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Continuity conditions and convergence properties in generalized metric spaces. (**)

Introduction.

In this paper we consider the class of H_σ-spaces, which are generalized metric spaces (see $[\mathbf{3}]_2$) and we show that many properties of convergence of sequences of the space correspond to appropriate conditions of continuity on the distance. In particular we show that when the distance is uniformly continuous a completion theorem analogous to the one existing in ordinary metric spaces holds.

The principal aim of the paper is to show that many properties of metric spaces don't essentially depend on the triangular axiom, but on properties of continuity induced by such an axiom; thus one can think that in many problems the triangular axiom can be replaced by the condition of uniform continuity of the distance. This is particular interesting if one considers that uniformly continuous distances are characterized by the following condition:

$$d(x, z) \leqslant \varphi\big(d(x, y)\big) + d(y, z) ,$$

where φ is a real function infinitesimal in zero (see $[\mathbf{3}]_1$).

1. – Let X be a set. In previous papers (see [**2**], $[\mathbf{3}]_1$, $[\mathbf{3}]_2$, [**4**]) we defined a *distance* on X to be a function $d: X \times X \to \boldsymbol{R}_+$ (1) satisfying quite general

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(1) With $\boldsymbol{R}_+$ we indicate the set of real non-negative numbers.

conditions equivalent to the supposition that the family of the discs $\{\sigma_d(x, \varepsilon) | |\varepsilon > 0\}$, with

$$\sigma_d(x, \varepsilon) = \{y \in X | d(x, y) < \varepsilon\},$$

satisfies the axioms for a local basis at a point of a topological space (²).

If d is a distance on X, the pair (X, d) takes the name of *H-space* (see [3]₂). Furthermore, if d verifies the conditions $(a_1), (a_2), \dots, (a_n)$, d is called a $(a_1 a_2 \dots a_n)$*-distance*, and the pair (X, d) an $H_{a_1 a_2 \dots a_n}$*-space*.

From now on we shall introduce in H-spaces various notions existing for ordinary metric spaces, implicitly accepting the usual definitions (³). In particular we shall speak about the *completion* of an $H_{a_1 a_2 \dots a_n}$-space, meaning a complete $H_{a_1 a_2 \dots a_n}$-space with a dense subspace isometric to (X, d).

If (X, d) is an H-space it will be useful to consider $X \times X$ an H-space also; to this purpose we shall introduce in $X \times X$ the distance $\bar{d}$ defined as follows:

$$\bar{d}((x, y), (x', y')) = \max (d(x, x'), d(y, y')),$$

with $x, x', y, y' \in X$. It is easy to verify that $\bar{d}$ induces the product topology in $X \times X$. The interest of this definition lies in the fact that every distance may be seen as a function between H-spaces.

From now on we shall consider only symmetric distances, that is distances satisfying the condition:

$$(\sigma) \qquad d(x, y) = d(y, x)$$

for all $x, y \in X$ (⁴). Note however that some propositions hold for general H-spaces also.

Proposition 1. *In an H_σ-space (X, d) the following conditions are equivalent:*

1) *d is continuous at the points of the diagonal* (⁵), (⁶),

2) *all convergent sequences are Cauchy,*

3) *if $\{x_n\}$ and $\{y_n\}$ are two convergent sequences at two points whose distance is zero, then $d(x_n, y_n) \to 0$.*

(²) So it is clear that every distance characterizes a topology. Conversely not all topologies are induced by some distance; this is true if and only if the topological space is first countable (see [4], Prop. 3.1).

(³) For all definitions which are not explicitly given we refer to [7].

(⁴) A topological space has a topology compatible with a (σ)-distance if and only if it is first countable and semistratifiable (see [5], Cor. 1.4).

(⁵) By the diagonal of X we mean the set $\Delta_X = \{(x, y) \subseteq X \times X | x = y\}$.

(⁶) A topological space has a topology compatible with a distance continuous at the points of the diagonal if and only if it is developable (see [2], Prop. 3).

Proof. 1) $\Rightarrow$ 2). Let $\{x_n\}$ be a sequence convergent to a point x. Since d is continuous at the point (x, x), for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d(x, y) < \delta$ and $d(x, z) < \delta$, then $d(y, z) < \varepsilon$. Since $\{x_n\} \to x$, in relation to δ there exists an integer h such that for every $n > h$ we have $d(x, x_n) < \delta$. Therefore, for every $m, n > h$, we have $d(x_m, x_n) < \varepsilon$, and so $\{x_n\}$ is Cauchy.

