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# On common fixed point of a sequence of maps in uniform spaces. (**)

## 1. - Introduction.

Recently Baidynath Roy [3] has proved the existence of a common fixed point for a sequence of mappings from a complete metric space into itself. His theorem is as follows

Theorem 1.1. Let $\{T_n\}$ be a sequence of maps each mapping a complete metric space $(E, d)$ into itself such that

(i) for any two maps $T_i$, $T_j$,

$$d(T_i x, T_j y) \leqslant \lambda d(x, y) ,$$

where $0 < \lambda < 1$, and $x, y \in E$ with $x \neq y$, and

(ii) there is a point $x_0$ in $E$ such that any two members of $\{x_n = T_n x_{n-1}\}$ are distinct. Then $\{T_n\}$ has a unique common fixed point.

In the present paper, we extend this theorem to uniform spaces in different directions.

## 2. - Preliminary definitions and results.

Let $(X, \mathscr{U})$ be a uniform space. For definitions of Cauchy net and a complete uniform space, see [2] (pp. 190-192). From Th. 15 ([2], p. 188) we see that

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the uniformity $\mathscr{U}$ on $X$ can be generated by the family $\mathscr{F}$ of all pseudometrics on $X$ which are uniformly continuous on $X \times X$. But we have observed that it is not necessary to take all the members of $\mathscr{F}$ to generate the uniformity $\mathscr{U}$ ([1], Th. 2.1).

For any pseudometric $p$ on $X$ and any $r > 0$, we write

$$V_{(p,r)} = \{(x, y); x, y \in X \text{ and } p(x, y) < r\} .$$

Let $\mathscr{P}$ be a family of pseudometrics on $X$ generating the uniformity $\mathscr{U}$. Denote by $\mathscr{V}$ the family of all sets of the form $\bigcap_{i=1}^{n} V_{(p_i,r_i)}$, where $p_i \in \mathscr{P}$, $r_i > 0$ ($i = 1, 2, \ldots, n$, the integer $n$ is not fixed).

Then clearly $\mathscr{V}$ is a base for the uniformity $\mathscr{U}$.

Let $V \in \mathscr{V}$. Then $V = \bigcap_{i=1}^{n} V_{(p_i,r_i)}$, where $p_i \in \mathscr{P}$, and $r_i > 0$ ($i = 1, 2, \ldots, n$). For each $\alpha > 0$, the set $\bigcap_{i=1}^{n} V_{(p_i,\alpha r_i)}$ belongs to $\mathscr{V}$. We denote this set by $\alpha V$.

We now give a list of some properties of the sets of the form $\alpha V$ which can be easily established.

(2.1) If $V \in \mathscr{V}$ and $\alpha, \beta$ are positive, then $\alpha(\beta V) = (\alpha\beta) V$.

(2.2) If $V \in \mathscr{V}$ and $\alpha, \beta$ are positive, then $\alpha V \subset \beta V$ when $\alpha < \beta$.

(2.3) Let $p$ be any pseudometric on $X$ and $\alpha, \beta$ be any two positive numbers. If $(x, y) \in \alpha V_{(p,r_1)} \circ \beta V_{(p,r_2)}$, then $p(x,y) < \alpha r_1 + \beta r_2$.

(2.4) If $V \in \mathscr{V}$ and $\alpha, \beta$ are positive, then $\alpha V \circ \beta V \subset (\alpha + \beta) V$.

Note 2.1. Let $p$ be any pseudometric on $X$ and $\alpha, \beta, \gamma$ any three positive numbers. If

$$(x, y) \in \alpha V_{(p,r_1)} \circ \beta V_{(p,r_2)} \circ \gamma V_{(p,r_3)} ,$$

then $p(x, y) < \alpha r_1 + \beta r_2 + \gamma r_3$.

(2.5) Let $x, y \in X$. Then for every $V$ in $\mathscr{V}$, there is a positive number $\lambda$ such that $(x, y) \in \lambda V$.

(2.6) Let $V \in \mathscr{V}$. Then there is a pseudometric $p$ on $X$ such that $V = V_{(p,1)}$. Let $x, y \in X$. Then by (2.5) there is a $\lambda > 0$ such that $(x, y) \in \lambda V$.

Write $A_{(x,y)} = \{\lambda;\ \lambda > 0 \text{ and } (x, y) \in \lambda V\}$.

We can now verify in (2.6) that

$$p(x, y) = \operatorname{Inf}\{\lambda;\ \lambda \in A_{(x,y)}\}\ .$$

We reafer to $p$ as the Minkowski's pseudometric of $V$ in analogy with the Minkowski's functional of a convex and a balanced set in a linear topological space.

