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Cyclic ideals in order. (**)

Introduction.

Let A be a Dedekind domain, K its quotient field and Σ a finite dimensional central K-algebra. An A-order is a subring Ω of Σ containing A such that $K \otimes_A \Omega \cong \Sigma$.

Assume that Γ and $\Lambda \subseteq \Gamma$ are two A-orders in Σ. It is well known that if μ is a unit in Σ, then $\Lambda\mu$ is Λ-projective. The primary purpose of this paper is to investigate the Λ-projectivity of modules of the form $n = \bigoplus_{i=1}^{r} \Lambda\mu_i$ where the μ_i are in Λ but not necessarily units in Σ. We shall be particularly interested in condition involving the knowledge that $\bigoplus_{i=1}^{r} \Gamma\mu_i$ is Γ-projective. Finally we will consider hereditary orders and matrix rings.

1. - Cyclic ideals and idempotents.

Assume Σ is a ring with unity and for μ in Λ, define $(0:\mu)_\Lambda = \{x \in \Lambda : x\mu = 0\}$.

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Lemma 1.1. *For* $\{\mu_i\}_{i=1}^r$ *a subset of* Λ, *let* $M = \bigoplus_{i=1}^n \Lambda\mu_i$. *The following conditions are equivalent.*

a) M *is* Λ*-projective.*

b) *For all* i, $(0:\mu_i)_\Lambda$ *contains an idempotent.*

c) *For all* i, $(0:\mu_i)_\Lambda$ *is* Λ*-projective.*

d) *For all* i, $(0, \mu_i)_\Lambda$ *is a direct summand of* Λ.

Proof. Define the map $\mu: \Lambda \to \Lambda\mu$ by right multiplication and consider the exact sequence

$$0 \to (0:\mu)_\Lambda \overset{i}{\to} \Lambda \overset{\mu}{\to} \Lambda\mu \to 0, \tag{1}$$

where i denotes the natural injection. Hence $\Lambda\mu$ is Λ-projective if and only if it is Λ-isomorphic to a direct summand of Λ and $(0:\mu)_\Lambda$ is the other direct summand. Since Λ has an identity, we have that $(0:\mu)_\Lambda$ is a direct summand of Λ if and only if it is generated by an idempotent.

The equivalence of a), b), c) and d) now follows from the fact that $M = \bigoplus_{i=1}^r \Lambda\mu_i$ is Λ-projective if and only if each summand is Λ-projective ([**4**], p. 382).

Theorem 1.2. *Let* Λ *be a ring and* M *a left* Λ*-module of the form* $M = \bigoplus_{i=1}^r \Lambda\mu_i$. *Then* M *is a projective* Λ*-module if and only if there is a left* Λ*-module of the form* $J = \bigoplus_{i=1}^r \Lambda l_i$ *such that the* l_i *are idempotents in* Λ *and* $\Lambda\mu_i$ *is* Λ*-isomorphic to* Λl_i *by the map* $l_i \to l_i\mu_i$, $i = 1, \ldots, r$.

Proof. For each i $(i = 1, \ldots, r)$ there is an exact sequence

$$0 \to (0:\mu_i)_\Lambda \to \Lambda \to \Lambda\mu_i \to 0. \tag{2}$$

Since finite direct sum is an exact functor, r sequences in (2) give rise to the exact sequence

$$0 \to \bigoplus_{i=1}^r (0:\mu_i)_\Lambda \to \Lambda^{(r)} \to \bigoplus_{i=1}^r \Lambda\mu_i = M \to 0.$$

By Lemma 1.1, M is Λ-projective if and only if $(0:\mu_i)_\Lambda$ is generated by an idempotent f_i in Λ $(i=1,\dots,r)$. Let $l_i = 1-f_i$. Then $\Lambda l_i \cong \Lambda/\Lambda f_i$ and $\Lambda\mu_i \cong \cong \Lambda/\Lambda f_i$. Set $J=\bigoplus_{i=1}^{r} \Lambda l_i$.

Conversely, if such a Λ-module J exists with $\Lambda l_i \cong \Lambda\mu_i$ $(i=1,\dots,r)$, then since Λl_i is Λ-projective, the $\Lambda\mu_i$, and hence $M=\bigoplus_{i=1}^{r}\Lambda\mu_i$, are Λ-projective.

To see that $\Lambda\mu_i$ is Λ-isomorphic to Λl_i, note that $\Lambda = \Lambda l_i \oplus \Lambda f_i$. Then

$$\Lambda\mu_i = \Lambda l_i\mu_i \oplus \Lambda f_i\mu_i = \Lambda l_i \mu_i\,.$$

Hence, $\lambda l_i \to \lambda l_i \mu_i$ is an isomorphism.

Lemma 1.3. *Let Λ be a ring with unity and M a left Λ-module of the form $M=\bigoplus_{i=1}^{r}\Lambda\mu_i$. Set $\mu=\mu_1+\dots+\mu_r$. The following statements are equivalent.*

(a) $M=\Lambda\mu$.

(b) *For every i, there is an x_i in Λ which is in every $(0:\mu_j)_\Lambda$ for $j\neq i$ and is not in $(0:\mu_i)_\Lambda$ and such that $x_i\mu_i=\mu_i$.*

(c) $\Lambda\mu\cap\Lambda\mu_i=\Lambda\mu_i$ *for each i.*

Proof:

(a) implies (b). If $M=\Lambda\mu$, then μ_i is an element of $\Lambda\mu$, for all i, and there is an element x_i in Λ such that $\mu_i = x_i(\mu_1+\dots+\mu_i+\dots+\mu_r)$.

Hence $\mu_i = x_i\mu_i+\dots+x_i\mu_i+\dots+x_i\mu_r$ and $x_i\mu_i=\mu_i$ since the sum is direct. Thus x_i is in $(0:\mu_j)_\Lambda$ for $j\neq i$, and x_i is not in $(0:\mu_i)_\Lambda$.