2) $\Rightarrow$ 3). Let $\{x_n\}$ and $\{y_n\}$ be two sequences such that $\{x_n\} \to x$, $\{y_n\} \to y$ and $d(x, y) = 0$. By the symmetry of d the filter of the neighborhoods of x is the same as the one for y; therefore we have $\{y_n\} \to x$. If we put $z_{2n} = x_n$ and $z_{2n+1} = y_n$, the sequence $\{z_n\}$ turns out to be convergent to x, so it is Cauchy; and so we have that for every $\varepsilon > 0$ there exists an integer h such that, for all $m, n > h$, $d(z_m, z_n) < \varepsilon$, in particular $d(z_{2n}, z_{2n+1}) < \varepsilon$, and so $d(x_n, y_n) \to 0$.

3) $\Rightarrow$ 1). Let us assume that d is not continuous at a point (x, x). Then we can find an $\varepsilon > 0$ such that for every n there exists a point (y_n, z_n) such that $d(x, y_n) < 1/n$, $d(x, z_n) < 1/n$ and $d(y_n, z_n) > \varepsilon$. Then the sequences $\{y_n\}$ and $\{z_n\}$ clearly converge to x but $\{d(x_n, y_n)\}$ does not converge to zero.

Definition 2. Let (X, d), (X', d') be two H_σ-spaces and let $Y \subset X$. A function $f\colon X \to X'$ is called *uniformly continuous relatively to* Y if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $A \subset X$ such that $A \cap Y \neq \emptyset$ and $\operatorname{diam} A < \delta$ we have $\operatorname{diam}(f(A)) < \varepsilon$ (7).

Consider the set of the sequences of an H_σ-space (X, d) and denote by $\mathscr{R}$ the relation defined as follows:

$$\{x_n\}\mathscr{R}\{y_n\} \qquad \text{if} \qquad d(x_n, y_n) \to 0 .$$

Furthermore, let us recall that (γ) names the following relation: (see [4], p. 104)

$$(\gamma) \qquad \forall \varepsilon > 0,\ \exists \delta > 0,\ \left.\begin{array}{l} d(x, y) < \delta \\ d(y, z) < \delta \end{array}\right\} \Rightarrow d(x, z) < \varepsilon .$$

Then the following proposition holds:

Proposition 3. *In an H_σ-space the following conditions are equivalent:*

1) *d is uniformly continuous relatively to the diagonal,*
2) *d is a (γ)-distance* (8),
3) *the relation $\mathscr{R}$ is transitive.*

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(7) Note that this condition entails both the uniform continuity of the restriction f/Y and the continuity of f at the points of Y.

(8) A topological space has a topology compatible with a $(\sigma\gamma)$-distance if and only if it is pseudo-metrizable (see [4], Prop. 8.7 and 8.8).

Proof. 1) $\Rightarrow$ 2). Let ε be an arbitrary positive number, and let $\delta > 0$ correspond to ε in the definition of uniform continuity relatively to the diagonal. Let $x, y, z \in X$ such that $d(x, y) < \delta$, $d(y, z) < \delta$; the set $\{(y, y), (x, z)\}$ has a diameter smaller than δ and has a non empty intersection with the diagonal, and so we have $d(x, z) < \varepsilon$, and condition (γ) holds.

2) $\Rightarrow$ 3). Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, be three sequences in X such that $d(x_n, y_n) \to 0$ and $d(y_n, z_n) \to 0$. Let $\varepsilon > 0$, and let $\delta > 0$ correspond to ε in the condition (γ). Owing to the given hypotesis there exists an integer h such that for every $n > h$ we have $d(x_n, y_n) < \delta$ and $d(y_n, z_n) < \delta$, and so by (γ) we have, for all $n > h$, $d(x_n, z_n) < \varepsilon$. This shows that $d(x_n, z_n) \to 0$.

3) $\Rightarrow$ 1). Let us suppose that d is not uniformly continuous relatively to the diagonal. Then there exists an $\varepsilon > 0$ such that for every integer n there exists an $A_n \subset X \times X$ with $A_n \cap \Delta_X \neq \emptyset$, $\operatorname{diam} A_n < 1/n$, $\operatorname{diam}\big(d(A_n)\big) > \varepsilon$. From this condition it follows that in every A_n there exist at least two points (x_n, x_n), (y_n, z_n) such that $d(x_n, y_n) < 1/n$, $d(x_n, z_n) < 1/n$, $d(y_n, z_n) > \varepsilon$. So we have that $\{x_n\}\mathscr{R}\{y_n\}$ and $\{x_n\}\mathscr{R}\{z_n\}$ but $\{y_n\}\mathscr{R}\{z_n\}$, being $d(x_n, z_n) > \varepsilon$ for every n.