In the following section we shall utilise the following four theorems which we have proved in [**1**]. Here $X$ is assumed to be a sequentially complete uniform space which is also a Hausdorff space, and $\mathscr{V}$, the family of all sets of the form $\bigcap_{i=1}^{n} V_{(p_i, r_i)}$, where $p_i \in \mathscr{P}$ and $r_i > 0$ $(i = 1, 2, \ldots, n$, the integer $n$ is not fixed), $\mathscr{P}$ being a fixed family of pseudometrics on $X$ generating the uniformity $\mathscr{U}$ for $X$.

Theorem 2.1. ([**1**], Th. 3.2). Let $T_1$ and $T_2$ be two operators on $X$ mapping $X$ into itself such that for any two members $V_1$, $V_2$ in $\mathscr{V}$ and $x$, $y$ in $X$

$$(T_1x,\ T_2y) \in \alpha V_1 \circ \beta V_2,$$

if $(x, T_1x) \in V_1$ and $(y, T_2y) \in V_2$, where $\alpha, \beta$ are independent of $x, y, V_1, V_2$, and $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$. Then $T_1$ and $T_2$ have a unique common fixed point in $X$.

Theorem 2.2. ([**1**], Th. 3.3). Let $T_1$ and $T_2$ be two operators on $X$ mapping $X$ into itself, such that for any $V_1$, $V_2$ in $\mathscr{V}$ and $x, y$ in $X$

$$(T_1x,\ T_2y) \in \alpha V_1 \circ \beta V_2\ ,$$

if $(y, T_1x) \in V_1$ and $(x, T_2y) \in V_2$, where $\alpha, \beta$ are independent of $x, y, V_1, V_2$ and $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$. Then $T_1, T_2$ have a unique common fixed point in $X$.

Theorem 2.3. ([**1**], Th. 3.4). Let $T_1$ and $T_2$ be two operators on $X$ mapping $X$ into itself such that for any three members $V_1$, $V_2$, $V_3$ in $\mathscr{V}$ and $x$, $y$ in $X$

$$(T_1x,\ T_2y) \in \alpha V_1 \circ \beta V_2 \circ y V_3,$$

if $(x, T_1x) \in V_1$, $(x, y) \in V_2$, $(y, T_2y) \in V_3$, where $\alpha, \beta, \gamma$ are independent of $x, y, V_1, V_2, V_3$, and $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha + \beta + \gamma < 1$. Then $T_1$, $T_2$ have a unique common fixed point in $X$.

Theorem 2.4. ([1], Th. 3.5) Let $T_1$ and $T_2$ be two operators on $X$ mappings $X$ into itself such that for any three members $V_1$, $V_2$, $V_3$ in $\mathscr{V}$ and $x$, $y$ in $X$

$$(T_1x,\ T_2y) \in \alpha V_1 \circ \beta V_2 \circ \gamma V_3\,,$$

if $(y, T_1x) \in V_1$, $(x, y) \in V_2$, $(x, T_2y) \in V_3$, where $\alpha, \beta, \gamma$ are independent of $x, y, V_1, V_2, V_3$ and $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha + \beta + \gamma < 1$. Then $T_1$, $T_2$ have a unique common fixed point in $X$.

## 3. - Results on fixed point of a sequence of operators.

In this section we assume that $(X, \mathscr{U})$ is a uniform space which is sequentially complete and also a Hausdorff space. Further we suppose that $\mathscr{P}$ is a fixed family of pseudometrics on $X$ generating the uniformity $\mathscr{U}$.

We denote by $\mathscr{V}$ the family of all sets of the form $\bigcap_{i=1}^{n} V_{(p_i, r_i)}$, $p_i \in \mathscr{P}$ and $r_i > 0$ $(i = 1, 2, \ldots, n$, the integer $n$ is not fixed).

By an operator on $X$ we mean a mapping of $X$ into itself.

Theorem 3.1. *Let $\{T_n\}$ be a sequence of operators on $X$ such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and any $V$ in $\mathscr{V}$, and $x, y \in X$ with $x \neq y$,*

$$(T_ix,\ T_jy) \in \alpha V \qquad \textit{if} \quad (x, y) \in V\,,$$

*where $0 < \alpha < 1$ and further there exists an $x_0 \in X$ such that in $\{x_n = T_nx_{n-1}\}$, $x_r \neq x_{r+1}$, $(r = 1, 2, \ldots)$. Then $\{T_n\}$ has a unique common fixed point in $X$.*

Proof. We first establish that $\{x_n\}$ is a Cauchy sequence in $X$.

Let $V \in \mathscr{V}$. Using (2.5), we find a positive number $\lambda$ such that $(x_0, x_1) \in \lambda V = W$, say.

Then $W \in \mathscr{V}$ and $\mu W \in \mathscr{V}$ for any $\mu > 0$. We have by the given condition

$$(x_1, x_2) = (T_1x_0,\ T_2x_1) \in \alpha W\,, \qquad (x_2, x_3) = (T_2x_1,\ T_3x_2) \in \alpha(\alpha W) = \alpha^2 W\,.$$

By induction $(x_n, x_{n+1}) \in \alpha^n W$.