(b) implies (c). If such x_i exist, then μ_i is in $\Lambda\mu$ for all i. Hence $\Lambda\mu_i$ is contained in $\Lambda\mu$ and $\Lambda\mu_i\cap\Lambda\mu=\Lambda\mu_i$.

(c) implies (a). $\Lambda\mu\cap\Lambda\mu_i=\Lambda\mu_i$ implies that μ_i is in $\Lambda\mu$ and so $M\subseteq\Lambda\mu$. The inclusion $N\supseteq\Lambda\mu$ is obvious.

2. - Change of rings.

Let $\Gamma\supseteq\Lambda$ be two A-orders in the central K-algebra Σ and again assume $M=\bigoplus_{i=1}^{r}\Lambda\mu_i$. The set $\{\mu_i\}$ is a generating set for M over Λ. We let ΓM be the Γ-module consisting of elements of the form $\sum_{j=1}^{m} \gamma_j x_j$ with x_j in Λ; then it is clear that $\{\mu_i\}$ is also a generating set for ΓM and $\Gamma M = \bigoplus_{i=1}^{r} \Gamma\mu_i$. Further, it is clear that the set $\{\mu_i\}$ can be a minimal generating set for M over Λ and not be a minimal generating set for ΓM over Γ.

This section will be concerned with conditions that insure M is Λ-projective when it is known that ΓM is Γ-projective. It is known that ΓM is Γ-projective if Γ is maximal ([**5**], p. 5).

By Theorem 1.2, $M = \bigoplus_{i=1}^{r} \Lambda\mu_i$ is Λ-projective if and only if M is Λ-isomorphic to a left Λ-module $J = \bigoplus_{i=1}^{r} \Lambda l_i$ with l_i an idempotent in Λ for all i. Call the set $\{l_1, l_2, \ldots, l_r\}$ the *set of idempotents associated with* M.

We will now consider a relationship between the idempotent associated with M and the idempotents associated with ΓM when M is Λ-projective.

We note first that if M is Λ-projective, then ΓM is Γ-projective. To see this, it is sufficient to consider the case where M is free, and for this it is sufficient to consider the case where $M = \Lambda$. But then $\Gamma M = \Gamma$, which is Γ-projective.

Theorem 2.1. *Let $\Gamma \supseteq \Lambda$ be two orders and let $M = \bigoplus_{i=1}^{r} \Lambda\mu_i$ be a left Λ-module.*

If M is Λ-projective,

(a) *There is a set of idempotents in Λ, $\{f_i\}$ such that $(0:\mu_i)_\Lambda = \Lambda f_i$ and $(0:\mu_i)\Gamma = \Gamma f_i$.*

(b) *For $l_i = 1 - f_i$, $\Lambda l_i \cong \Lambda\mu_i$ as Λ-modules and $\Gamma l_i \cong \Gamma\mu_i$ as Γ-modules.*

Conversely, if ΓM is Γ-projective, then letting y_i be the idempotents such that $(0:\mu_i)\Gamma = \Gamma y_i$, we have that M is Λ-projective if and only if there is a set $\{\gamma_i\}$, $i = 1, \ldots, r$, in Γ such that $\gamma_i y_i$ is idempotent in Λ with $\Lambda\gamma_i y_i = (0:\mu_i)_\Lambda$.

Proof. If $\Lambda\mu$ is Λ-projective, then by Lemma 1.1, $(0:\mu)_\Lambda$ is generated by an idempotent f of Λ. Set $l = 1 - f$. Clearly $(0:\mu)_\Lambda = (0:l)_\Lambda$. Now let γ be in $(0:\mu)\Gamma$; then $\gamma = \gamma l + \gamma f$ in $\Gamma = \Gamma l \oplus \Gamma f$. By ([**3**], sect. 4), there is an $a \neq 0$ in Λ such that $a\Gamma \subseteq \Lambda$. Then $a\gamma$ is in $(0:\mu)_\Lambda = (0:l)_\Lambda$, so $a\gamma l = 0$. Since Γ is Λ-torsion free, $\gamma l = 0$, so $\gamma = \gamma f$ and $(0:\mu)\Gamma = \Gamma f$.

Part (a) follows with the above ideas applied to each i.

By Theorem 1.2, $\Lambda l_i \cong \Lambda\mu_i$ and the isomorphism extends to $\Gamma l_i \cong \Gamma\mu_i$. Hence (b) is true.

If M is Λ-projective, let $\{l_1, l_2, \dots, l_r\}$ be the idempotents associated with M and define $f_i = 1 - l_i$. By the first paragraph of this proof, $(0 : \mu_i)\Gamma = \Gamma f_i$ and $\Gamma f_i = \Gamma y_i$. Let $\{\gamma_i\}$ be the set of elements such that $f_i = \gamma_i y_i$. Conversely if such a set of γ_i exists, then by Lemma 1.1, M is Λ-projective.

We will now approach the problem from a slightly different viewpoint. We assume that ΓM is Γ-projective and we let $\{l_1, \dots, l_r\}$ be the set of idempotents associated with ΓM. Set $f_i = 1 - l_i$. We consider the set $\{\Lambda l_i\}$. Since l_i is not necessarily in Λ, Λl_i may not be Λ-projective; however we can prove the following.

THEOREM 2.2. *Let $\Gamma \supseteq \Lambda$ be two A-orders in Σ and let $M = \bigoplus_{i=1}^{r} \Lambda \mu_i$ be a left Λ-module. Assume ΓM is Γ-projective, and let $\{l_1, \dots, l_r\}$ be the set of idempotents associated with ΓM. Then the following statements are equivalent.*

(a) *M is Λ-projective.*

(b) *For $i = 1, \dots, r$, there is an idempotent x_i in Λ such $\Lambda l_i \cong \Lambda x_i$.*

(c) *Λl_i is Λ-projective, $i = 1, \dots, r$.*

PROOF. Set $f_i = 1 - l_i$, $i = 1, \dots, r$.