From now on we shall say that the distance of an H-space (X, d) satisfies *condition* (δ) if d is a continuous function at every point.

Recalling that every H-space is first countable, condition (δ) can be characterized as follows (see [7], Theor. 5.3.4).

Proposition 4. *The distance d of an H_σ-space is continuous if and only if, given two sequences $\{x_n\}$ and $\{y_n\}$ respectively convergent to x and y, we have $d(x_n, y_n) \to d(x, y)$.*

When a $(\sigma\delta)$-distance satisfies condition:

$$(\alpha) \qquad d(x, y) = 0 \Rightarrow x = y\,,$$

a theorem of uniqueness of completion holds.

Proposition 5. *If an $H_{\alpha\sigma\delta}$-space (X, d) has two completions (Y, d'), (Z, d''), and if i and j are the isometric embeddings of X respectively in Y and Z, then there exists an unique isometry $h\colon Y \to Z$ such that $h \circ i = j$.*

Proof. Let $y \in Y$; being Y first countable and $i(X)$ dense in Y there exists a sequence $\{x_n\}$ in X such that $\{i(x_n)\}$ is convergent to y. The sequence $\{j(x_n)\}$ is Cauchy and so it converges to a point $z \in Z$. Put $h(y) = z$. The point z so defined does not depend on the sequence $\{x_n\}$. For, if $\{x'_n\}$ is a sequence such that $i(x_n') \to y$, we have $\{x_n'\}\mathscr{R}\{x_n\}$ and so $\{j(x_n')\}\mathscr{R}\{j(x_n)\}$; but $\{j(x_n')\}$ is Cauchy and so it converges to a point $z' \in Z$; by the continuity of d'' we have:

$$d''(z, z') = \lim d''(j(x_n), j(x_n')) = 0$$

and, owing to (α), it follows that $z = z'$. The function h so defined is an isometry, because if $\{x_n\}$, $\{x_n'\}$ are two sequences such that $i(x_n) \to y$ and $i(x_n') \to y'$, by Proposition 4 we have:

$$d'(y, y') = \lim d'(i(x_n), i(x_n')) = \lim d''(j(x_n), j(x_n')) = d''(h(y), h(y')) .$$

A distance on X is *uniformly continuous* if it verifies the following condition:

$$(\mu) \qquad \forall \varepsilon > 0, \ \exists \delta > 0, \ \left.\begin{array}{l} d(x, z) < \delta \\ d(y, t) < \delta \end{array}\right\} \Rightarrow |d(x, y) - d(z, t)| < \varepsilon .$$

Condition (μ) may be characterized as follows:

Proposition 6. *The distance d of an H_σ-space (X, d) is uniformly continuous if and only if, given four sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{t_n\}$ such that $d(x_n, z_n) \to 0$ and $d(y_n, t_n) \to 0$, we have $|d(x_n, y_n) - d(z_n, t_n)| \to 0$.*

Proof. Let d be uniformly continuous and let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{t_n\}$ be such that $d(x_n, z_n) \to 0$ and $d(y_n, t_n) \to 0$. Let ε be an arbitrary positive number, let $\delta > 0$ correspond to ε in condition (μ); in relation to δ there exists an integer h such that for every $n > h$ we have $d(x_n, z_n) < \delta$ and $d(y_n, t_n) < \delta$. Then we have, for every $n > h$, $|d(x_n, y_n) - d(z_n, t_n)| < \varepsilon$ and so $|d(x_n, y_n) - d(z_n, t_n)| \to 0$.

Conversely, if d is not uniformly continuous, then there exists an $\varepsilon > 0$ such that for every n there are four points x_n, y_n, z_n, t_n such that $d(x_n, z_n) < 1/n$, $d(y_n, t_n) < 1/n$ and $|d(x_n, y_n) - d(z_n, t_n)| > \varepsilon$. From this it follows that $d(x_n, z_n) \to 0$, $d(y_n, t_n) \to 0$, while the sequence $\{|d(x_n, y_n) - d(z_n, t_n)|\}$ does not converge to zero.

