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Uniqueness structure and parametrization for a class of a functional equation's solutions. (**)

1. – An interesting question in functional equations is to uniquely define a solution of a functional equation when its values are known in a set of points U.

It is clear the smaller is the set U, the more interesting the problem is. This question was deeply studied for the continuous solutions of the functional equation

$$f[G(x, y)] = H[f(x), f(y); x, y]$$

(see for instance **[1]**-**[9]**$_3$, **[10]**$_1$, **[10]**$_2$).
In this paper we shall generalize some of the results known till now.

In **[9]**$_4$ and **[9]**$_5$, after having defined a « global [local] uniqueness structure » for a functions' class, we got some sufficient conditions to ensure such a structure.

We shall use the same notations as in these papers, and we shall recall only those results with which we shall work.

2. – Let E be a Hausdorff space without isolated points and N a set. Denote by $\mathscr{H}$ a class of functions $f: E \to N$, and, for every $f_1, f_2 \in \mathscr{H}$, let $S = \{x \in E: f_1(x) = f_2(x)\}$.

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Definition ([**9**]$_4$. $\mathscr{H}$ has a global [local] uniqueness structure if there exists a nowhere dense set $U \subset E$, such that, for every $f_1, f_2 \in \mathscr{H}$, $S \supset U$ implies $S = E$ [$S^0 \neq \emptyset$]. Every set U with such a property is a global [local] uniqueness set for $\mathscr{H}$.

Theorem A ([1]). *Let N be a Hausdorff space, E a connected and locally connected space and $\mathscr{H} \subset \mathscr{C}(E, N)$. Suppose that, for every $f_1, f_2 \in \mathscr{H}$ with $S \neq \emptyset$ and $E - S \neq \emptyset$, there exist a topological space T_1, a connected subset $T \subset T_1$ and a function $F\colon T_1 \times E \to E$, continuous in each variable, such that:*

1) *for every $x \in E$, $F^x(T_1) = E$;*

2) *for every $x \in E - S$, $F^x(T) \subset E - S$ and $\bigcup_{x \in E - S} F^x(T) = E - S$;*

3) *for every $x \in S$, $F^x(T_1 - T) \subset E - S$.*

Then, if V is a closed nowhere dense set such that $E - V$ isn't connected and $x_0 \in E - V$, $U = V \cup \{x_0\}$ is a global uniqueness set for $\mathscr{H}$.

Theorem B ([2]). *Let N be a Hausdorff space, E a metric space in which balls are connected, and $\mathscr{H} \subset \mathscr{C}(E, N)$. Suppose that, for every $f_1, f_2 \in \mathscr{H}$ with $E - S \neq \emptyset$, there exist a connected space T, a constant k $(0 < k < 1)$ and a function $F\colon T \times E \to E$, continuous in each variable, such that:*

a) *$F(t, y) \in E - S$ iff $(t, y) \in T \times (E - S)$;*

b) *for every $x \in S$ and $y \in E$, there exists $t \in T$ such that $O_t^+(y)$ or $O_t^-(y)$ converges to x so that $O_t^+(y) \cap D(y, kd(x, y)) \neq \emptyset$ or respectively $O_t^-(y) \cap D(y, kd(x, y)) \neq \emptyset$.*

Then $\mathscr{H}$ has a global uniqueness structure. Furthermore, if V is a closed nowhere dense set such that $E - V$ isn't connected and $x_0 \in E - V$, then $U = V \cup \{x_0\}$ is a global uniqueness set for $\mathscr{H}$.

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(1) this Theorem is an obvious consequence of Corollary 1 in [**9**]$_5$ and of Theorem 1 in [**9**]$_4$, when we consider the case $\Omega = T_1 \times E$.

(2) this Theorem is an obvious consequence of Theorem 1 in [**9**]$_4$ and Theorem 5 in [**9**]$_5$.

3. – Let us consider the functional equations' class

$$(*) \qquad f[G(x, y)] = H[f(x), f(y); x, y],$$

where $G: E \times E \to E$, $f: E \to N$, $H: N \times N \times E \times E \to N$.

Throughout this paper we suppose also N a Hausdorff space and in particolar $\mathcal{H} = \{f: E \to N,\ f \text{ continuous solution of } (*)\}$.
Now we can prove the following

Theorem 1. *Let us suppose that*

a) *E is connected and locally connected;*

b) *for every $u \in E$, G_u and G^u are continuous and surjective;*

c) *H is injective in the first and second variable;*

d) *for every $f_1, f_2 \in \mathcal{H}$, S is connected.*

Then, if V is a closed nowhere dense set the complement of which is not connected and $x_0 \in E - V$, $V \cup \{x_0\}$ is a global uniqueness set for $\mathcal{H}$.