(a) implies (*b*). Let M be Λ-projective, and let $\{x_1, \dots, x_r\}$ be the set of idempotents associated with M. Set $y_i = 1 - x_i$. By Theorem 2.1 $\Gamma y_i = \Gamma f_i$ since $(0 : \mu_i)_\Lambda = \Lambda y_i$. Thus $\Gamma = \Gamma l_i \oplus \Gamma f_i$ implies $\Gamma x_i \cong \Gamma l_i x_i \oplus \Gamma f_i x_i = \Gamma l_i x_i$; hence Γl_i is Γ-isomorphic to Γx_i by the map $\theta_i : l_i \to l_i x_i$.
Restricting θ_i to Λl_i gives the desired isomorphism.

(b) implies (c). This is clear since x_i is an idempotent in Λ and Λx_i is Λ-projective.

(c) implies (a). Since $\Gamma M = \bigoplus_{i=1}^{r} \Gamma \mu_i$, $(0 : \mu_i)_\Lambda = (0 : \mu_i)\Gamma$, so

$$(0 : \mu_i)_\Lambda = \Lambda \cap (0 : \mu_i)_\Gamma = \Lambda \cap (0 : l_i)_\Gamma = (0 : l_i)_\Lambda .$$

Thus for each i, there is a commutative diagram

$$\begin{array}{ccccccc} 0 \to & (0 : l_i)_\Lambda & \xrightarrow{j} & \Lambda & \xrightarrow{(l_i)_r} & \Lambda l_i & \to 0, \\ & \downarrow \alpha_i & & \downarrow \beta & & \downarrow \eta_i & \\ 0 \to & (0 : \mu_i)_\Lambda & \xrightarrow{j} & \Lambda & \xrightarrow{(\mu_i)_r} & \Lambda \mu_i & \to 0 , \end{array}$$

where j is the injection map, $(l_i)_r$ and $(\mu_i)_r$ denote right multiplication by l_i and μ_i respectively, α_i and β are the identity maps, and η_i will now be defined.

Since Λl_i is Λ-projective, there is a Λ-homomorphism $h_i\colon \Lambda l_i \to \Lambda$ such that $(l_i)_r h_i = 1$. Define $\eta_i = (\mu_i)_r \beta h_i$. From the hypothesis of (c) it is easy to show that η_i is an isomorphism. Hence $\Lambda\mu_i$ and M are Λ-projective.

The next result will be very helpful in checking when Λl_i is Λ-projective.

Theorem 2.3. *Let $\Gamma \supseteq \Lambda$ be two A-orders in Σ and let l be an idempotent in Γ. The following conditions are equivalent.*

(a) *Λl is Λ-projective.*

(b) *There is a γ in Γ such that $\gamma(1-l) = \gamma f$ is idempotent in Λ with $(0:l)_\Lambda = \Lambda\gamma f$ for $1-l=f$.*

(c) *Λl is Λ-isomorphic to Λx for x an idempotent in Λ.*

Proof. (a) is equivalent to (b). By ([**3**], sec. 4), there is an element a in A such that al is in Λ. Now Γl is Γ-isomorphic to Γal, so Γal is Γ-projective. Hence taking al as μ_i and l as l_i in Theorem 2.2, we have that Λl is Λ-projective if and only if Λal is Λ-projective. By Theorem 2.1, Λal is Λ-projective if and only if there is a γ in Γ such that $\gamma(1-l)$ is an idempotent in Λ and $(0:al) = (0:l)_\Lambda = \Lambda\gamma(1-l) = \Lambda\gamma f$ for $1-l=f$.

(a) is equivalent to (c). Certainly, if Λl is Λ-isomorphic to Λx for x an idempotent in Λ, then Λl is Λ-projective. Conversely, if Λl is Λ-projective, consider the exact sequence $0 \to (0:l)_\Lambda \xrightarrow{i} \Lambda \underset{h}{\overset{g}{\rightleftarrows}} \Lambda l \to 0$, where $gh=1$. Then $\Lambda = h(\Lambda l) \oplus (0:l)_\Lambda$, so $(0:l)_\Lambda$ is generated by an idempotent y. Let $x = 1-y$. Then $\Lambda = \Lambda x \oplus \Lambda y = \Lambda x \oplus (0:l)_\Lambda$ so Λl is Λ-isomorphic to Λx.

3. - Applications to hereditary orders.

An order is said to be *left hereditary* if it is left hereditary as a ring; i.e., every left ideal is left projective over the ring.

It is known that if an order A Dedekind domain is left (resp. right)hereditary, then it is right (resp. left) hereditary, ([**2**], p. 5). Thus we may speak of an hereditary order without any confusion.

Theorem 3.1. *Let Λ be an hereditary A-order in Σ. Then if α is an idempotent in any order Γ containing Λ, there is an idempotent β in Λ such that $\Gamma\alpha = \Gamma\beta$.*

Proof. Let α be an idempotent in some order $\Gamma \subseteq \Lambda$. By ([**3**]; sect. 4), there is a k in A such that $k\Gamma \subseteq \Lambda$. Then $k\alpha$ and $k(1-\alpha)$ are in Λ. Consider the sequence

$$0 \to (0:k(1-\alpha))_\Lambda \to \Lambda \to \Lambda k(1-\alpha) \to 0$$

which is split exact since Λ is hereditary. Since Λ is A-torsion free, $(0:k(1-\alpha))_\Lambda = (0:(1-\alpha))_\Lambda$ and is generated by an idempotent β. By Theorem 2.1, $(0:(1-\alpha))_\Gamma = \Gamma\beta$; but $(0:(1-\alpha))_\Gamma = \Gamma\alpha$, so $\Gamma\alpha = \Gamma\beta$.

