ define $p_a(x) = (\sum_{i=1}^{\infty} |x_i|^a)^{1/a}$ then p_a is a norm on X. The family $\mathscr{P} = \{p_a;\ a > 1\}$ of norms generates a topology τ on X making (X, τ) a linear topological space. If $1 < a < b$, then $p_b(x) \leqslant p_a(x)$ for all $x \in X$; so it follows that the family $\mathscr{V} = \{V_a;\, a > 1\}$, where $V_a = \{x;\, x \in X$ and $p_a(x) < 1\}$, is a local base at 0. Since $p_a(x) > 0$ for $x \neq 0$ it follows that each V_a possesses the property (I). Again, for each $a > 1$, (X, p_a) is a B-space and from § 26, 7(12) ([1], Ch. V, p. 358) we see that V_a possesses the property (II). Hence the space (X, τ) is uniformly convex and possesses the property (P). It is easy to see that (X, τ) satisfies the first axiom of countability. Therefore (X, τ) is reflexive by Theorem 2.3.

Example 2.4. Let X denote the set of all complex valued functions $x(t)$ measurable on $[0, 1]$ such that

$$\int_0^1 |x(t)|^a \, dt < +\infty \qquad \text{for all } a > 1 .$$

Then X is a linear space. For $x \in X$ and $a > 1$, define

$$p_a(x) = \{\int_0^1 |x(t)|^a \, \mathrm{d}t\}^{1/a} .$$

Then p_a is a norm on X under the convention that $x = y$ iff $x(t) = y(t)$ a.e. on $[0, 1]$. The family $\mathscr{P} = \{p_a;\, a > 1\}$ of norms generates a topology τ on X relative to which X is a linear topological space. For $1 < a < b$ we have $p_a(x) \leqslant p_b(x)$ for all $x \in X$. So the family $\mathscr{V} = \{V_a;\, a > 1\}$, where $V_a = \{x;\, x \in X$ and $p_a(x) < 1\}$, is a local base at 0. Clearly (X, p_a) is a B-space for each $a > 1$. From § 26, 7(12) ([1], Ch. V, p. 358) it follows that V_a possesses the property (II) and that V_a possesses the property (I) follows from $p_a(x) > 0$ for $x \neq 0$. Further it is easy to verify that (X, τ) satisfies first axiom of countability. Hence the space (X, τ) is reflexive.

References.

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[2] K. YOSIDA, *Functional Analysis*, Springer-Verlag, Berlin 1968.

Summary.

In the present paper the concept of uniform convexity is extended to linear topological spaces. It has been shown that a uniformly convex linear topological space is locally convex. Further it has been established that a uniformly convex linear topological space possessing the property (P) und satisfyng the ist axiom of countability is reflexive. Some examples of uniformly convex linear topological spaces are given.

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