Since

$$(x_n, x_{n+1}) \in \alpha^n W\,, \qquad (x_{n+1}, x_{n+2}) \in \alpha^{n+1} W\,,$$

we have [by (2.4)]

$$(x_n, x_{n+2}) \in \alpha^n W \circ \alpha^{n+1} W \subset (\alpha^n + \alpha^{n+1})\, W.$$

Similarly

$$(x_n, x_{n+3}) \in (\alpha^n + \alpha^{n+1} + \alpha^{n+2}) W .$$

Let $n$ and $m$ $(> n)$ be any two positive integers. Then

$$(x_n, x_m) \in (\alpha^n + \alpha^{n+1} + \ldots + \alpha^{m-1}) W \subset \frac{\alpha^n}{1-\alpha} W = \frac{\lambda\alpha^n}{1-\alpha} V .$$

Since $0 < \alpha < 1$, we can choose a positive integer $n_0$ such that $[\lambda\alpha^n/(1-\alpha)] < 1$ for $n \geqslant n_0$. Then $(x_n, x_m) \in V$ for $n \geqslant n_0$. Thus $\{x_n\}$ is a Cauchy sequence in $X$.

Since $X$ is sequentially complete, there is an element $\xi$ in $X$ such that $\xi = \mathscr{L}t_{n\to\infty} x_n$.

Next since $x_n \to \xi$, as $n \to \infty$, for every $V \in \mathscr{V}$, there is a positive integer $N$ such that $(x_{n-1}, \xi) \in V$, when $n \geqslant N$. Since $x_n \neq x_{n+1}$ $(n = 1, 2, \ldots)$ we can choose $r > N$ such that $x_{r-1} \neq \xi$. So for a fixed $m$

$$(x_r, T_m\xi) = (T_r x_{r-1}, T_m\xi) \in \alpha V .$$

Thus $(x_r, \xi) \in V$ and $(x_r, T_m\xi) \in \alpha V$. So $(\xi, T_m\xi) \in V \circ \alpha V \subset (1+\alpha) V$.

Since $V$ is arbitrary, $T_m\xi = \xi$. Hence $\xi$ is a fixed point of $\{T_n\}$.

Let $\eta \in X$ and $T_n\eta = \eta$ $(n = 1, 2, \ldots)$. If possible let $\eta \neq \xi$. Since $X$ is a Hausdorff space, we can find a member $V$ of $\mathscr{V}$ such that $\eta \notin V[\xi]$. Then

$$(\xi, \eta) \notin V . \tag{1}$$

By (2.5) there is a $\lambda > 0$ such that $(\xi, \eta) \in \lambda V = W$, say. Now $(\xi, \eta) = (T_n\xi, T_m\eta) \in \alpha W$. Similarly $(\xi, \eta) \in \alpha(\alpha W) = \alpha^2 W$.
In this way we have

$$(\xi, \eta) \in \alpha^\nu W = \lambda\alpha^\nu V \qquad (\nu = 2, 3, \ldots) . \tag{2}$$

Take $\nu$ so large that $0 < \lambda\alpha^\nu < 1$. Then (2) gives $(\xi, \eta) \in V$, which contradicts (1). Hence $\eta = \xi$ and the proof is complete.

Theorem 3.2. *Let $\{T_n\}$ be a sequence of operators on $X$ such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and for any two members $V_1$, $V_2$ in $\mathscr{V}$ and $x$, $y$ in $X$*

$$(T_i x, T_j y) \in \alpha_{ij} V_1 \circ \alpha_{ij} V_2 ,$$

*if $(x, T_i x) \in V_1$ and $(y, T_j y) \in V_2$, where $\alpha_{ij}$ is independent of $x$, $y$, $V_1$, $V_2$ and*

$0 < \alpha_{ij} \leqslant \alpha < 1$ *and* $\sum_{k=1}^{\infty} [\prod_{i=1}^{k} \alpha_{i,i+1}/(1-\alpha_{i,i+1})]$ *is convergent. Then* $\{T_n\}$ *has a unique common fixed point.*

Proof. Let $x_0$ be an arbitrary but fixed point of $X$. Define the sequence $\{x_n\}$ by

$$x_n = T_n x_{n-1} \qquad (n = 1, 2, \ldots) .$$

We write

$$a_k = \prod_{i=1}^{k} \frac{\alpha_{i,i+1}}{1-\alpha_{i,i+1}} \qquad (k = 1, 2, \ldots) .$$

Now we prove the theorem by the following steps.

(I) The sequence $\{x_n\}$ converges to a point $\xi$ in $X$.

Let $V \in \mathscr{V}$. Suppose $p$ is the Minkowski's pseudometric of $V$. Let $x, y \in V$. Write $p(x, T_i x) = r_i$, $p(y, T_j y) = r_j$ and take $\varepsilon > 0$. Then $(x, T_i x) \in (r_i + \varepsilon) V$, $(y, T_j y) \in (r_j + \varepsilon) V$.