Proposition 3.2. *Let Σ_n be the $n\times n$ matrices over K. Let Λ be an A-order in Σ_n such that Γ contains the matrix units l_{ii}, $1\leqslant i\leqslant n$. Assume that there is a set $\{\gamma_i\}$ in Γ such that $\gamma_i l_{ii} = \beta_i$ is an idempotent in Λ with $\Gamma l_{ii} = \Gamma\beta_i$. Then Λ contains a set of matrix units g_{ii} such that $\Lambda = \Lambda g_{11}\oplus \dots \oplus \Lambda g_{nn}$.*

Proof. We first note that

$$\Lambda \cap \Gamma\beta_i = \Lambda \cap (0:(1-\beta_i))\Gamma = \Lambda\beta_i = \Lambda \cap \Gamma l_{ii}$$

and

$$\Lambda = \Lambda\cap\Gamma = \Lambda \cap (\Gamma l_{11}\oplus \dots \oplus \Gamma l_{nn}) = \Lambda\cap(\Gamma\beta_1\oplus\dots\oplus\Gamma\beta_n)\,.$$

If x is in $\Lambda\cap(\Gamma\beta_1\oplus\dots\oplus\Gamma\beta_n)$, then $x = y_1+\dots+y_n$ is in Λ with y_i in $\Gamma\beta_i$. Clearly $x\beta_i$ is in Λ, and $x\beta_i = y_i$. Further, $y_i = \lambda_i\beta_i$ with λ_i in Γ, so y_i is in $\Lambda\cap\Gamma\beta_i$. Thus x is in $(\Lambda\cap\Gamma\beta_1)\oplus\dots(\Lambda\cap\Gamma\beta_n)$.

On the other hand, if x is in $(\Lambda\cap\Gamma\beta_1)\oplus\dots\oplus(\Lambda\cap\Gamma\beta_n)$, then $x = y_1 + {} + \dots + y_n$, y_i in Λ and $y_i = \lambda_i\beta_i$ for λ_i in Γ. Hence x is in $\Lambda\cap(\Gamma\beta_1\oplus\dots\oplus\Gamma\beta_n)$. Therefore

$$\Lambda = (\Lambda\cap\Gamma\beta_1)\oplus\dots\oplus(\Lambda\cap\Gamma\beta_n) = \Lambda\beta_1\oplus\dots\oplus\Lambda\beta_n\,.$$

We note that $\Sigma_n = \bigoplus_{i=1}^{n}\Sigma_n l_{ii} = \bigoplus_{i=1}^{n} K\otimes_A \Gamma l_{ii} = \bigoplus_{i=1}^{n} K\oplus_A \Lambda\beta_i = \bigoplus_{i=1}^{n}\Sigma_n\beta_i\,.$

We claim that the β_i are a set of matrix units in Σ_n and will serve for a set of matrix units in Λ.

Since Σ_n is simple and has $\Sigma_n\beta_i$ as minimal left ideals for all i, the $\Sigma_n\beta_i$ are all isomorphic as Σ_n-modules ([7], p. 45). So there are Σ_n-isomorphisms $f_{1j}\colon \Sigma_n\beta_1\to\Sigma_n\beta_j$.

Further we define Σ_n-isomorphisms $f_{ij} = f_{1j}f_i$, where $f_{i1} = f_{1i}^{-1}$. Then set $g_{ij} = f_{ij}\beta_i$. We have $g_{ij}g_{mn} = g_{ij}f_{mn}\beta_m = f_{mn}f_{ijm}$ which is zero unless $j = m$ since g_{ij} is in $\Sigma_n\beta_j$ and $\beta_j\beta_m = 0$ for $j\neq m$. Also if $j = m$, $g_{im}\cdot g_{mn} = f_{mn}(g_{im}\beta_m) = {} = f_{mn}g_{im} = f_{mn}f_{im}\beta_i = f_{1n}f_{m1}f_{1m}f_{i1}\beta_i = f_{1n}f_{i1}\beta_i = f_{in}\beta_i = g_{in}$. Thus, the $\{g_{ij}\}$ $(i = 1,\dots,n;\ j = 1,\dots,n)$ are a set of matrix units in Σ_n and $g_{ii} = f_{ii}\beta_i = {} = f_{ii}f_{i1}\beta_i = \beta_i$.

We say that an order Λ in Σ_n has *isolated positions* if every element $\sum_{i,j=1}^{n} a_{ij}l_{ij}$ in Λ, a_{ij} in A is such that $a_{ij}l_{ij}$ is in Λ for all i, j.

Theorem 3.3. *Let A be a discrete valuation ring. Every hereditary A-order Λ in Σ_n has isolated positions.*

Proof. The A-order A_n ($n \times n$ matrices with positions in A) is maximal and every other maximal A-order in Σ_n is isomorphic to it by an inner automorphism ([2], p. 11). It is clear that since A_n contains the l_{ii}, each maximal A-order does. Therefore applying Theorem 3.2, we have that Λ contains a set of matrix units $\{g_{ij}\}$. Let Γ be any maximal A-order containing Λ. Since A is a principal ideal domain, both Γ and Λ are free A-modules ([**5**], p. 92). Let π be a prime element of A and let P have an A-basis $\{\pi^{\mu_{ij}} l_{ij}\}$. Let $\{\lambda_{ij}\}$ be an A-basis for Λ and write it in terms of the A-basis for Γ, say

$$\lambda_{ij} = \sum_{p,q} a_{pq}^{ij}\, \pi^{\alpha_{pq}}\, l_{pq}\,.$$

Then

$$l_{mi}\, \lambda_{ij}\, l_{jn} = a_{mn}^{ij}\, \pi^{\alpha_{mn}}\, l_{mn}$$

is in Λ; so Λ has isolated positions.

4. - Matrix orders.

We will now devote most of our attention to matrix orders and apply the results of the preceding sections to them.

An ideal I of an order Λ is *full* in Λ if $I \oplus_A K \cong \Lambda \oplus_A K$. If I is a full left ideal over Λ and has a generating set $\{\mu_i\}$, $i = 1, \ldots, r$, over Λ such that $\Lambda\mu_i l_{kk} \neq 0$ if and only if $\Lambda\mu_j l_{kk} = 0$ for all $j \neq i$, then say that the μ_i *group columns*.