Proof. We prove that the hypotheses of Theorem A are satisfied with $T_1 = E$, $T = S$ and $F = G$. The hypothesis 1) is satisfied because G^u is surjective. Let now $x \in S$ and $t \in E - S$: the hypothesis 3) is satisfied because H is injective in the first variable. Analogously, if $x \in E - S$ and $t \in S$, by the injectivity of H in the second variable, we have $F(t, x) \in E - S$; but $t, x \in S$ imply $F(t, x) \in S$ and so, by the surjectivity of G_u we have $F_t(E - S) = E - S$ for every $t \in E - S$. Therefore the hypothesis 2) is satisfied.

Remark 1. If $F(V \times V) \not\subset V$, then V itself is a global uniqueness set. $S \supset V$ implies indeed $S \supset F(V \times V)$ and therefore there exists $x_0 \in F(V \times V) - V$; now, if f_1 and f_2 are equal on V, they are equal on $V \cup \{x_0\}$ and therefore are identical.

Remark 2. An analogous theorem is also true for the more general class of functional equation

$$f[G(x, y)] = K[x, y, t, u, f(x), f(y), f(t), f(u), f(L(t, u))]$$

if there exist two points $t_0, u_0 \in S$ such that $L(t_0, u_0) \in S$.

This remark is also true for the following Theorem 2.

As we have seen, to apply Theorem 1 we have to ensure S is a connected set. The following lemma gives a sufficient condition under which S is connected.

Lemma 1 (3). *Consider the functional equation* (*) *where E is a complete metric space and H is an injective function in the second variable. If there exists k, $0<k<1$, such that, for every $x, y \in E$, $x \neq y$,*

$$O_x(y) \cap D(x, kr) \cap D(y, kr) \neq \emptyset \ (^4)\,, \qquad \textit{where } r = \mathrm{d}(x, y)$$

then S is connected (moreover arcwise connected).

Proof. By the injectivity of H, for every $x, y \in S$, $O_x(y) \subset S$. The hypothesis on $O_x(y)$ implies therefore that, for every $x, y \in S$, there exists $z \in S$ such that $d(z, x)$, $d(z, y) < kd(x, y)$. Let $\varphi\colon S \times S \to S$ a function which associate such a z to every pair (x, y). We shall show in a classic way how to construct a continuous function $\gamma\colon [0, 1] \to S$ such that $\gamma(0) = x$, $\gamma(1) = y$. Let, for every $n \geqslant 0$, $D_n = \{m/2^n, 0 \leqslant m \leqslant 2^n\}$ and $D = \bigcup_0^\infty D_n$. We define γ_n on D_n in this iterative way.

Let $\gamma_0(0) = x$ and $\gamma_0(1) = y$. Now, after having defined γ_n on D_n, we define γ_{n+1} on D_{n+1} in this way

$$\gamma_{n+1/D_n} = \gamma_n\,, \qquad \gamma_{n+1}\big((2h+1)/2^{n+1}\big) = \varphi\big(\gamma_n(h/2^n), \gamma_n\big((h+1)/2^n\big)\,.$$

Then, for every $t \in D$, we put $\gamma(t) = \lim_n \gamma_n(t)$.

Now we prove that γ is uniformly continuous in D. If x_k and x_{k+1} are two consecutive elements of D_n, we have indeed

$$d(x_k, x_{k+1}) = 1/2^n\,, \qquad d\big(\gamma(x_k), \gamma(x_{k+1})\big) < k^n d(x, y)\,;$$

moreover, if $t, u \in D \cap [x_k, x_{k+1}]$ then $d\big(\gamma(t), \gamma(u)\big) < \big(2/(1-k)\big) k^n d(x, y)$ (5). Let now $\delta > 0$ and $\bar{n}$ an integer such that $\delta < 1/2^{\bar{n}+1}$; if $t, u \in D$ and $d(t, u) < \delta$, they have to be in an interval $[x_k, x_{k+1}]$ where $x_k, x_{k+1} \in D_{\bar{n}}$ and are conse-

(3) A similar result, in a particular case, can be found in [**6**].

(4) Here $O_x(y) = \{z \in E\colon G_x^n(z) = G_x^m(y)$ for a pair of integers $m, n \geqslant 0\}$ and G_x^k is the k-iterate of G_x.

(5) See an analogous argument in [**9**]$_3$, Theorem 6.

cutive. Therefore

$$d(\gamma(t), \gamma(u)) < \frac{2}{1-k} k^{\bar{n}} d(x, y) .$$

Since the second term can be choosen arbitrary small if we take $\bar{n}$ sufficiently large, γ is uniformly continuous on D. A classical theorem lets us conclude that $\gamma: D \to E$ can be uniquely extended to a continuous function $\gamma: [0, 1] \to E$ (moreover to an uniformly continuous function). The proof is now complete.