So by the given condition

$$(T_i x, T_j y) \in \alpha_{ij}(r_i + \varepsilon) V_0 \alpha_{ij}(r_j + \varepsilon) V ,$$

$$\therefore\ p(T_i x, T_j y) < \alpha_{ij}(r_i + \varepsilon) + \alpha_{ij}(r_j + \varepsilon) .$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$p(T_i x, T_j y) \leqslant \alpha_{ij} p(x, T_i x) + \alpha_{ij} p(y, T_j y) . \tag{3}$$

Let $\lambda$ be a positive number with $\lambda \geqslant p(x_0, x_1)$.

Now [by (3)]

$$p(x_1, x_2) = p(T_1 x_0, T_2 x_1) \leqslant \alpha_{12} p(x_0, x_1) + \alpha_{12} p(x_1, x_2) .$$

So

$$p(x_1, x_2) \leqslant \frac{\alpha_{12}}{1-\alpha_{12}}\, p(x_0, x_1) \leqslant a_1 \lambda ,$$

$$p(x_2, x_3) = p(T_2x_1, T_3x_2) \leqslant \alpha_{23}p(x_1, x_2) + \alpha_{23}p(x_2, x_3),$$

$$\therefore p(x_2, x_3) \leqslant \frac{\alpha_{23}}{1-\alpha_{23}} p(x_1, x_2) \leqslant a_2\lambda .$$

In general

$$p(x_n, x_{n+1}) \leqslant \left[\prod_{i=1}^{n} \frac{\alpha_{i,i+1}}{1-\alpha_{i,i+1}}\right] \lambda = a_n\lambda .$$

Let us take any two positive integers $n$ and $m$ ($> n$).

Now

$$p(x_n, x_m) \leqslant p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \ldots + p(x_{m-1}, x_m)$$

$$\leqslant (a_n + a_{n+1} + \ldots + a_{m-1})\lambda = [\sum_{k=n}^{m-1} a_k]\lambda .$$

Since $\sum_{k=1}^{\infty} a_k$ is convergent, we can choose a positive integer $n_0$ such that $[\sum_{k=n}^{m-1} a_k]\lambda < 1$, when $n \geqslant n_0$.

So $p(x_n, x_m) < 1$, when $m > n \geqslant n_0$. This gives that

$$(x_n, x_m) \in V_{(p,1)} = V \quad \text{for } m > n \geqslant n_0 .$$

So $\{x_n\}$ is a Cauchy sequence in $X$.

Since $X$ is sequentially complete, there is a point $\xi$ in $X$ such that $\xi = \mathscr{L}t_{n\to\infty}\, x_n$.

(II) $\xi$ is a common fixed point of $\{T_n\}$.

Take $V$ and $p$ as above. For any positive integer $n$, we have

$$p(\xi, T_m\xi) \leqslant p(\xi, x_n) + p(T_nx_{n-1}, T_m\xi)$$

$$\leqslant p(\xi, x_n) + \alpha_{n,m}p(x_{n-1}, x_n) + \alpha_{n,m}p(\xi, T_m\xi) .$$

Letting $n \to \infty$, we have $p(\xi, T_m\xi) \leqslant \alpha p(\xi, T_m\xi)$.

Since $0 < \alpha < 1$, we get $p(\xi, T_m\xi) = 0$. So $(\xi, T_m\xi) \in V$.

Since $V$ is arbitrary and $X$ is a Hausdorff space, we have $\xi = T_m\xi$.

Hence $\xi$ is a fixed point of $T_n$ $(n = 1, 2, \ldots)$.

(III) $\xi$ is the unique common fixed point of $\{T_n\}$.

If possible, let $\eta$ be a fixed point common to $T_n$ $(n = 1, 2, \ldots)$. Take any $V \in \mathscr{V}$.

Then $(\eta, T_1\eta) = (\eta, \eta) \in V$, and $(\xi, T_2\xi) = (\xi, \xi) \in V$.

So we get

$$(\xi, \eta) = (T_1\xi, T_2\eta) \in \alpha_{12} V_0 \alpha_{12} V \subset 2\alpha_{12} V .$$

Since $V$ is arbitrary, $\xi = \eta$. Hence the theorem.

