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A note on a class of operators. (**)

1. – In the present Note we extend Luecke's class of operators [2] and obtain several results for the extended class of operators.

Let T be an operator (a bounded linear transformation on a complex Hilbert space H. Let $\sigma(T)$, con $\sigma(T)$, $\overline{W(T)}$ denote the spectrum, convex-hull of the spectrum and closure of the numerical range of T respectively. If X is a subset of the complex plane then ∂X denotes the boundary of X. In [2] Luecke has introduced a new class of operators as follows.

An operator $T \in R$ if and only if $\|(T - z)^{-1}\| = 1/d(z, \overline{W(T)}$ for all $z \notin \overline{W(T)}$, where $d(z, \overline{W(T)}) = \operatorname{Inf}\{|z - \mu|: \mu \in \overline{W(T)}\}$. We say an operator T is of class Y if and only if $\|(T - z)^{-1}\| \geqslant 1/d(z, \text{con } \sigma(T))$ for all $z \notin \text{con } \sigma(T)$. Firstly we prove that Y is an extension of R.

Let $T \in R$ then $\overline{W(T)} = \text{con } \sigma(T)$. Thus $\|(T - z)^{-1}\| = 1/d(z, \text{con } \sigma(T))$, and hence $T \in Y$. Since Y contains all quasi-nilpotent operators, it follows that R is properly contained in Y. Thus Y provides an extension of R. Following the same technique due to Luecke [2], it can be shown that $T \in Y$ if and only if ∂ con $\sigma(T) \subseteq \sigma(T)$ using this characterization of class Y, we show here by producing an example that the inverse of a non-singular operator in Y need not be in Y.

Example. Let T be a nonsingular operator such that $\sigma(T) = CU\{\frac{1}{2}\}$ where the notation C is used for the unit circle. Clearly ∂ con $\sigma(T) = C \subseteq \sigma(T)$. Therefore by the characterization given above, $T \in Y$. Now as $\sigma(T^{-1}) = CU(\bar{2})$, ∂ con $\sigma(T^{-1})$ is not contained in $\sigma(T^{-1})$ and so $T^{-1} \notin Y$.

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2. – Next, we give another characterization of the operators of class Y. Let $\widetilde{\sigma(T)}$ denotes the hen-spectrum of T, defined by Fujii [**1**]. Fujii has proved $\sigma(T) \subseteq \widetilde{\sigma(T)} \subseteq \operatorname{con} \sigma(T) \subseteq \overline{W(T)}$.

Further more we need the following idea due to Fujii [**1**]. T is an operator satisfying the condition (H_1) if

$$\|(T-z)^{-1}\| \leqslant 1/d(z, \widetilde{\sigma(T)}) \quad \text{for any} \quad z \notin \widetilde{\sigma(T)} .$$

In the following theorem we characterize the operators belonging to Y as follows.

Theorem 1. *$T \in Y$ if and only if* $\operatorname{con} \sigma(T) = \widetilde{\sigma(T)}$.

Proof. Let $T \in Y$ then $\partial \operatorname{con} \sigma(T) \subseteq \sigma(T)$ so that $\sigma(T)$ contains the convex curve $\partial \operatorname{con} \sigma(T)$. Therefore $\operatorname{con} \sigma(T) \subseteq \widetilde{\sigma(T)} \subseteq \operatorname{con} \sigma(T)$ and hence $\operatorname{con} \sigma(T) = \widetilde{\sigma(T)}$.

Conversely, let $\operatorname{con} \sigma(T) = \widetilde{\sigma(T)}$. Then $\partial \operatorname{con} \sigma(T) = \partial \widetilde{\sigma(T)} \subseteq \sigma(T)$ so that $\partial \operatorname{con} \sigma(T) \subseteq \sigma(T)$. Hence $T \in Y$.

Next we prove the following

Theorem 2. *If A is an operator and B is a normal operator with $\overline{W(A)} \subset \sigma(B)$ and $\widetilde{\sigma(B)} \neq \operatorname{con} \sigma(B)$. Then $T = A \oplus B \notin Y$ but T satisfies the condition (H_1).*

Proof. Since by hypothesis, $\overline{W(A)} \subset \sigma(B)$. Therefore T satisfies the condition (H_1) by [**1**]. As $T = A \oplus B$, it follows that $\sigma(T) = \sigma(A) \cup \sigma(B) = \sigma(B)$. Also $\widetilde{\sigma(T)} = \widetilde{\sigma(A)} \cup \widetilde{\sigma(B)} = \widetilde{\sigma(B)} \neq \operatorname{con} \sigma(B) = \operatorname{con} \sigma(T)$. Therefore $T \notin Y$ by Theorem 1.

We now give another method to construct non-trivial examples of operators in Y.

Theorem 3. *If A is an operator on H, then $A \oplus N \in Y$ on $H \oplus K$ whenever N is a normal operator on K with* $\sigma(A) = \operatorname{con} \sigma(N)$.

Proof. Let $T = A \oplus N$. Then $\sigma(T) = \sigma(A) \cup \sigma(N)$. Therefore $\operatorname{con} \sigma(T) = \operatorname{con} \sigma(N)$. Let $z \notin \sigma(A)$. Then

$$\|(A-z)^{-1}\| \geqslant 1/d(z, \sigma(A)) = 1/d(z, \operatorname{con} \sigma(N)) = 1/d(z, \operatorname{con} \sigma(T)) .$$

Also

$$\|(N-z)^{-1}\| = 1/d(z, \sigma(N)) = 1/d(z, \operatorname{con} \sigma(N)) = 1/d(z, \operatorname{con}(T)) .$$

Therefore

$$\|(T-z)^{-1}\| = \text{Max}\,\{\|(A-z)^{-1}\|,\ \|(N-z)^{-1}\| \geqslant\} 1/d(z, \text{con}\,\sigma(T))\ .$$

Thus $T \in Y$.

We say that an operator T belongs to class Y locally if $\|(T-z)^{-1}\| \geqslant 1/d(z, \text{con}\,\sigma(T))$ for all $z \in U - \text{con}\,\sigma(T)$, where U is an open set containing $\text{con}\,\sigma(T)$. In the following theorem we prove

Theorem 4. *If $T \in Y$ locally, then $T \in Y$.*

Proof. Let $z_0 \in \partial\,\text{con}\,\sigma(T)$. Then there exists a sequence $z_n \subset U - \text{con}\,\sigma(T)$ such that $z_n \to z_0$ and $|z_n - z_0| = d(z_n, \text{con}\,\sigma(T))$.

Therefore $\|(T - z_n)^{-1}\| \geqslant 1/d(z_n, \text{con}\,\sigma(T)) = 1/|z_n - z_0| \to \infty$ which shows $z_0 \in \sigma(T)$. Hence $T \in Y$.

Lastly, we remark that Y contains nilpotent operators and it excludes all operators with countable spectra having more than one point, it follows Y is independent of the class of convexoid operators. In other words we get that the class (H_1) due to Fujii and class Y are independent of each other.

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References.

[1] M. Fujii, *On some examples of non-normal operators*, Proc. Japan Acad. **49** (1973), 118-123.

[2] G. R. Luecke, *A class of operators*, Pacific J. Math. **41** (1972), 153-156.

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