For $H_{\sigma\mu}$-spaces one may prove a completion theorem analogous to the one on pseudometric spaces (see [**8**], Cor. 24.5), as can be seen in the following proposition.

Proposition 7. *Every $H_{\sigma\mu}$-space admits a completion.*

Proof. Let (X, d) be an $H_{\sigma\mu}$-space and let $\tilde{X}$ be the set of Cauchy sequences of X. If $\tilde{x} = \{x_n\}$ and $\tilde{y} = \{y_n\}$ are two Cauchy sequences of X, then the sequence $\{(x_n, y_n)\}$ of $\tilde{X} \times \tilde{X}$ is Cauchy also. Moreover, d being uniformly continuous the, sequence $\{d(x_n, y_n)\}$ is Cauchy and so it converges in $\boldsymbol{R}_+$. Thus let us consider the function $\tilde{d}\colon X \times X \to \boldsymbol{R}_+$ defined as follows:

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim d(x_n, y_n)\ .$$

$\tilde{d}$ is obviously symmetric. Our purpose is to prove that $\tilde{d}$ is uniformly continuous. Choose an arbitrary $\varepsilon > 0$, and let $0 < \varepsilon' < \varepsilon$ be. Since d is uniformly continuous, there exists a $\delta > 0$ corresponding to $\varepsilon > 0$ such that

$$d(x, z) < \delta\ , \quad d(y, t) < \delta \ \Rightarrow\ |d(x, y) - d(z, t)| < \varepsilon'\ .$$

Let $\tilde{x} = \{x_n\}$, $\tilde{y} = \{y_n\}$, $\tilde{z} = \{z_n\}$, $\tilde{t} = \{t_n\}$ be four arbitrary elements of $\tilde{X}$ such that

$$\tilde{d}(\tilde{x}, \tilde{z}) = \lim d(x_n, z_n) < \delta\ ; \qquad \tilde{d}(\tilde{y}, \tilde{t}) = \lim d(y_n, t_n) < \delta\ .$$

Then there will exist an integer h such that for every $n > h$ we have $d(x_n, z_n) < \delta$, $d(y_n, t_n) < \delta$ and so $|d(x_n, y_n) - d(z_n, t_n)| < \varepsilon'$. And so we have

$$|\tilde{d}(\tilde{x}, \tilde{y}) - \tilde{d}(\tilde{z}, \tilde{t})| = \lim |d(x_n, y_n) - d(z_n, t_n)| \leqslant \varepsilon' < \varepsilon\ .$$

Thus the function $\tilde{d}$ is uniformly continuous and $(\tilde{X}, \tilde{d})$ is an $H_{\sigma\mu}$-space. The function $i\colon X \to \tilde{X}$ which associates to every $x \in X$ the constant sequence $\{x_n\}$, with $x_n = x$, is obviously an isometry between X and $i(X)$. Let us now prove that $i(X)$ is dense in $\tilde{X}$. Let $\tilde{x} = \{x_n\}$; for every $\varepsilon > 0$ there will exist an integer h such that $d(x_m, x_n) < \varepsilon$ for all $m, n > h$. Therefore we have, for every $n > h$, $\tilde{d}(\tilde{x}, i(x_n)) = \lim_{m \to +\infty} d(x_m, x_n) \leqslant \varepsilon$, and this shows that $\tilde{x}$ belongs to the closure of $i(X)$.

Let us prove that $(\tilde{X}, \tilde{d})$ is complete. Let $\{\tilde{x}_n\}$ be a Cauchy sequence in $\tilde{X}$. Since $i(X)$ is dense in $\tilde{X}$, for every n there exists some $x_n \in X$ such that $\tilde{d}(\tilde{x}_n, i(x_n)) < 1/n$. The sequence $\{x_n\}$ in X is Cauchy. For, $\{\tilde{x}_n\}$ being Cauchy, for every $\varepsilon > 0$ there exists an integer h such that

$$m, n > h \Rightarrow \tilde{d}(\tilde{x}_m, \tilde{x}_n) < \frac{\varepsilon}{2}\ .$$

