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Formulae for Laguerre and Hermite polynomials. (**)

1. - Introduction.

Brafman [1], Hardy [5], Hille [6], Waston [10], Sharma and Manocha [8], Carlitz $[2]_2$, Sharma and Abiodun [9] have obtained new formulae for Laguerre and Hermite polynomials. By using Watson's method [10] and operational techniques, we obtain various extension of Hille-Hardy formula [7] and Mehler formula. The formulae obtained in this paper are believed to be new.

The Laguerre polynomials may be defined as follows:

$$\sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-z)^k}{k!} = L_n^{(\alpha)}(z) . \tag{1.1}$$

The bilinear expansion

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) z^n = \tag{1.2}$$

$$= (1-z)^{-1} \exp\left[-\frac{(x+y)z}{1-z}\right] (xyz)^{-\frac{1}{2}\alpha} I_\alpha \left[\frac{2\sqrt{ryz}}{1-z}\right] ,$$

$|z|<1$ is known as Hille-Hardy formula.

In my previous paper [9], I gave the following generalization of (1.2).

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$$(1.3)\qquad \sum_{n=0}^{\infty} \frac{(n+m)!}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(x)\, L_{n+m}^{(\beta)}(y)\, t^n =$$

$$= \frac{\Gamma(m+\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)} \exp[y](1-t)^{-(m+\beta+1)} \cdot$$

$$\cdot\, \psi_2 \left[m+\beta+1;\ \alpha+1,\ \beta+1;\ \frac{-xt}{1-t},\ \frac{-y}{1-t} \right],$$

where $|t|<1$ and $\mathrm{R}(\beta+1)>0$ and ψ_2 is defined as follows.

$$(1.4)\qquad \psi_2[\alpha;\ \beta_1,\ \beta_2;\ x,\ y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\beta_1)_m (\beta_2)_n} x^m y^n .$$

I shall give another proof of (1.3) by using the following formulae ([3], p. 270, Ex. 27 and 29):

$$(1.5)\qquad L_n^{(\alpha)}(x) = \frac{\exp[x]\, x^{-\frac{1}{2}\alpha}}{n!} \int_0^{\infty} \exp[-\omega]\, \omega^{n+\frac{1}{2}\alpha} J_\alpha(2\sqrt{x\omega})\, d\omega ,$$

valid for $\mathrm{R}(n+\alpha+1)>0$.

$$(1.6)\qquad \sum_{n=0}^{\infty} \frac{t^n L_n^{(\alpha)}(x)}{\Gamma(\alpha+n+1)} = \exp[t](xt)^{-\frac{1}{2}\alpha} J_\alpha(2\sqrt{xt}) ,$$

and Erdelyi ($[4]_1$, p. 187, eq. (43)).

$$(1.7)\qquad \int_0^{\infty} t^{\lambda-1} J_{2\mu_1}(2\sqrt{a_1 t})\, J_{2\mu_2}(2\sqrt{a_2 t}) \ldots J_{2\mu_n}(2\sqrt{a_n t}) \exp[-pt]\, dt =$$

$$= \frac{\Gamma(\lambda+M)\, p^{-\lambda-M} a_1^{\mu_1} a_2^{\mu_2} \ldots a_n^{\mu_n}}{\Gamma(2\mu_1+1)\,\Gamma(2\mu_2+1) \ldots \Gamma(2\mu_n+1)} \cdot$$

$$\cdot\, \psi_2 \left[\lambda+M;\ 2\mu_1+1,\ 2\mu_2+1,\ \ldots,\ 2\mu_n+1;\ \frac{-a_1}{p},\ \frac{-a_2}{p},\ \ldots,\ \frac{-a_n}{p} \right],$$

where $M=\mu_1+\mu_2+\ldots+\mu_n$, $\mathrm{R}(p)>0$, $\mathrm{R}(\lambda+M)>0$.