Corollary 3.2.1. *Let $\{T_n\}$ be a sequence of operators on $X$ and $\nu$ a positive integer such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and for any two members $V_1$ and $V_2$ in $\mathscr{V}$ and $x, y \in X$*

$$(T_i^\nu x, T_j^\nu y) \in \alpha_{ij} V_1 \circ \alpha_{ij} V_2 ,$$

*if $(x, T_i^\nu x) \in V_1$ and $(y, T_j^\nu y) \in V_2$, where $\alpha_{ij}$ is independent of $x, y, V_1, V_2$, and $0 < \alpha_{ij} \leqslant \alpha < 1$, and $\sum_{k=1}^{\infty} [\prod_{i=1}^{k} \alpha_{i,i+1}/(1-\alpha_{i,i+1})]$ is convergent. Then $\{T_n\}$ has a unique common fixed point in $X$.*

Theorem 3.3. *Let $\{T_n\}$ be a sequence of operators on $X$ such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and for any two members $V_1$, $V_2$ in $\mathscr{V}$ and $x, y$ in $X$*

$$(T_i x, T_j y) \in \alpha_{ij} V_1 \circ \alpha_{ij} V_2 ,$$

*if $(x, T_i x) \in V_1$ and $(y, T_j y) \in V_2$, where $\alpha_{ij}$ is independent of $x, y, V_1, V_2$ and $0 < \alpha_{ij} < 1$. Further suppose that for at least one pair of operators $T_r$, $T_s$ in $\{T_n\}$ satisfying the above condition, $0 < \alpha_{rs} < \frac{1}{2}$. Then $\{T_n\}$ has a unique common fixed point in $X$.*

Proof. Without any loss of generality, we can take $r = 1, s = 2$. Then in virtue of the given condition, it readily follows from Th. 2.1 that $T_1$ and $T_2$ have a unique common fixed point in $X$. Let $\xi \in X$ be the common fixed point of $T_1$, $T_2$.

Let $V$ be any member of $\mathscr{V}$ and $p$ the Minkowski's pseudometric of $V$. Then for any $x, y$ in $X$ we obtain

$$p(T_i x, T_j y) \leqslant \alpha_{ij} p(x, T_i x) + \alpha_{ij} p(y, T_j y) .$$

Now let us take any member $T_i \neq T_1$, $T_2$ of $\{T_n\}$.

Then

$$\begin{aligned} p(\xi, T_r\xi) &= p(T_1\xi, T_r\xi) \\ &\leqslant \alpha_{1r}p(\xi, T_1\xi) + \alpha_{1r}p(\xi, T_r\xi) \\ &= \alpha_{1r}p(\xi, T_r\xi) . \end{aligned}$$

Since $0 < \alpha_{1r} < 1$, $p(\xi, T_r\xi) = 0$. So $(\xi, T_r\xi) \in V$.

Since $V$ is arbitrary and $X$ is a Hausdorff space, $\xi = \mathrm{T}_r\xi$.

The uniqueness of $\xi$ can be shown exactly as in Th. 3.2.

Note 3.3.1. $\xi$ is the only fixed point of each of the members of the sequence $\{T_n\}$.

Corollary 3.3.1. *Let $\{T_n\}$ be a sequence of operators on $X$ and $\nu$ a positive integer such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and for any two members $V_1$, $V_2$ in $\mathscr{V}$ and $x, y$ in $X$*

$$(T_i^\nu x, T_j^\nu y) \in \alpha_{ij} V_1 \circ \alpha_{ij} V_2 ,$$

*if $(x, T_i^\nu x) \in V_1$ and $(y, T_j^\nu y) \in V_2$ where $\alpha_{ij}$ is independent of $x, y, V_1, V_2$ and $0 < \alpha_{ij} < 1$. Further suppose that for at least one pair of operators $T_r$, $T_s$ in $\{T_n\}$ satisfying the above condition, $0 < \alpha_{rs} < \frac{1}{2}$. Then $\{T_n\}$ has a unique common fixed point in $X$.*

Theorem 3.4. *Let $\{T_n\}$ be a sequence of operators on $X$ such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and for any two members $V_1$, $V_2$ in $\mathscr{V}$ and $x, y$ in $X$*

$$(T_i x, T_j y) \in \alpha_{ij} V_1 \circ \alpha_{ij} V_2 ,$$

*if $(y, T_i x) \in V_1$ and $(x, T_j y) \in V_2$, where $\alpha_{ij}$ is independent of $x, y, V_1, V_2$ and $0 < \alpha_{ij} < 1$. Further suppose that for at least one pair of operators $T_r$, $T_s$ in $\{T_n\}$ satisfying the above condition, $0 < \alpha_{rs} < \frac{1}{2}$.*

*Then $\{T_n\}$ has a unique common fixed point in $X$.*

Proof. Without any loss of generality we can take $r = 1$, $s = 2$. Then it follows from Th. 2.2 that $T_1$ and $T_2$ have a unique common fixed point $\xi$ in $X$.

Let $V$ be any member of $\mathscr{V}$ and $p$ the Minkowski's pseudometric of $V$. Then for any $x, y$ in $X$ we obtain,

$$p(T_i x, T_j y) \leqslant \alpha_{ij}p(y, T_i x) + \alpha_{ij}p(x, T_j y) .$$

We deduce that $\xi$ is a fixed point common to each member of $\{T_n\}$.

