Riv. Mat. Univ. Parma (4) 2 (1976), 147-153

S. SRIVASTAVA and K. L. SINGH (*)

Construction of fixed point for densifying maps. (II) (**)

1. - Introduction.

In recent years many authors have proved various fixed point theorems for nonlinear operators in Banach and Hilbert spaces. If D is a closed, bounded convex subset of a Banach space X, then a contraction mapping T ($\|Tx - Ty\| \leqslant \alpha\|x-y\|$, $0<\alpha<1$, $\forall x, y \in D$) has a unique fixed point in D, but a nonexpansive mapping ($\|Tx-Ty\| \leqslant \|x-y\|$) need not. If X is either uniformly convex [7] Banach space or reflexive Banach space with normal structure [1], then the existence of a fixed point for nonexpansive mappings have been proven [1], [2].

It was Krasnoselsky [6] who proved the existence of fixed points for sum of two nonlinear operators, namely the following theorem.

Theorem A. Let C be a closed, bounded and convex subset of a Banach space X. Let $A: C \to C$ be a nonexpansive and $B: C \to C$ be completely continuous such that $Ax + By \in C$ for all $x, y \in C$. Then $T = A + B$ has a fixed point in C.

By putting the weaker condition $Ax + Bx \in C$ in place of $Ax + By \in C$, the above theorem is given in [5], [10] and [7].

The aim of this paper is to prove some fixed point theorems for sum of two nonlinear operators which generalizes all the results proved till now on this topic. Also first time in this paper an attempt has been made to relate the potential operator with densifying mappings.

(*) Indirizzo degli Autori: S. SRIVASTAVA, Department of Mathematics, Bowie State College, Bowie Maryland, 20715 U.S.A.; K. L. SINGH, Department of Mathematics, Texas A. and M. University, College Station, Texas 77843, U.S.A.

(**) Ricevuto: 18-VII-1974.

2. - Preliminary definitions and results.

Let X be a real Banach space and D be a open, bounded subset of X. The measure of noncompactness of D denoted by $\gamma(D)$ is defined as $\inf\{\varepsilon > 0 \mid D \subseteq \bigcup_{i=1}^{n} \delta_i$ such that diameter $\delta_i < \varepsilon\}$. Closely related to the concept of measure of noncompactness is k-set contraction defined [8] in the following way.

Definition 1.1. Let D be a bounded subset of X and $T\colon D \to X$ be continuous. T is said to be *k-set contraction* if for some $k > 0$, if for any bounded subset A of D

$$\gamma(T(A)) \leqslant k\gamma(A) .$$

Definition 1.2. Let D be a bounded subset of X and $T\colon D \to X$ be continuous. T is said to be *densifying* if for any bounded subset A of D

$$\gamma(T(A)) < k\gamma(A) . \tag{1}$$

If in (1) we have $(T(A)) < \gamma(A)$; then T is called 1-*set contraction.*

Remark 1.1. The sum of two k-set contractions, the composition of two k-set contractions is again a k-set contraction [9].

Let X be a real Banach space, X^* its dual space and T a nonlinear (or rather, not necessarily linear) operator mapping X into X^*.

Definition 1.3. The operator $T\colon X \to X^*$ is *strongly monotone* if there exists a constant $\alpha > 0$ such that

$$(Tx - Ty, x - y) \geqslant \alpha(\|x - y\|)^2 \quad \text{for all } x, y, \in X .$$

Definition 1.4. The operator $T\colon X \to X^*$ is *hemicontinuous* if it is continuous from line segment of X to the weak topology of X^*.

Theorem B ([4], pp. 64). Let X be a reflexive Banach space and X its dual. Let $T\colon X \to X^*$ be hemicontinuous strongly monotone operator mapping X into X^*. Then the operator T is bijective and T^{-1} is continuous from the strong topology of X^* to the strong topology of X.

Theorem 2.1. *Let A be a hemicontinuous densifying operator defined everywhere on the Hilbert space H and satisfying*

$$(u-v,\, Au-Av) \leqslant \{1-(1+1/\alpha)^2\}\cdot(\|u-v\|^2) \tag{1}$$

for all $u, v \in H$, where $\alpha < -\frac{1}{2}$ is fixed. Suppose that $L(x)$ is a densifying map defined on the ball $D(\|x\| \leqslant r)$ and such that the operator $((1/\alpha)+1)^2\{L(x)+A(\theta)\}$ acts from D into itself, where θ is the zero vector. Then the equation $x = Ax + Lx$ has at least one solution in D.