Theorem 4.1. *Let Σ_n be the K-algebra of $n \times n$ matrices over K. Let Λ be an A-order in Σ and let I be a full left ideal of $I = \bigoplus_{i=1}^{r} \Lambda\mu_i$; then the μ_i group columns, I is Λ-projective if and only if I is Λ-isomorphic to a Λ-module J of the form $J = \bigoplus_{i=1}^{r} \Lambda l_i$ with the l_i idempotents and the Λl_i isomorphic to $\Lambda\mu_i$ as Λ-modules. Further, the μ_i can be corresponded to the l_i such that $\Lambda\mu_i l_{kk} = 0$ if and only if $\Lambda\mu_i l_{kk} = 0$;* i.e., *μ_i and l_i group the same columns.*

Proof. Let i be an index such that $\mu_i l_{kk} \neq 0$ for some k. Then μ_i must have a non-zero element in the k-th column, say in the (m, k) position. By

([3], sect. 4) and the proof of Theorem 3.3, we see that for every pair (n, m) there is an element α_{nm} such that $\pi^{\alpha_{nm}} l_{nm}$ is in Λ. Thus $\pi^{\alpha_{nm}} l_{nmi}$ has a non-zero element in the (n, k) position for all n. If there is a j such that $\mu_j l_{kk} \neq 0$, then in the same way we see that $\Lambda \mu_j l_{kk}$ has a non-zero element in every (n, k) position. Hence since $\Lambda \mu_i \cap \Lambda \mu_j = (0)$ when $i \neq j$, we must have $i = j$.

If $\Lambda \mu_j l_{kk} = 0$ for all $j \neq i$, then since I has dimension n^2 over A, $\Lambda \mu_i l_{kk} \neq (0)$; for otherwise the dimension of the direct sum is not n^2.

The existence of the ideal J follows from Theorem 1.2.

Let f_i be the idempotent such that $\Lambda f_i = (0 : \mu_i)_\Lambda$ and $l_i = 1 - f_i$. As in Theorem 1.2, the map $l_i \to l_i \mu_i$ is an isomorphism from Λl_i to $\Lambda \mu_i l_i$. Consider $\Lambda \cong \Lambda \mu_i \oplus \Lambda f_i$ and let m be such that $\Lambda \mu_i l_{mm} = (0)$. Then $\Lambda \mu_i l_{mm} \oplus \Lambda f_i l_{mm} \cong \cong \Lambda l_{mm}$ implies that $\Lambda f_i l_{mm} \cong \Lambda l_{mm}$. Now considering $\Lambda \cong \Lambda l_i \oplus \Lambda f_i$, we have $\Lambda l_{mm} = \Lambda l_i l_{mm} \oplus \Lambda f_i l_{mm}$. Since $\Lambda f_i l_{mm} \cong \Lambda l_{mm}$ and $\dim_A \Lambda l_{mm} = \dim_A \Lambda f_i l_{mm} + + \dim_A \Lambda l_i l_{mm}$, we have $\dim_A \Lambda l_i l_{mm} = (0)$ and $\Lambda l_i l_{mm} = (0)$. Thus $\Lambda \mu_i l_{kk} = (0)$ implies that $\Lambda l_i l_{kk} = (0)$.

On the other hand, let m be such that $\Lambda \mu_i l_{mm} \neq (0)$. Then as before we have $\Lambda \mu_i l_{mm} \oplus \Lambda f_i l_{mm} = \Lambda l_{mm}$ and $\Lambda l_i l_{mm} \oplus \Lambda f_i l_{mm} = \Lambda l_{mm}$. Thus $\dim_A \Lambda f_i l_{mm} \neq \neq \dim_A \Lambda l_{mm}$, so $\dim_A \Lambda l_i l_{mm} \neq 0$ and $\Lambda l_i l_{mm} \neq 0$. Hence $\Lambda l_i l_{kk} \neq (0)$ implies $\Lambda l_i l_{kk} \neq (0)$.

Conversely, if such an ideal J exists, then as in Theorem 1.2, I is Λ-projective.

Henceforth A is a discrete valuation ring with π a prime element in A. Σ_n is the $n \times n$ matrices in K. Γ will be the maximal order of the form

$$\Gamma = \begin{bmatrix} A & A & \cdots & A \\ A & A & \cdots & A \\ \vdots & \vdots & & \vdots \\ A & A & \cdots & A \end{bmatrix}.$$

The sub-order Λ of Γ will be taken of the form

$$\Lambda = \begin{bmatrix} \pi^{r_{11}} A & \pi^{r_{12}} A & \cdots & \pi^{r_{1n}} A \\ \pi^{r_{21}} A & \pi^{r_{22}} A & \cdots & \pi^{r_{2n}} A \\ \vdots & \vdots & & \vdots \\ \pi^{r_{n1}} A & \pi^{r_{n2}} A & \cdots & \pi^{r_{nn}} A \end{bmatrix},$$

where since 1 is in Λ, $r_{ii} = 0$, for all i, and $r_{ij} \geqslant 0$ since $\Lambda \subseteq \Gamma$. The left ideal I of Λ is of the form

$$I = \begin{bmatrix} \pi^{r_{11}+s_{11}}A & \pi^{r_{12}+s_{12}}A & \cdots & \pi^{r_{1n}+s_{1n}}A \\ \pi^{r_{21}+s_{21}}A & \pi^{r_{22}+s_{22}}A & \cdots & \pi^{r_{2n}+s_{2n}}A \\ \vdots & \vdots & & \vdots \\ \pi^{r_{n1}+s_{n1}}A & \pi^{r_{n2}+s_{n2}}A & & \pi^{r_{nn}+s_{nn}}A \end{bmatrix}.$$

By the form of Γ, if $x_j = \min_i (r_{ij} + s_{ij})$, then

$$\Gamma I = \begin{bmatrix} \pi^{x_1}A & \pi^{x_2}A & \cdots & \pi^{x_n}A \\ \pi^{x_1}A & \pi^{x_2}A & \cdots & \pi^{x_n}A \\ \vdots & \vdots & & \vdots \\ \pi^{x_1}A & \pi^{x_2}A & & \pi^{x_n}A \end{bmatrix}.$$

Thus ΓI is cyclic over Γ and generated by an element μ which has $\delta_{ij}\pi^{x_i}$ in the (i, j) position where δ_{ij} is the Kronecker delta, $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

We will now define a numerical invariant which will be used as a condition in our later theorems.