To prove (1.3), we start with the left side of (1.3): by (1.5), (1.6) and (1.7) we have

$$\sum_{n=0}^{\infty} \frac{(n+m)!}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(x)\, L_{n+m}^{(\beta)}(y)\, t^n =$$

$$= \gamma^{-\frac{1}{2}\beta} \exp[y] \int_0^{\infty} \exp[-\omega]\, \omega^{m+\frac{1}{2}\beta} J_\beta(2\sqrt{y\omega}) \left[\sum_{n=0}^{\infty} \frac{(\omega t)^n}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(x) \right] d\omega$$

$$= y^{-\frac{1}{2}\beta}(xt)^{-\frac{1}{2}\alpha} \exp[y] \int_0^{\infty} \exp[-(1-t)\,\omega]\, \omega^{m+\frac{1}{2}(\beta-\alpha)} J_\alpha(2\sqrt{x\omega t})\, J_\beta(2\sqrt{y\omega})\, d\omega$$

$$= \frac{\Gamma(m+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \exp[y](1-t)^{-(m+\beta+1)} \cdot$$

$$\cdot\, \psi_2 \left[m+\beta+1;\, \alpha+1, \beta+1;\, \frac{-xt}{1-t}, \frac{-y}{1-t} \right].$$

This completes the proof of (1.3).

Replace x in (1.3) by $x\omega$, multiply both sides by $\omega^{\lambda-1}$ and take the Laplace transform with respect to ω using the known results, Erdelyi ([**4**]$_1$, p. 191, eq. (33)) and ([**4**]$_1$, p. 223, eqs. (12) and (13)). A similar operation on the variable y will finally give us

$$(1.8)\quad \sum_{n=0}^{\infty} \frac{(\beta+m+1)_n}{n!}\, {}_2F_1[-n, \lambda+1; \alpha+1; x]\, {}_2F_1[-(n+m), \varrho+1; 1+\beta; y]\, t^n =$$

$$= (1-y)^{-(\varrho+1)}(1-t)^{-(m+\beta+1)} F_2 \left[m+\beta+1;\, \lambda+1, \varrho+1;\, \alpha+1, \beta+1;\, \frac{-x}{1-t}, \frac{-y}{(1-y)(1-t)} \right],$$

for the definition of Appell function F_2, see ([**4**]$_2$, p. 224, eq. (7)).

The proof of (1.3) suggests the following formula ($|t|<1$) due to Carlitz [**2**].

$$(1.9)\quad \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} L_{n+m}^{(\alpha)}(x)\, t^n = \frac{\exp[x]\,\Gamma(\alpha+m+1)}{\Gamma(1+\alpha)} \cdot$$

$$\cdot\, (1+t)^{1+\alpha+m}\, {}_1F_1 \left[\alpha+1+m;\, 1+\alpha;\, \frac{-x}{1-t} \right].$$

Now *we prove the following generalization of* (1.2).

$$(1.10)\qquad \sum_{m,n=0}^{\infty} \frac{(m+n+p)!\,x^n y^m}{\Gamma(1+\beta+n)\Gamma(1+\gamma+m)} L_{m+n+p}^{(\alpha)}(a)\, L_n^{(\beta)}(b)\, L_m^{(\gamma)}(c) =$$

$$= \frac{\exp[a]\,\Gamma(p+\alpha+1)(1-x-y)^{-p-\alpha-1}}{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)} \cdot$$

$$\cdot\, \psi_2\left[p+\alpha+1;\, \alpha+1, \beta+1, \gamma+1;\, \frac{-a}{1-x-y}, \frac{-bx}{1-x-y}, \frac{-cy}{1-x-y}\right],$$

$R(\alpha+1) > 0$, $|x| + |y| < 1$.

Proof. We start with the l.h.s. of (1.10): by (1.5), (1.6) and (1.7),

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(m+n+p)!\,x^n y^m}{\Gamma(1+\beta+n)\,\Gamma(1+\gamma+m)} L_{m+n+p}^{(\alpha)}(a)\, L_n^{(\beta)}(b)\, L_m^{(\gamma)}(c) =$$

$$= \exp[a]\, a^{-\frac{1}{2}\alpha/2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{x^n y^m}{\Gamma(1+\beta+n)\,\Gamma(1+\gamma+m)} L_n^{(\beta)}(b)\, L_m^{(\gamma)}(c)\, \cdot$$

$$\cdot \int_0^{\infty} \exp[-\omega]\, \omega^{m+n+p+\alpha/2} J_\alpha(2\sqrt{a\omega})\, d\omega$$

$$= \exp[a]\, a^{-\alpha/2} \int_0^{\infty} \exp[-\omega]\, \omega^{p+\alpha/2} J_\alpha(2\sqrt{a\omega}) \left[\sum_{n=0}^{\infty} \frac{(x\omega)^n}{\Gamma(1+\beta+n)} L_n^{(\beta)}(b)\right] \cdot$$