Remark 3. If E is a compact metric space and, for every $x, y \in E$, $x \neq y$,

$$O_x(y) \cap D(x, r) \cap D(y, r) \neq \emptyset , \qquad r = d(x, y) ,$$

then S is connected.

On the contrary, we should have $S = S_1 \cup S_2$ with $S_1, S_2 \neq \emptyset$, compact and $S_1 \cap S_2 = \emptyset$. But, if $\delta = \text{dist}(S_1, S_2)$, there exists $x_i^* \in S_i$ such that $0 < d(x_1^*, x_2^*) = \delta$. But, by our hypothesis, there exists $z \in O_{x_1^*}(x_2^*)$ such that $d(z, x_2^*)$, $d(z, x_1^*) < \delta$ and this is impossible.

Remark 4. The result we got in the Remark 3 is still true if we weaken the hypothesis on the compactness of E, asking that E is the union of a sequence of compact sets E_n where $E_n \subset E_{n+1}$ for every n and $G(E_n \times E_n) \subset E_n$. In this case $S_n = S \cap E_n$ is connected and S is also connected because it is the union of a monotonic sequence of connected sets.

From Lemma 1 and Theorem B we have the following

Theorem 2. *Let us suppose*:

1) *E is a complete metric space in which balls are connected*;

2) *G is a continuous function in each variable*;

3) *H is injective in the second variable*;

4) *there exists k, $0 < k < 1$, such that , for every $x, y \in E$, $x \neq y$, $O_x^+(y) \cap D(x, kr) \cap D(y, kr) \neq \emptyset$ or $O_x^-(y) \cap D(x, kr) \cap D(y, kr) \neq \emptyset$ $(r = d(x, y))$.*

Then, if V is a closed nowhere dense set the complement of which is not connected and $x_0 \in E - V$, $V \cup \{x_0\}$ is a global uniqueness set for $\mathcal{H}$.

We have only to assume $T = S$ and $F = G/S \times E$.

Remark 5. The hypotheses on E, $O_x^+(y)$ and $O_x^-(y)$ can be weakened as in Remark 3 and 4.

We shall now give some conditions under which we can get a partial or total parametrization of the solutions' class of $(*)$.

Definition. A set $W \subset E$ is *G-invariant* if $G(W \times W) \subset W$.

It's obvious that if a set W is G-invariant, we can study the functional equation $(*)$ only on W instead of studying it on E. In particular S is G-invariant and, if the hypotheses of Lemma 1 are satisfied, S is also a con-connected set.

Thinking of the previous results we can prove immediately the following

Corollary 1. *Let us suppose that there exists a finite or infinite family of closed G-invariant sets Z_n, $n \geqslant 0$, such that Z_n is nowhere dense in Z_{n+1} and, for every $n \geqslant 1$, the functional equation $(*)$, with $E = Z_n$, satisfies the hypotheses of Theorem 1 or 2.*

If, for every $n \geqslant 0$, $Z_n \subset Z_{n+1}$ and $Z_{n+1} - Z_n$ is not connected, then, for every $f_1, f_2 \in \mathscr{H}$, from $S \supset Z_0 \cup \big(\bigcup_{n \geqslant 0} \{y_n\}\big)$, where $y_n \in Z_{n+1} - Z_n$, it follows $S \supset \bigcup_{n \geqslant 0} Z_n$.

Moreover, if $E = \bigcup_{n \geqslant 0} Z_n$, then $Z_0 \cup \big(\bigcup_{n \geqslant 0} \{y_n\}\big)$ is a global uniqueness set for $\mathscr{H}$.

Proof. We can indeed prove by induction that $S \supset Z_n$ for every $n \geqslant 0$.

Example. Let $E = R^3$, Z_2 is the spherical surface with center in 0 and radius 1, Z_1 is the circle intersection of Z_2 with a plane through the origin and $Z_0 = \{x_0, y_0\}$ where $x_0, y_0 \in Z_1$. Then, if $y_1 \in Z_2 - Z_1$, $y_2 \in R^3 - Z_2$, Z_1, Z_2 are G-invariant and the functional equation $(*)$ satisfies the hypotheses of Theorem 1 or 2 on Z_1, Z_2 and R^3, the set $\{x_0\} \cup \{y_0\} \cup \{y_1\} \cup \{y_2\}$ is a global uniqueness set for $\mathscr{H}$.