Let M and N be torsion free A-modules such that $M \otimes_A K$ is A-isomorphic to $N \otimes_A K$. Let T be an A-linear transformation of M onto N; then T has a unique extension $T^*\colon M \otimes_A K \to N \otimes_A K$ as K modules. Define the *module index* $[M:N]$ by $[M:N] = (\det(T^*))$.

Notice that changing the A-basis of M or N alters T only by a unimodular transformation, so that $[M:N]$ is independent of the choice of basis.

We state without proof the following properties of the module index. These follows directly from the definition or from [**1**].

Theorem 4.2. *Let* M_1, M_2, M_3 *be such that* $M_1 \otimes_A K \cong M_2 \otimes_A K \cong M_3 \otimes_A K$.

1) $[M_1 : M_1] = A$;
2) $[M_1 : M_2][M_2 : M_3] = [M_1 : M_3]$;
3) *If* $[M_1 : M_2] = A$ *and* $M_1 \supseteq M_2$, *then* $M_1 = M_2$;
4) *If* $M_1 \supseteq M_2$, *then* $[M_1 : M_2] \subseteq A$.

A. Fröhlich used the module index to give criteria for projectivity of modules in orders of finite dimensional commutative separable K-algebras in [**7**]. Specifically he showed that if Γ is the unique maximal A-order in Σ, Λ any other A-order in Σ, M a Λ-module such that $K \otimes_A M \cong \Sigma^{(r)}$, then M is Λ-projective if and only if $[\Gamma M : M] = [\Gamma : \Lambda]^r$. This criteria was extended to noncommutative algebras in $[1]_1$ and $[1]_2$. The following will give further extensions to matrix orders.

Now returning to the preceding notation, we have that

$$[\Gamma : \Lambda] = \prod^{\sum_{i,j} r_{ij}} \tag{2}$$

and

$$[\Gamma I : I] = \prod^{\sum_{i,j} r_{ij}+s_{ij}-x_j} \tag{3}$$

directly from the definition.

Lemma 4.2. *Given any j, $1 \leqslant j \leqslant n$, then $s_{ij} \leqslant s_{jj}$ for every i and $r_{ij} + s_{ij} = \min_k (r_{ik} + s_{ik} + r_{kj})$.*

Proof. The left ideal I must be closed under left multiplication, so since

$$\pi^{r_{mn}} l_{mn} \pi^{r_{pq}+s_{pq}} l_{pq} = \pi^{r_{mn}+s_{pq}+r_{pq}} l_{nq} d_{np}$$

where l_{ij} are the matrix units and d_{ij} is the Kronecker delta, we have $r_{mq} + s_{mq} \leqslant \min_i (r_{mi} + s_{iq} + r_{iq})$ for all (m, q). Since $r_{ii} = 0$ for all i, $r_{mq} + s_{mq} = r_{mm} + r_{mq} + s_{mq}$ so that

$$r_{mq} + s_{mq} = \min_i (r_{mi} + s_{iq} + r_{iq}) \tag{4}$$

for all (m, q). If in (4) we set $q = i$, $r_{mq} + s_{mq} \leqslant r_{mq} + s_{qq}$. Hence for every (m, q), $s_{mq} \leqslant s_{qq}$.

Lemma 4.3. *If for each j, $1 \leqslant j \leqslant n$, Il_{jj} is of the form $\Lambda a_j l_{jj}$ with a_j in Λ, then Il_{jj} is Λ-projective, I is Λ-projective and $[\Gamma : \Lambda] = [\Gamma I : I]$. Further $[\Lambda : I] = ((a_1 \dots a_n)^n)$.*

Proof. If $Il_{jj} = \Lambda a_j l_{jj}$ with a_j in Λ, then Il_{jj} is clearly Λ-projective since it is a direct summand of Λa_j. We are identifying a in Λ with the matrix in Σ having a in every position on the main diagonal and zeros elsewhere. If $a_j = \mu_j \pi^{\nu_j}$ for μ_j a unit in Λ, we can multiply by μ_j^{-1} and without loss of generality consider a_j of the form π^{ν_j}. Then I is cyclic with generator

$$\mu = \begin{bmatrix} \pi^{\nu_1} & 0 & \cdots & 0 \\ 0 & \pi^{\nu_2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \pi^{\nu_n} \end{bmatrix}.$$

Then the map $\lambda \to \lambda\mu$ is a Λ-isomprhism of Λ to I which extends to a Γ-isomorphism of Γ to ΓI. Then by the definition of module index and usual properties of determinants we have $[\Gamma:\Lambda] = [\Gamma I:I]$. Again from the definitions, $[\Lambda:I] = ((a_1 \dots a_n)^n)$. I is Λ-projective since $I = \bigoplus_{i=1}^{n} \Lambda l_{ii}$ and is thus a finite direct sum of projectives.

Set $p_{ij} = r_{ij} + s_{ij} - s_{ij}$ for all i and j.