Now we show the uniqueness of $\xi$. Let $\eta \in X$ and $T_n \eta = \eta$ for $n = 1, 2, \ldots$. Take $V$ and $p$ as above.

Then

$$\begin{aligned} p(\xi, \eta) &= p(T_1 \xi, T_2 \eta) \\ &\leqslant \alpha_{12} p(\eta, T_1 \xi) + \alpha_{12} p(\xi, T_2 \eta) \\ &= 2\alpha_{12} p(\xi, \eta) . \end{aligned}$$

Since $0 < 2\alpha_{12} < 1$, $p(\xi, \eta) = 0$. So, $(\xi, \eta) \in V$. $V$ being arbitrary, $X$ a Hausdorff space, $\xi = \eta$.

Note 3.4.1. If each $\alpha_{ij}$ in Th. 3.4 satisfies the condition $0 < \alpha_{ij} < \frac{1}{2}$, then $\xi$ is the only fixed point of each of $\{T_n\}$.

Corollary 3.4.1. *Let $\{T_n\}$ be a sequence of operators on $X$ and $\nu$ be a positive integer such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and for any two members $V_1$, $V_2$ in $\mathscr{V}$ and $x, y$ in $X$,*

$$(T_i^\nu x, T_j^\nu y) \in \alpha_{ij} V_1 \circ \alpha_{ij} V_2 ,$$

*if $(y, T_i^\nu x) \in V_1$ and $(x, T_j^\nu y) \in V_2$, where $\alpha_{ij}$ is independent of $x$, $y$, $V_1$, $V_2$ and $0 < \alpha_{ij} < \frac{1}{2}$. Then $\{T_n\}$ has a unique common fixed point in $X$.*

Theorem 3.5. *Let $\{T_n\}$ be a sequence of operators on $X$ such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and for any three members $V_1$, $V_2$, $V_3$ in $\mathscr{V}$ and $x, y \in X$,*

$$(T_i x, T_j y) \in \alpha_{ij} V_1 \circ \beta_{ij} V_2 \circ \alpha_{ij} V_3 ,$$

*if $(x, T_i x) \in V_1$, $(x, y) \in V_2$, $(y, T_j y) \in V_3$, where $\alpha_{ij}$, $\beta_{ij}$ are independent of $x$, $y$, $V_1$, $V_2$, $V_3$ and $0 < \alpha_{ij} \leqslant \alpha < 1$, $0 < \beta_{ij} < 1$ and $\sum_{k=1}^{\infty} [\prod_{i=1}^{k} (\alpha_{i,i+1} + \beta_{i,i+1})/(1 - \alpha_{i,i+1})]$ is convergent. Then $\{T_n\}$ has a unique common fixed point in $X$.*

Proof. Let $x_0$ be an arbitrary but fixed point of $X$. Define the sequence $\{x_n\}$ and take $V$ and $p$ as in Th. 3.2. Let

$$a_n = \prod_{i=1}^{n} \left( \frac{\alpha_{i,i+1} + \beta_{i,i+1}}{1 - \alpha_{i,i+1}} \right) \qquad (n = 1, 2, \ldots) .$$

Write

$$p(x, T_i x) = r_i, \quad p(x, y) = r, \quad p(y, T_j y) = r_j$$

and take $\varepsilon > 0$. Then

$$(x, T_i x) \in (r_i + \varepsilon) V, \quad (x, y) \in (r + \varepsilon) V, \quad (y, T_j y) \in (r_j + \varepsilon) V.$$

So by the given condition

$$(T_i x, T_j y) \in \alpha_{ij}(r_i + \varepsilon) V \circ \beta_{ij}(r + \varepsilon) V \circ \alpha_{ij}(r_j + \varepsilon) V,$$

in view of Note 2.1, we obtain

$$p(T_i x, T_j y) < \alpha_{ij}(r_i + \varepsilon) + \beta_{ij}(r + \varepsilon) + \alpha_{ij}(r_j + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary

$$p(T_i x, T_j y) \leqslant \alpha_{ij} p(x, T_i x) + \beta_{ij} p(x, y) + \alpha_{ij} p(y, T_j y).$$

Now take a positive number $\lambda$ with $\lambda \geqslant p(x_0, x_1)$.