$$\cdot \left[\sum_{m=0}^{\infty} \frac{(y\omega)^m}{\Gamma(1+\gamma+m)} L_m^{(\gamma)}(c)\right] d\omega$$

$$= \exp[a]\, a^{-\alpha/2} (bx)^{-\beta/2} (cy)^{-\gamma/2} \int_0^{\infty} \exp[-(1-x-y)\omega]\, \omega^{p+\frac{1}{2}(\alpha-\beta-\gamma)} \cdot$$

$$\cdot J_\alpha(2\sqrt{a\omega})\, J_\beta(2\sqrt{bx\omega})\, J_\gamma(2\sqrt{cy\omega})\, d\omega$$

$$= \frac{\exp[a]\,\Gamma(p+\alpha+1)(1-x-y)^{-p-\alpha-1}}{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)} \cdot$$

$$\cdot\, \psi_2\left[p+\alpha+1;\, \alpha+1, \beta+1, \gamma+1;\, \frac{-a}{1-x-y}, \frac{-bx}{1-x-y}, \frac{-cy}{1-x-y}\right].$$

This completes the proof under the conditions stated with (1.10)

The (1.10) can be extended as follows:

$$(1.11)\quad \sum_{m,n,p=0}^{\infty} \frac{(m+n+p+r)!\,x^n y^m z^p}{\Gamma(1+\beta+n)\,\Gamma(1+\gamma+m)\,\Gamma(1+\delta+p)}\cdot L^{(\alpha)}_{m+n+p+r}(a)\,L^{(\beta)}_n(b)\,L^{(\gamma)}_m(c)\,L^{(\delta)}_p(e) =$$

$$= \frac{\exp[a]\,\Gamma(r+\alpha+1)(1-x-y-z)^{-r-a-1}}{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)\,\Gamma(\delta+1)}\cdot \psi_2\left[r+\alpha+1;\ \alpha+1,\beta+1,\gamma+1,\delta+1;\ \frac{-a}{1-x-y-z},\ \frac{-bx}{1-x-y-z},\ \frac{-cy}{1-x-y-z},\ \frac{-ez}{1-x-y-z}\right],$$

where $\mathrm{R}(\alpha+1)>0,\ |x|+|y|+|z|<1$.

We prove the following formula

$$(1.12)\quad \sum_{n=0}^{\infty} \frac{L^{(\alpha)}_n(x)\,L^{(\alpha)}_n(y)}{(1+\alpha)_n(1+\alpha)_n}\,t^{2n} = I_0(2t)\,{}_0F_3[-;1,1+\alpha,1+\alpha;xyt^2]+$$

$$+\sum_{r=1}^{\infty}\frac{(-1)^r}{r!(1+\alpha)_r}(x^r+y^r)\,t^r I_r(2t)\cdot{}_0F_3[-;r+1,1+\alpha+r,1+\alpha;xyt^2].$$

Proof. We start with the left side of (1.12): by (1.1)

$$\varphi(t) = \sum_{n=0}^{\infty}\frac{L^{(\alpha)}_n(x)\,L^{(\alpha)}_n(y)}{(1+\alpha)_n(1+\alpha)}\,t^{2n} =$$

$$= \sum_{n=0}^{\infty} t^{2n}\sum_{r=0}^{n}\frac{(-1)^r x^r}{r!(n-r)!(1+\alpha)_r}\sum_{s=0}^{n}\frac{(-1)^s y^s}{s!(n-s)!(1+\alpha)}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{r!\,s!(1+\alpha)_r(1+\alpha)_s}\sum_{n\geqslant\max(r,s)}\frac{t^{2n}}{(n-r)!(n-s)!}.$$

For $r\geqslant s$ the inner sum on the extreme right is equal to

$$\sum_{n=0}^{\infty}\frac{t^{2n+2r}}{n!(n+r-s)} = t^{r+s}\sum_{n=0}^{\infty}\frac{t^{2n+r-s}}{n!(n+r-s)} = t^{r+s} I_{r-s}(2t),$$

in the usual notation for Bessel functions of purely imaginary argument.