Proof. Consider the equation $F(x) = x - A(x) + A(\theta)$. Using condition (1) we have

$$\begin{aligned}(u-v,\, Fu-Fv) &= (u-v,\, u-Au-v+Av) = (u-v,\, u-v-(Au-Av)) \\ &= \|u-v\|^2 - (u-v,\, Au-Av) \geqslant \|u-v\|^2 - \{1-(1+1/\alpha)^2\}\cdot\|u-v\|^2 \\ &= \{1-(1-(1+1/\alpha)^2)\}\|u-v\|^2 = (1+1/\alpha)^2\cdot\|u-v\|^2 .\end{aligned} \tag{2}$$

It follows from (2) that F is strongly monotone operator. Now using the monotonicity and hemicontinuity we infer from Theorem B that F has an inverse operator R defined everywhere on H.

Also from (1) we have

$$\|u-v\|\cdot\|Fu-Fv\| \geqslant (1+1/\alpha)^2\|u-v\|^2 .$$

Therefore

$$\|u-v\| \leqslant \left(\frac{\alpha+1}{\alpha}\right)^2\cdot\|Fu-Fv\|. \tag{3}$$

If $F(u) = z_1$, $F(v) = z_2$, then $u = R(z_1)$, $v = R(z_2)$.

Substituting this in (3) we get

$$\|R(z_1)-R(z_2)\| \leqslant (1/\alpha+1)^2\|z_1-z_2\| . \tag{4}$$

Furthermore, since $F(\theta) = \theta$, we have $R(\theta) = \theta$. Now we consider the equation $x = R\{L(x)+A(\theta)\}$. Since $L(x)+A(\theta)$ is densifying continuous operator and R by (4) satisfies a Lipschitz condition with constant $k < 1$. Thus $L(x)+A(\theta)$ is a k-set contraction with $k < 1$. By putting $z_1 = [L(x)+A(\theta)]$, $z_2 = \theta$ in (4) we have

$$\|R\{L(x)+A(\theta)\}\| \leqslant \left(\frac{\alpha+1}{\alpha}\right)^2 \|L(x)+A(\theta)\|. \tag{5}$$

Since by the hypothesis the operator $((1/\alpha)+1)^2\{L(x)+A(\theta)\}$ acts from D into D, it follows from (5) that $R\{L(x)+A(\theta)\}$ also acts from D into D. It follows from Darbo's Theorem [3] that the equation $x=R\{L(x)+A(\theta)\}$ has at least one solution $x_0\in D$.We apply F to the equation $x_0=R\{L(x_0)+A(\theta)\}$ to obtain

$$Fx_0=FR\{L(x_0)+A(\theta)\}\,,\qquad Fx_0=L(x_0)+A(\theta)$$

or

$$x_0-A(x_0)+A(\theta)=L(x_0)+A(0)\,.$$

Thus $x_0=L(x_0)+A(x_0)$. Thus the theorem.

Theorem 2.2. *Let D be a bounded, closed and convex set in a Hilbert space H. Let $T\colon D\to D$ be a densifying map and $A(x)$ be also a densifying map defined on D with values in H. Suppose $\overline{T}(x)=T(x)+A(x)$ acts from $D\to D$ and $I-\overline{T}$ is convex (weakly linear semi continuous on D). Then there exists at least one $x_0\in D$ such that: $x_0=T(x_3)+A(x_0)$.*

Proof. For any fixed numbers $k<1$ define the mapping $F(x)=k[T(x)+$ $+A(x)]$. Then the mapping $F(x)$ is a k-set contraction with $k<1$. Therefore by Darbo's Theorem [**3**] $F(x)$ has a fixed point $x_k\in D$, i.e. $x_k=k[T(x_k)+$ $+A(x_k)]=F(x_k)$.

Let k_n be a sequence of numbers such that $0<k_n<1$ and $k_n\to 1$.

Then: $x_k-T(x_k)-A(x_k)=(k-1)[T(x_k)+A(x_k)]$. Clearly T and A map bounded sets into bounded sets, hence $\|T(x_k)+A(x_k)\|\leqslant M$, a constant for all $k<1$.

Therefore

$$\|x_{k_n}-T(x_{k_n})-A(x_{k_n})\|=\|k_n-1\|\cdot\|T(x_{k_n})+A(x_{k_n})\|\leqslant\|(k_n-1)\|\cdot M\to 0\,.$$

Thus $\inf\limits_{x\in D}\|x-Tx-Ax\|=0$.