Then

$$p(x_1, x_2) = p(T_1 x_0, T_2 x_1) \leqslant \alpha_{12} p(x_0, x_1) + \beta_{12} p(x_0, x_1) + \alpha_{12} p(x_1, x_2),$$

or

$$p(x_1, x_2) \leqslant \frac{\alpha_{12} + \beta_{12}}{1 - \alpha_{12}} p(x_0, x_1) \leqslant a_1 \lambda,$$

$$p(x_2, x_3) = p(T_2 x_1, T_3 x_2) \leqslant \alpha_{23} p(x_1, x_2) + \beta_{23} p(x_1, x_2) + \alpha_{23} p(x_2, x_3),$$

or

$$p(x_2, x_3) \leqslant \frac{\alpha_{12} + \beta_{12}}{1 - \alpha_{12}} \cdot \frac{\alpha_{23} + \beta_{23}}{1 - \alpha_{23}} \lambda = a_2 \lambda.$$

In general

$$p(x_n, x_{n+1}) \leqslant \left[ \prod_{i=1}^{n} \frac{\alpha_{i,i+1} + \beta_{i,i+1}}{1 - \alpha_{i,i+1}} \right] \lambda = a_n \lambda.$$

Let $n$ and $m$ $(>n)$ be any two positive integers. Then

$$p(x_n, x_m) \leqslant p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \dots + p(x_{m-1}, x_m)$$
$$\leqslant [a_n + a_{n+1} + \dots + a_{m-1}]\lambda = [\sum_{k=n}^{m-1} a_k]\,\lambda\,.$$

Since by hypothesis, $\sum_{k=1}^{\infty} a_k$ is convergent, we can find a positive integer $n_0$ such that $\lambda \sum_{k=n}^{m-1} a_k < 1$, for $n \geqslant n_0$.
So $p(x_n, x_m) < 1$, for $m > n \geqslant n_0$.
Then we complete the proof as in Th. 3.2..

Corollary 3.5.1. *Let $\{T_n\}$ be a sequence of operators on $X$ and $\nu$ a positive integer such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and any three members $V_1$, $V_2$, $V_3$ in $\mathscr{V}$ and $x, y$ in $X$*

$$(T_i^\nu x,\ T_j^\nu y) \in \alpha_{ij} V_1 \circ \beta_{ij} V_2 \circ \alpha_{ij} V_3\,,$$

*if $(x, T_i^\nu x) \in V_1$, $(x, y) \in V_2$, $(y, T_j^\nu y) \in V_3$, where $\alpha_{ij}$, $\beta_{ij}$ are independent of $x, y$, $V_1, V_2, V_3$ and $0 < \alpha_{ij} \leqslant \alpha < 1$, $0 < \beta_{ij} < 1$, and $\sum_{k=1}^{\infty} [\prod_{i=1}^{k} (\alpha_{i,i+1} + \beta_{i,i+1})/(1-\alpha_{i,i+1})]$ is convergent. Then $\{T_n\}$ has a unique common fixed point in $X$.*

Theorem 3.6. *Let $\{T_n\}$ be a sequence of operators on $X$ such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and for any three members $V_1$, $V_2$, $V_3$ in $\mathscr{V}$ and $x, y$ in $X$*

$$(T_i x,\ T_j y) \in \alpha_{ij} V_1 \circ \beta_{ij} V_2 \circ \alpha_{ij} V_3\,,$$

*if $(x, T_i x) \in V_1$, $(x, y) \in V_2$, $(y, T_j y) \in V_3$, where $\alpha_{ij}$, $\beta_{ij}$ are independent of $x, y$, $V_1$, $V_2$, $V_3$ and $0 < \alpha_{ij} < 1$, $\beta_{ij} > 0$. Further suppose that for at least one pair of operators $T_r$, $T_s$ satisfying the above condition, $2\alpha_{rs} + \beta_{rs} < 1$. Then $\{T_n\}$ has a unique common fixed point in $X$.*

Proof. Without any loss of generality we take $r=1$, $s=2$. Then it follows from Th. 2.3 that $T_1$, $T_2$ have a unique common fixed point $\xi$ in $X$.
Let $V$ be any member of $\mathscr{V}$ and $p$ the Minkowski's pseudometric of $V$. Then for any $x, y$ in $X$ we obtain

$$p(T_i x,\ T_j y) \leqslant \alpha_{ij} p(x,\ T_i x) + \beta_{ij} p(x, y) + \alpha_{ij} p(y,\ T_j y)\,.$$

Let us take any member $T_i \neq T_1,\ T_2$ of $\{T_n\}$.

Then

$$p(\xi, T_i\xi) = p(T_1\xi, T_i\xi)$$
$$\leqslant \alpha_{1i} p(\xi, T_1\xi) + \beta_{1i} p(\xi, \xi) + \alpha_{1i} p(\xi, T_i\xi) = \alpha_{1i} p(\xi, T_i\xi)\,.$$

Since $0 < \alpha_{1i} < 1$, $p(\xi, T_i\xi) = 0$. So $(\xi, T_i\xi) \in V$.

Since $V$ is arbitrary and $X$ is a Hausdorff space, $\xi = T_i\xi$.