For $r \leqslant s$ we find that the inner sum is equal to $t^{r+s} I_{s-r}(2t)$.
Thus we have

$$\varphi(t) = \sum_{r=0}^{\infty} \sum_{s=0}^{r} \frac{(-1)^{r+s} x^r y^s t^{r+s}}{r!\,s!(1+\alpha)_r(1+\alpha)_s} I_{r-s}(2t) + \sum_{s=1}^{\infty} \sum_{r=0}^{s} \frac{(-1)^{r+s} x^r y^s t^{r+s}}{r!\,s!(1+\alpha)_r(1+\alpha)_s} I_{s-r}(2t)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (xt)^r}{r!(1+\alpha)_r} I_r(2t)\, {}_0F_3[-;\, r+1,\, 1+\alpha+r,\, 1+\alpha;\, xyt^2] +$$

$$+ \sum_{r=1}^{\infty} \frac{(-1)^r (yt)^r}{r!(1+\alpha)_r} I_r(2t)\, {}_0F_3[-;\, r+1,\, 1+\alpha+r, 1+\alpha;\, xyt^2]$$

$$= I_0(2t)\, {}_0F_3[-;\, 1,\, 1+\alpha,\, 1+\alpha;\, xyt^2] +$$

$$+ \sum_{r=1}^{\infty} \frac{(-1)^r}{r!(1+\alpha)_r} (x^r + y^r) t^r I_r(2t)\, {}_0F_3[-;\, r+1,\, 1+\alpha+r,\, 1+\alpha;\, xyt^2] .$$

This completes the proof of (1.12).
In case we take $y = 0$ in (1.12),

$$(1.13) \qquad \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{n!(1+\alpha)_n} t^n = I_0(2\sqrt{t} + \sum_{r=1}^{\infty} \frac{(-1)^r}{r!(1+\alpha)_r} x(\sqrt{t})^r I_r(2\sqrt{t}) .$$

2. – In this section we obtain formulae for Hermite polynomials.
The Hermite polynomial $H_n(x)$ may be defined by

$$(2.1) \qquad \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \exp[2xt - t^2] .$$

The bilinear expansion

$$(2.2) \qquad \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{t^n}{n!} = (1 - 4t^2)^{-\frac{1}{2}} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

is known as Mehler's formula [7]. We give below various extensions of (2.2).

We shall require the formula ([7], p. 190)

$$(2.3)\qquad \int_{-\infty}^{\infty} x^n \exp[-x^2+2iyx]\,\mathrm{d}x = \frac{\sqrt{\pi}}{(-2i)^n}\exp[-y^2]H_n(y)\,.$$

We first prove the formula:

$$(2.4)\qquad \sum_{m,n=0}^{\infty} H_{m+n+p}(a)H_m(b)H_n(c)\frac{x^m y^n}{m!n!} = (1-4x^2-4y^2)^{-\frac{1}{2}(1+p)}\cdot$$

$$\cdot\exp\left[\frac{-4a^2(x^2+y^2)+4a(bx+cy)-4(bx+cy)^2}{1-4x^2-4y^2}\right]H_p\left\{\frac{a-2bx-2cy}{\sqrt{(1-4x^2-4y^2)}}\right\}.$$

Proof. Using (2.1), (2.3) and interchanging the order of integration and summation, which is justified due to the absolute convergence of the integral and series involved, we have after some simple manipulation that

$$(2.5)\qquad \sum_{m,=n0}^{\infty} H_{m+n+p}(a)H_m(b)H_n(c)\frac{x^m y^n}{m!\,n!} =$$

$$= \frac{(-2i)^p}{\sqrt{\pi}}\exp[a^2]\int_{-\infty}^{\infty}\sum_{m=0}^{\infty} H_m(b)\frac{(-2ixu)^m}{m!}\cdot$$

$$\cdot\sum_{n=0}^{\infty} H_n(c)\frac{(-2iyu)^n}{n!}u^p\exp[-u^2+2iau]\,\mathrm{d}u$$

$$= \frac{(-2i)^p\exp[a^2]}{\sqrt{\pi}}\int_{-\infty}^{\infty} u^p\exp[-a^2(1-4x^2-4y^2)+2iu(a-2bx-2cy)]\,\mathrm{d}u\,.$$