Corollary 2. *Let E a Hausdorff topological vector space on R and $\{u_n\}$, $n > 0$, a topological basis of E(⁶). If, for every $n > 0$, the subspace $V_n = V(u_1, u_2, \ldots, u_n)$ generated by the first n vectors $u_1, u_2, \ldots, u_n$ is G-invariant and the functional equation $(*)$ satisfies the hypotheses of Theorem 1 or 2 on V_n, then $\{0\} \cup \big(\bigcup_{n \geqslant 0} \{u_n\}\big)$ is a global uniqueness set for $\mathscr{H}$.*

Proof. As $\{u_n\}$ is a topological basis of E and S is a closed set, we have only to prove that, for every $n > 0$, $V_n \subset S$.

(⁶) A set $\{u_\alpha\}_{\alpha \in I}$ is said *a topological basis of E* if the vectors u_α are linearly independent and the subspace generated by $\{u_\alpha\}_{\alpha \in I}$ is dense in E.

We go on by induction. The property is true for $n = 1$ because $V_1 = V(u_1)$ is homeomorphic to R, $\{0\} \subset S \cap V_1$ is a set the complement of which is not connected in V_1 and $S \cap V_1 \supset \{0\} \cup \{u_1\}$.

Let us suppose $S \supset V_n$. As V_n is nowhere dense in V_{n+1}, $V_{n+1} - V_n$ is not connected and $u_{n+1} \in S \cap (V_{n+1} - V_n)$; then we have $S \supset V_{n+1}$. The proof is so complete.

Corollary 3. *Let E be a Hausdorff topological vector space on R and $\{u_\alpha\}$ a topological basis of E. If every finite dimension subspace V, generated by $\{u_\alpha\}$ is G-invariant and functional equation* (*) *satisfies the hypotheses of Theorem* 1 *or* 2, *then* $\{0\} \cup \big(\bigcup_\alpha u_\alpha\big)$ *is a global uniqueness set for* $\mathscr{H}$.

Proof. This proof is analogous to that of Corollary 2. We go on by induction, and we show that S contains every finite dimension subspace.

Remark 6. Note that in Corollary 3 we consider all finite dimension subspaces while in Corollary 2 we have only to consider those generated by vectors $\{u_i\}$, $1 \leqslant i \leqslant n$.

Remark 7. Let ξ be a homeomorphism of a Hausdorff real topological vector space E_1 on a topological space E such that, using the same notations as in Corollary 2 or 3, $\xi(V_n)$ $[\xi(V)]$ are G-invariant and the functional equation (*) satisfies the hypotheses of Theorem 1 or 2 on them.

Then $\xi(0) \cup \big(\bigcup_n \xi(u_n)\big)$ $[\xi(0) \cup \big(\bigcup_\alpha \xi(u_\alpha)\big)]$ is a global uniqueness set for $\mathscr{H}$. In this case it is indeed sufficient to consider the functional equation

$$g\big(G_1(u, v)\big) = H\big(g(u), g(v); \xi(u), \xi(v)\big)$$

where $G_1\colon E_1 \times E_1 \to E_1$ is defined in such a way:

$$G_1(u, v) = \xi^{-1} G\big(\xi(u), \xi(v)\big) \qquad \text{and } g = f \cdot \xi \,.$$

For this equation the hypotheses of Theorem 1 or 2 are satisfied. Therefore, if $S_1 = \{u \in E_1\colon g_1(u) = g_2(u)\} \supset \{0\} \cup \big(\bigcup_\alpha \{u_\alpha\}\big)$, it follows $S_1 = E_1$ and then $S = E$.

Example. An easy case in which we can apply the Corollary 1 or 2 is the following one (7).
Let $E = R^n$ and $G\colon R^n \times R^n \to R^n$ defined in this way:

(7) See an analogous result by Ng, [**8**]$_2$.

if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, $G(x, y) = (z_1, \ldots, z_n)$ is such that, if $x_k = y_k = 0$, $i \leqslant k \leqslant n$, then $z_k = 0$, $i \leqslant k \leqslant n$.

If $i_1, \ldots, i_n$ are the unit vectors and $u_1, \ldots, u_n$ are linearly independent vectors such that $V_k = V(u_1, \ldots, u_k) = V(i_1, \ldots, i_k)$, $1 \leqslant k \leqslant n$, then V_k are G-invariant.

It's sufficient now to suppose that G satisfies the hypotheses of Theorem 1 or 2 in every V_k, $1 \leqslant k \leqslant n$ to prove that $\{0\} \cup (\bigcup_{k=1}^{n} \{u_k\})$ is a global uniqueness set.

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Sunto.

In questa Nota, partendo da alcuni risultati sulle «strutture d'unicità» in classi di funzioni, si studia il problema di individuare univocamente una soluzione continua della equazione funzionale $f[G(x, y)] = H[f(x), f(y); x, y]$ *a partire dalla conoscenza dei suoi valori su un prefissato insieme di punti* U.

I risultati ottenuti generalizzano quelli attualmente noti.

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