Furthermore, by the uniform continuity of d, there is a $\delta > 0$ corresponding to $\varepsilon/2$ such that

$$\left.\begin{array}{l}\tilde{d}(\tilde{x}, \tilde{z}) < \delta\\ \tilde{d}(\tilde{y}, \tilde{t}) < \delta\end{array}\right\} \Rightarrow |\tilde{d}(\tilde{x}, \tilde{y}) - \tilde{d}(\tilde{z}, \tilde{t})| < \frac{\varepsilon}{2}\ .$$

Let k be an integer such that $1/k < \min[\delta, 1/h]$. Then, for every $m, n > k$ we have

$$\tilde{d}(\tilde{x}_m, i(x_m)) < \frac{1}{m} < \delta, \qquad \tilde{d}(\tilde{x}_n, i(x_n)) < \frac{1}{n} < \delta,$$

and so

$$|\tilde{d}(\tilde{x}_m, \tilde{x}_n) - \tilde{d}(i(x_m), i(x_n))| < \frac{\varepsilon}{2}.$$

Thus, for every $m, n > k$, we have:

$$d(x_m, x_n) = \tilde{d}(i(x_m), i(x_n)) < \tilde{d}(\tilde{x}_m, \tilde{x}_n) + \frac{\varepsilon}{2} < \varepsilon,$$

and from this, one can see that $\{x_n\}$ is Cauchy.

Let us prove that $\tilde{x}_n \to x$. From the uniform continuity of $\tilde{d}$ it follows that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\tilde{d}(\tilde{x}, \tilde{y}) < \delta, \quad \tilde{d}(\tilde{y}, \tilde{z}) < \delta \;\Rightarrow\; \tilde{d}(\tilde{x}, \tilde{z}) < \varepsilon,$$

for every $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}$. Being $\tilde{x}$ Cauchy, corresponding to $\delta' < \delta$ there exists an integer h such that

$$m, n > h \Rightarrow d(x_m, x_n) < \delta' < \delta.$$

It follows that if $n > h$ we have

$$\tilde{d}(\tilde{x}, i(x_n)) = \lim_{m \to +\infty} d(x_m, x_n) \leqslant \delta' < \delta.$$

Let k be an integer such that $1/k < \min[\delta, 1/h]$. For $n > k$ we have

$$\tilde{d}(\tilde{x}, i(x_n)) < \delta, \qquad \tilde{d}(\tilde{x}_n, i(x_n)) < \frac{1}{n} < \delta,$$

and so $\tilde{d}(\tilde{x}, \tilde{x}_n) < \varepsilon$. So it is proved that $\tilde{x}_n \to x$.

Corollary 8. *An $H_{\alpha\sigma\mu}$-space (X, d) has a completion (X', d') such that if (X'', d'') is another completion of (X, d) there is an isometry $h: X' \to X''$ such that $h \circ i = j$, where i and j are the isometric embeddings of X in X' and X'' respectively.*

Proof. After repeating the construction of $(\tilde{X}, \tilde{d})$ as in Proposition 7, let us consider the relation $\mathscr{R}$ defined on $\tilde{X}$ as above. Obviously we have $\tilde{x}\mathscr{R}\tilde{y}$ if and only if $\tilde{d}(\tilde{x}, \tilde{y}) = 0$. From Proposition 3 we have that $\mathscr{R}$ is an equivalence relation; from Proposition 6 it follows that $\mathscr{R}$ is compatible with $\tilde{d}$, that is if $\tilde{x}\mathscr{R}\tilde{y}$ and $\tilde{z}\mathscr{R}\tilde{t}$ then $\tilde{d}(\tilde{x}, \tilde{z}) = \tilde{d}(\tilde{y}, \tilde{t})$. Then if we put $X' = X/\mathscr{R}$, we can consider the $H_{\lambda\sigma\mu}$-space (X', d') where

$$d'([\tilde{x}], [\tilde{y}]) = \tilde{d}(\tilde{x}, \tilde{y}) .$$

It is obvious that X' is complete. Moreover the function $i'\colon X \to X'$ defined by $i'(x) = [i(x)]$ is an isometric embedding such that $i'(X)$ is dense in X'. So (X', d') is a completion of (X, d).

In order to verify that (X'', d'') satisfies the desired properties it is sufficient to recall Proposition 5.

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Sunto.

Ponendosi nella classe degli H-spazi, che sono spazi metrici generalizzati, si fa vedere che ad alcune proprietà di convergenza corrispondono opportune condizioni di continuità della distanza. In particolare si dimostra che quando la distanza è uniformemente continua vale un teorema di completamento analogo a quello degli ordinari spazi metrici.

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