Put $u^2(1-4x^2-4y^2)=V^2$, (2.5) reduces to

$$(2.6)\qquad (1-4x^2-4y^2)^{-\frac{1}{2}(1+p)}\exp\left[\frac{a^2(1-4x^2-4y^2)-(a-2bx-2cy)^2}{(1-4x^2-4y^2)}\right]\cdot$$

$$\cdot\frac{(-2i)^p}{\sqrt{\pi}}\exp\left[\frac{(a-2bx-2cy)}{(1-4x^2-4y^2)^{\frac{1}{2}}}\right]^2\int_{-\infty}^{\infty} V^p\exp\left[-V^2+2iV\frac{(a-2bx-2ay)}{(1-4x^2-4y^2)^{\frac{1}{2}}}\right]\mathrm{d}V =$$

$$= (1-4x^2-4y^2)^{-\frac{1}{2}(1+p)}\exp\left\{\frac{-4a^2(x^2+y^2)+4a(bx+cy)-4(bx+cy)^2}{(1-4x^2-4y^2)}\right\}\cdot$$

$$\cdot H_p\left\{\frac{(a-2bx-2cy)}{\sqrt{(1-4x^2-4y^2)}}\right\},$$

on applying (2.3). This completes the proof of (2.4). If $p = 0$ and $b = 0$ in (2.4), it reduces to (2.2). If $p = 0$ then (2.4) reduces to a known result due to Carlitz [**2**]$_1$.

Similarly we can prove the following generalizations of (2.2).

$$(2.7) \qquad \sum_{m,n,p=0}^{\infty} H_{m+n+p+q}(a) H_m(b) H_n(c) H_p(d) \frac{x^m y^n z^p}{m!\, n!\, p!} =$$

$$= (1 - 4x^2 - 4y^2 - 4z^2)^{-\frac{1}{2}(1+q)} \cdot$$

$$\cdot \exp \left\{ \frac{-4a^2(x^2+y^2+z^2) + 4a(bx+cy+dz) - 4(bx+cy+dz)^2}{1 - 4x^2 - 4y^2 - 4z^2} \right\} \cdot$$

$$\cdot H_q \left\{ \frac{a - 2bx - 2cy - 2dz}{\sqrt{(1 - 4x^2 - 4y^2 - 4z^2)}} \right\},$$

$$(2.8) \qquad \sum_{n_1, n_2, \ldots, n_k = 0}^{\infty} H_{n_1+n_2+n_3+\ldots+n_k+q}(a) H_{n_1}(b_1) H_{n_2}(b_2) \ldots H_{n_k}(b_k) \cdot$$

$$\cdot \frac{x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_k^{n_k}}{n_1!\, n_2!\, n_3! \ldots u_k!} =$$

$$= \left[1 - 4 \sum_{r=1}^{k} x_r^2 \right]^{-\frac{1}{2}(1+q)} \exp \left\{ \frac{-4a^2 \sum_{r=1}^{k} x_r^2 + 4a \sum_{r=1}^{k} b_r x_r - 4 \left(\sum_{r=1}^{k} b_r x_r \right)^2}{\left(1 - 4 \sum_{r=1}^{k} x_r^2 \right)} \right\} \cdot$$

$$\cdot H_q \left\{ \frac{a - 2 \sum_{r=1}^{k} b_r x_r}{\sqrt{\left[1 - 4 \sum_{r=1}^{k} x_r^2 \right]}} \right\},$$

$$(2.9) \qquad \sum_{m,n,p=0}^{\infty} H_{n+p}(a) H_{p+m}(b) H_{m+n+k}(c) \frac{x^m y^n z^p}{m!\, n!\, p!} =$$

$$= (1 - 4x^2 - 4y^2 - 4z^2 + 16xyz)^{-\frac{1}{2}(1+k)} \cdot$$

$$\cdot \exp \left\{ \sum a^2 - \frac{\sum a^2 - 4 \sum a^2 x^2 - 4 \sum abz + 8 \sum abxy}{1 - 4x^2 - 4y^2 - 4z^2 + 16xyz} \right\} \cdot$$

$$\cdot H_k \left\{ \frac{c(1-4z^2) - 2(b - 2az)x - 2(a - 2bz)y}{\sqrt{(1-4z^2)(1-4z^2-16xyz)}} \right\},$$

where $\sum a^2$, $\sum a^2x^2$, $\sum abz$ and $\sum abxy$ are symmetric functions in the indicated variables.

In particular (2.7), (2.8) and (2.9) reduce to known results due to Carlitz **[2]**$_1$ by taking $q = 0$ and $k = 0$.

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Summary.

In this paper we have obtained some generating functions for Laguerre and Hermite polynomials. These formulae are various types of extensions of the well known Hille-Hardy formula and Mehler formula.

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