Let $\eta \in X$ and $T_i\eta = \eta$, $(i = 1, 2, \ldots)$. Then

$$p(\xi, \eta) = p(T_1\xi, T_2\eta)$$
$$\leqslant \alpha_{12} p(\xi, T_1\xi) + \beta_{12} p(\xi, \eta) + \alpha_{12} p(\eta, T_2\eta) = \beta_{12} p(\xi, \eta)\,.$$

Since $0 < \beta_{12} < 1$, $p(\xi, \eta) = 0$. So $(\xi, \eta) \in V$. $V$ being arbitrary, $\xi = \eta$.

Note 3.6.1. $\xi$ is the only fixed point of each of the members of the sequence $\{T_n\}$.

Corollary 3.6.1. *Let $\{T_n\}$ be a sequence of operators on $X$ and $\nu$ a positive integer such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and for any three members $V_1, V_2, V_3$ in $\mathscr{V}$ and $x, y$ in $X$*

$$(T_i^\nu x, T_j^\nu y) \in \alpha_{ij} V_1 \circ \beta_{ij} V_2 \circ \alpha_{ij} V_3,$$

*if $(x, T_i^\nu x) \in V_1$, $(x, y) \in V_2$, $(y, T_j^\nu y) \in V_3$, where $\alpha_{ij}, \beta_{ij}$ are independent of $x, y, V_1, V_2, V_3$ and $0 < \alpha_{ij} < 1$, $\beta_{ij} > 0$. Further suppose that for at least one pair of operators $T_r$, $T_s$ satisfying the above condition, $2\alpha_{rs} + \beta_{rs} < 1$. Then $\{T_n\}$ has a unique common fixed point in $X$.*

Theorem 3.7. *Let $\{T_n\}$ be a sequence of operators on $X$ such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and for any three members $V_1, V_2, V_3$ in $\mathscr{V}$ and $x, y$ in $X$*

$$(T_i x, T_j y) \in \alpha_{ij} V_1 \circ \beta_{ij} V_2 \circ \alpha_{ij} V_3,$$

*if $(y, T_i x) \in V_1$, $(x, y) \in V_2$, $(x, T_j y) \in V_3$, where $\alpha_{ij}, \beta_{ij}$ are independent of $x, y, V_1, V_2, V_3$ and $0 < \alpha_{ij} < 1$, $\beta_{ij} > 0$. Further suppose that for at least one pair of operators $T_r$, $T_s$ satisfying the above condition, $2\alpha_{rs} + \beta_{rs} < 1$. Then $\{T_n\}$ has a unique common fixed point in $X$.*

Proof. Without any loss of generality, we take $r=1$, $s=2$. Then it follows from Th. 2.4 that $T_1$, $T_2$ have a unique common fixed point $\xi$ in $X$.

Let $V\in\mathscr{V}$ and $p$ be the Minkowski's pseudometric of $V$. Then for any $x, y$ in $X$ we deduce that

$$p(T_i x, T_j y)\leqslant \alpha_{ij}p(y, T_i x) + \beta_{ij}p(x, y) + \alpha_{ij}p(x, T_j y)$$

and show that $\xi$ is a fixed point of each member of $\{T_n\}$.

Then we show the uniqueness of $\xi$.

Let $\eta\in X$ and $T_i\eta=\eta$, $(i=1, 2, \ldots)$. Take $V$ and $p$ as above. Then

$$\begin{aligned} p(\xi, \eta) &= p(T_1\xi, T_2\eta) \\ &\leqslant \alpha_{12}p(\eta, T_1\xi) + \beta_{12}p(\xi, \eta) + \alpha_{12}p(\xi, T_2\eta) \\ &= (2\alpha_{12}+\beta_{12})p(\xi, \eta)\,. \end{aligned}$$

Since $0<2\alpha_{12}+\beta_{12}<1$, $p(\xi, \eta)=0$, So $(\xi, \eta)\in V$. $V$ being arbitrary, $\xi=\eta$.

Note 3.7.1. If each $\alpha_{ij}$ in Th. 3.7. satisfies the condition $0<2\alpha_{ij}+$ $+\beta_{ij}<1$, then $\xi$ is the only fixed point of each member of $\{T_n\}$.

Corollary 3.7.1. *Let $\{T_n\}$ be a sequence of operators on $X$ and $v$ be a positive integer such that for any two operators $T_i$, $T_j$ in $\{T_n\}$ and for any three members $V_1$, $V_2$, $V_3$ in $\mathscr{V}$ and $x, y$ in $X$*

$$(T_i^v x, T_j^v y)\in\alpha_{ij}V_1\circ\beta_{ij}V_2\circ\alpha_{ij}V_3,$$

*if $(y, T_i^v x)\in V_1$, $(x, y)\in V_2$, $(x, T_j^v y)\in V_3$, where $\alpha_{ij}, \beta_{ij}$ are independent of $x, y, V_1, V_2, V_3$, and $\alpha_{ij}>0$, $\beta_{ij}>0$, $2\alpha_{ij}+\beta_{ij}<1$. Then $\{T_n\}$ has a unique common fixed point in $X$.*

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