Now D being closed, bounded convex subset of a Hilbert space H is also weakly compact. Now since D is weakly compact and $I-\overline{T}$ is weakly lower semicontinuous on D. Therefore $\|x-\overline{T}x\|$ has its infimum on D. i.e. there exists a point $x_0\in D$ such that: $\|x_0-\overline{T}_{x_0}\|=\inf\limits_{x\in D}\|x_0-\overline{T}x_0\|$. But $\inf\|(I-$ $-\overline{T})x\|=\inf\|x-Tx-Ax\|=0$.

Therefore $\|x_0-\overline{T}_0\|=0$. This implies that $\bar{x}_0=\overline{T}x_0$, i.e. $Tx_0+Ax_0=x_0$. Thus the theorem.

Theorem 2.3. *Let D be a nonempty bounded closed convex set containing the origin as interior point in a reflexive Banach space X. Let $A: D \to D$ and $B: D \to D$ be a densifying maps. If $(I - A - B)$ is convex on D. Then there exists at least one $x_0 \in D$ such that $Ax_0 + Bx_0 = x_0$.*

Proof. Since a closed, bounded, convex, subset of a reflexive Banach space is weakly compact, therefore D is weakly compact. Since a convex continuous real valued function in a Banach space is lower semicontinuous. Therefore $\|x - Ax - Bx\|$ is weakly lower semicontinuous. Hence $\|x - \overline{T}x\|$, where $Tx = Ax + Bx$ has its infimum on D i.e. there exists $x_0 \in D$ such that $\|(I - \overline{T})x_0\| = \inf \|(I - T)x\|$.

We need only to show that $\inf \|(I - \overline{T})x\| = 0$. But this follows as in previous theorem by taking $k\overline{T}$, for $0 < k < 1$.

Corollary. *If we take $B = 0$, then we get a result due to Sadovsky* [**10**].

Definition 3.1. Let D be a subset of a Banach space X. Let $f: D \to X$ be continuous map. We call f *locally almost nonexpansive* (LANE) if and only if for all $x \in D$ and $\varepsilon > 0$ there exists a weak neighborhood Nx of x in D such that for all $u, v \in Nx$, $\|f(u) - f(v)\| \leqslant \|u - v\| + \varepsilon$.

Theorem 3.1. *Let G_1, G_2 be two closed, bounded, convex sets of a reflexive Banach space X. Let $f_1: G_1 \to G_2$ be locally almost nonexpansive (LANE) map and $f_2: G_2 \to X$ be a nonexpansive map. Then $f_2 \circ f_1$ is LANE.*

Proof. It is enough to show that for any weakly convergant sequence $\{x_n\}$, $\{y_n\}$, $x_n \to x$ (x_n converges weakly to x), $y_n \to y$ and any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$\|f_2 \circ f_1(x_n) - f_2 \circ f_1(y_n)\| \leqslant \|x_n - y_n\| + \varepsilon , \qquad \text{for any } n \geqslant N .$$

Now $\|f_2 \circ f_1(x_n) - f_2 \circ f_1(y_n)\| \leqslant \|f_1(x_n) - f_1(y_n)\|$, since f_2 is nonexpansive.
Thus $\|f_2 \circ f_1(x_n) - f_2 \circ f_1(y_n)\| \leqslant \|f_1(x_n) - f_1(y_n)\|$.
But f_1 LANE implies

$$\|f_1(x_n) - f_1(y_n)\| \leqslant \|x_n - y_n\| + \varepsilon .$$

Therefore $f_2 \circ f_1$ is LANE.

Theorem 3 2 *Let G_1, G_2 be closed, bounded, convex weakly compact subset of a Banach space X, and $f_1: G_1 \to X$ be a LANE and weakly continuous* *map. Let* $f_2\colon G_2 \to X$ *be a LANE map. Suppose* $f_1(G_1) \subset G_2$. *Then* $f = f_2 \circ f_1$ *is a LANE*

Remark 3.1. In the proof of Theorem 3.2 we will use the following theorem.

Theorem C [**9**]. Let G be a closed, bounded, convex and weakly compact subset of a Banach space X and $f\colon G \to X$ be a continuous map. Then f is LANE if and only if $x_n, y_n \in G$, $x_n \to x$, $y_n \to y$ then there exists subsequences $x_{n_i} \to x$, $y_{n_i} \to y$ such that for any $\varepsilon > 0$ there exists an integer N such that

$$\|f(x_{n_i}) - f(y_{n_i})\| \leqslant \|x_{n_i} - y_{n_i}\| + \varepsilon , \qquad \text{for } n_i \geqslant N .$$

Proof of Theorem 3.2. Suppose $\{x_n\}$, $\{y_n\} \subset G_1$ such that $x_n \to x$ and $y_n \to x$.