Theorem 4.4. *If $p_{ij} \geqslant 0$ for all i, j, then $[\Gamma I:I] = [\Gamma:\Lambda]$ implies that I is of the form $I = \bigoplus_j \Lambda a_j l_{jj}$ with a_j in A.*

Proof. By Lemma 4.2, $s_{ij} \leqslant s_{jj}$ for all i and j, so we have $0 \leqslant p_{ij} \leqslant r_{ij}$. By equation (2), $[\Gamma:\Lambda] = \Pi^{\sum_{ij} r_{ij}}$. We claim that $[\Gamma I:I] = \Pi^{\sum_{ij} p_{ij}}$. Since $p_{ij} \geqslant 0$ for all (i, j), then $r_{ij} + s_{ij} \geqslant s_{jj}$. Then $\min_k (r_{kj} + s_{kj}) = s_{jj}$, so by equation (3), we have $[\Gamma I:I] = \Pi^{\sum_{ij} p_{ij}}$. By our hypothesis, $\Pi^{\sum_{ij} p_{ij}} = \Pi^{\sum_{ij} r_{ij}}$, so $p_{ij} = r_{ij}$ for all (i, j). Hence $s_{jj} = s_{ij}$ and we take $a_j = \pi^{s_{jj}}$.

We will now consider Λ to be an hereditary order. By Theorem 3.3, every hereditary order in Σ_n has isolated positions. In [**8**], J. Murtha proves the following which we assume without proof.

Theorem 4.5. *Assume the preceding notation. An A-order Λ is hereditary if and only if*

(a) $r_{ii} = 0$, $1 \leqslant i \leqslant n$,

(b) *there is a k, $1 \leqslant k \leqslant n$, such that* $r_{ki} = 0$, $1 \leqslant i \leqslant n$;

(c) $r_{ij} \leqslant r_{ik}$ *for some fixed k,* $1 \leqslant i \leqslant n$;

(d) $r_{ij} \leqslant r_{ik} + r_{kj}$, $1 \leqslant i, j, k \leqslant n$;

(e) $r_{ij} + r_{ji} \leqslant 1$, $1 \leqslant i, j \leqslant n$;

(f) $r_{ij} \leqslant 1$, $1 \leqslant i, j \leqslant n$.

Lemma 4.6. *For every pair (i, j), $s_{ij} = s_{jj}$ or $s_{ij} = s_{jj} - 1$.*

Proof. Since I is closed under left multiplication by elements of Λ, so

$$\pi^{r_{ji}} l_{ji} \pi^{r_{ik} + s_{ik}} l_{ik} = \pi^{r_{ji} + r_{ik} + s_{ik}} l_{jk}$$

implies

$$r_{ji} + r_{ik} + s_{ik} \geqslant r_{jk} + s_{jk} \,. \tag{5}$$

By Theorem 4.5 (f), equation (5) implies $2+s_{ik}\geqslant s_{jk}$. Thus $2+s_{ij}\geqslant s_{jj}\geqslant s_{ij}$. From (5) with $k=j$, we use Theorem 4.5 (e) to obtain $1+s_{ij}\geqslant s_{jj}$. By Lemma 4.2 this means $s_{jj}=s_{ij}$ or $s_{jj}=s_{ij}+1$.

Lemma 4.7. *If $s_{ij}=s_{ij}+1$, then $r_{ij}=1$ or $r_{ji}=1$ but not both.*

Proof. Let $s_{jj}=s_{ij}+1$ for the pair (i,j). If $r_{ij}=1$, we are done by Theorem 4.5 (e). If $r_{ij}=0$ then from the closure of I under left multiplication

$$\pi^{r_{ji}}\, l_{ji}\, \pi^{r_{ij}+s_{ij}}\, \mathscr{L}_{ij}=\pi^{r_{ji}+s_{ij}}\, l_{jj}\,,$$

we have $r_{ji}+s_{ij}\geqslant s_{jj}=s_{ij}+1$.
Thus $r_{ji}=1$ by Theorem 4.5 (f).

Lemma 4.8. *If $p_{ij}\geqslant 0$ for all pairs (i,j), then $r_{ij}+s_{ij}-\min_k(r_{kj}+s_{kj})\leqslant r_{ij}$ for all (i,j). If for any pair (i,j), $s_{ij}=s_{jj}-1$. Then $[\Gamma I\colon I]\leqslant[\Gamma\colon\Lambda]$.*

Proof. Since $p_{ij}\geqslant 0$ for all pairs (i,j), $r_{ij}+s_{ij}\geqslant s_{jj}$ for all (i,j), and so $\min_k(r_{kj}+s_{kj})=s_{jj}$. Thus $p_{ij}\leqslant r_{ij}$. By equations 2) and 3), $[\Gamma I\colon I]\leqslant[\Gamma\colon\Lambda]$. If $[\Gamma I\colon I]=[\Gamma\colon\Lambda]$, then by Theorem 4.4, I has the form $I=\bigoplus_j \Lambda a_j l_{jj}$ with a_j in Λ. Thus $s_{ij}=s_{jj}$ for all (i,j) which is a contradiction. Thus $[\Gamma I\colon I]\leqslant$ $\leqslant[\Gamma\colon\Lambda]$.

Corollary 4.9. *With the same hypothesis as in Theorem 4.8, if $s_{ij}=s_{jj}$ for all (i,j), then $I=\bigoplus_j \Lambda a_j l_{jj}$ for a_j in Λ.*

For a fixed j, $1\leqslant j\leqslant n$, let $N_j(i)$ denote the number of (i, j) positions such that $s_{jj}=s_{ij}$ if at least one (p,j) posit is such that $s_{jj}=s_{pj}+1$. Set $N_j(i)=0$ otherwise. Let $N(I)=\sum_j N_j(i)$.

Theorem 4.10. *If there is an (i,j) position such that $p_{ij}<0$, then*

(a) *there is an (i,j) position such that $r_{ij}=0$ and $s_{ij}+1=s_{jj}$;*
(b) $[\Gamma I\colon I]\gneqq[\Gamma\colon\Lambda]$;
(c) $[\Gamma I\colon I]=[\Gamma\colon\Lambda]+N(I)$;
(d) $N(I)\gneqq 0$.

If $[\Gamma I\colon I]=[\Gamma\colon\Lambda]$, then I has the form $I=\bigoplus_i \Lambda a_i l_{ii}$ for a_i in Λ.

Proof. If there is a pair (i,j) such that $p_{ij} \underset{+}{\stackrel{<}{}} 0$, then for that pair $r_{ij}+s_{ij}-s_{jj} \underset{+}{\stackrel{<}{}} 0$, so that $r_{ij}+s_{ij} \underset{+}{\stackrel{<}{}} s_{jj}$. If $s_{ij}=s_{jj}$, then $r_{ij} \underset{+}{\stackrel{<}{}} 0$ which is a contradiction. Thus $s_{ij}+1=s_{jj}$ by Lemma 4.6. If $r_{ij}=1$, then since $r_{ij}+ + s_{ij} \underset{+}{\stackrel{<}{}} s_{jj}$ we have $1+s_{ij} \underset{+}{\stackrel{<}{}} s_{ij}+1$ which is ridiculous. Thus $r_{ij}=0$ and $s_{ij}+1=s_{jj}$.

Consider the j-th column and hold j fixed for the moment. If $r_{kj}+s_{kj} \underset{+}{\stackrel{<}{}} s_{ij}$, then $r_{kj}+s_{kj} \underset{+}{\stackrel{<}{}} s_{jj}-1$. By Lemma 4.6, $s_{kj}+1=s_{jj}$ or $s_{kj}=s_{jj}$. In the first case, we have $r_{kj}+s_{kj} \underset{+}{\stackrel{<}{}} s_{kj}+1-1$, which implies $r_{kj} \underset{+}{\stackrel{<}{}} 0$. In the second case, we have $r_{kj}+s_{jj} \underset{+}{\stackrel{<}{}} s_{jj}-1$ or $r_{kj} \underset{+}{\stackrel{<}{}} -1$. Thus both cases are impossible, and we must have $s_{ij} \leqslant r_{kj}+s_{kj}$ for all pairs (k,j). Further, $s_{ij}=\min_k (r_{kj}+s_{kj})$ because $r_{ij}=0$.

For any p, $1 \leqslant p \leqslant n$, consider $r_{pj}+s_{pj}-\min_k (r_{kj}+s_{kj}) = r_{pj}+s_{pj}-s_{ij}$. By Lemma 4.6, $s_{pj}=s_{jj}$ or $s_{pj}+1=s_{jj}$. In the first case we have $r_{pj}+ + s_{pj}-s_{ij} = r_{pj}+s_{jj}-s_{jj}+1 = r_{pj}+1 \underset{+}{\stackrel{>}{}} r_{pj}$. In the second case, we get $r_{pj}+s_{pj}-s_{ij} = r_{pj}+s_{ij}-1-s_{jj}+1 = r_{pj}$.

Thus for those (p,j) positions such that $s_{pj}=s_{jj}$, we obtain a contribution of π to $[\Gamma I : I]$ which does not appear in $[\Gamma : \Lambda]$. Thus $[\Gamma I : I] = [\Gamma : \Lambda] + + N_\Lambda(I)$.

If no position (p,k) occurs such that $s_{pk}+1=s_{kk}$, then the index for that column does not change since $s_{ij}=s_{jj}$ for all i.

Since by hypothesis, $p_{ij} \underset{+}{\stackrel{<}{}} 0$ for some (i,j), $N_\Lambda(I) \geqslant 1$, so $[\Gamma I : I] \underset{+}{\stackrel{>}{}} [\Gamma : \Lambda]$.

Thus finally, if $[\Gamma I : I] = [\Gamma : \Lambda]$, then $p_{ij} \geqslant 0$, so by Lemma 4.4, I is of the form $I = \bigoplus_j \Lambda a_j l_{jj}$ for a_j in A.

We collect what we have proven into the following

Theorem 4.11. *Let Λ be an hereditary A-order in Σ_n and I a left ideal such that Λ and I have isolated positions. Let Γ be a maximal order containing Λ in Σ_n. Then $[\Gamma I : I] = [\Gamma : \Lambda]$ if and only if I is Λ-projective and of the form $I = \bigoplus_j \Lambda_{a_j} l_{jj}$ with a_j in A.*

For our final result, we will no longer require that Λ be hereditary, but we will assume the notation of this section.

Theorem 4.12. *Assume $\Gamma I = I$. Then I is Λ-projective.*

Proof. We know that ΓI has an A-basis of the form

$$\{\pi^{\min_k (r_{kj}+s_{kj})} l_{kj}\}_{k,j=1}^{m}.$$

Thus ΓI is cyclic and generated by

$$\mu = \{\delta_{kj}\ \pi^{\min_k(r_{kj}+s_{kj})}\ l_{kj}\},$$

where δ_{kj} is the Kronecker delta. Since $\Gamma I = \Gamma$, $\mu = \{\delta_{ij}\}$. So we have $\min_k (r_{kj} + s_{kj}) = 1$.

Since I has isolated positions $I = \sum_{i=1}^{n} \Lambda\mu_i$ where

$$\mu_i = \begin{bmatrix} 0 & \cdots & 0 & \pi^{r_{1i}+s_{1i}} \Lambda & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \pi^{r_{2i}+s_{2i}} \Lambda & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \pi^{r_{ni}+s_{na}} \Lambda & 0 & \cdots & 0 \end{bmatrix}$$

and $\Gamma\mu_i = \Gamma l_{ii}$. Thus since l_{ii} is in Λ, $1 - l_{ii}$ is in Λ for all i. Further $(0 : \mu_i) = \Lambda(1 - l_{ii})$. Thus $\Lambda\mu_i$ is Λ-projective and $I = \bigoplus_{i=1}^{n} \Lambda\mu_i$ is Λ-projective since a finite direct sum of projectives is projective.

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Summary.

This paper considers A-orders in finite dimensional central algebras where A is a Dedekind domain. The theorems concern conditions on projectivity for direct sums of cyclic modules. Change of rings theorems are proven which give conditions for an order to be projective given that it is a sub-order of a projective order and viceversa. These results are applied to hereditary orders and especially matrix orders.

Finally, theorems are given with conditions on numerical invariants such as the module index.

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