Now $\|f_2 \circ f_1(x_n) - f_2 \circ f_1(y_n)\| = \|f_2(f_1(x_n)) - f_2(f_1(y_n))\|$.

Suppose $f_1(x_n) \to u$, $f_1(y_n) \to v$. Then there exists $f_1 x_{n_j}, f_1 y_{n_j}$ such that for any $\varepsilon > 0$ there exists an integer N such that

$$\|f_2(f_1(x_{n_j})) - f_2(f_1(y_{n_j})\| \leqslant \|f_1(x_{n_j}) - f_1(y_{n_j})\| + \varepsilon/2 ,$$

because f_2 is LANE. Now

$$f_1(x_n) \in G_1 \Rightarrow f_1(x_{n_j}) \in G_1 \quad \text{such that} \quad f_1(x_{n_j}) \to u ,$$
$$f_1(y_n) \in G_1 \Rightarrow f_1(y_{n_j}) \in G_1 \quad \text{such that} \quad f_1(y_{n_j}) \to v.$$

Suppose $u \neq v$, then $u - v \neq 0$. Therefore $u - v = \eta > 0$.

$$\|u - v\| = \|F(u - v)\| \quad \text{where } F \in X^*;\ \|F\| = 1 .$$
$$\begin{aligned}\|u - v\| = \|Fu - Fv\| &= \|Fu - F(f_1(x_{n_j}) + F(f_1(x_{n_j}))) - \\ &\quad - F(f_1(y_{n_j})) + F(f_1(y_{n_j})) - F(v)\| \\ &\leqslant \|Fu - F(f_1(x_{n_j}))\| + \|F(f_1(x_{n_j})) - F(f_1(y_{n_j}))\| + \\ &\quad + \|F(f_1(y_{n_j})) - F(v)\| \\ &\leqslant \varepsilon/3 + \|F\| + \|f_1(x_{n_j}) - f_1(y_{n_j})\| + \varepsilon/3 \\ &\leqslant \varepsilon/3 + \varepsilon/3 + \|x_{n_j} - y_{n_j}\| + \varepsilon/3.\end{aligned}$$

Since f_1 is LANE

$$\|u - v\| = \|x_{n_j} - y_{n_j}\| + \varepsilon .$$

References.

[1] L. P. BELLUCE and P. KIRK, *Some fixed point theorems in metric and Banach spaces*, Canad. Math. Bull. **12** (1969), 481-491.

[2] N. E. BROWDEN, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. USA **54** (1965), 1041-1044.

[3] G. DARBO, *Punti uniti in trasformazione a condominio non compatto*, Rend. Sem. Mat. Univ. Padova **24** (1955), 84-92.

[4] D. G. DE FIGUEIREDO, *Topics in nonlinear functional analysis*, Lecture Series 48, University of Maryland 1967.

[5] M. FURI and A. VIGNOLI, *On L-nonexpansive mappings and fixed points*, Rend. Accad. Naz. Lincei **482** (1970), 195-198.

[6] M. A. KRASNOSELSKY, *Two remarks about the method of successive approximations*, Uspehi Mat. Nauk **19** (1965), 123-127.

[7] M. A. KRASNOSELSKY and P. P. ZABREIKO, *Construction of a new fixed point theorem*, Dokl. Akad. Nauk SSSR. **176** (1967), 1223.

[8] K. KWIATOWSKI, *Topology* (I), New York 1966.

[9] R. D. NUSSBAUM, *The fixed point index and fixed point theorem for k-set contractions*, Ph. D. Thesis, University of Chicago 1969.

[10] B. N. SADOVSKY, *On a fixed point principle*, Functional Anal. Appl. **1** (1967), 74-76.

Summary.

In recent years many authors have proved various fixed point theorems for nonlinear operators in Banach and Hilbert spaces. If D is a closed, bounded convex subsets of a Banach space X, then a contraction mapping T, has unique fixed point in D, but a nonexpansive mapping need not. If X is either uniformly convex Banach space or reflective Banach space with normal structure then the existence of a fixed point for nonexpansive mappings have been proven. It was Krasnoselsky who proved the existence of fixed points for sum of two nonlinear operators.

The aim of this paper is to prove some fixed point theorems for sum of two nonlinear operators which generalizes all the results proved till now on this topic. Also first time in this paper an attempt has been made to relate the potential operator with densifying mappings.

* * *