Riv. Mat. Univ. Parma (4) 3 (1977), 53-57

MOHD. SAEED KHAN (*)

A fixed point theorem for metric spaces. (**)

Let (X, d) be a complete metric space, and let $T: X \to X$ satisfy

$$d(Tx, Ty) \leqslant \alpha d(x, y) ,$$

where $0 \leqslant \alpha < 1$ and $x, y \in X$. By Banach's fixed point theorem T has a unique fixed point.

Many extensions and generalizations of Banach's fixed point theorem were derived in recent years. For related results see [**1**]$_1$, [**1**]$_2$, [**2**]. In this Note, we shall prove theorems about fixed points using rational expression.

1. - Theorem 1. *Let (X, d) be a complete metric space and let $T: X \to X$ satisfy*

$$d(Tx, Ty) \leqslant \alpha \frac{d(x, Tx)\, d(x, Ty) + d(y, Ty)\, d(y, Tx)}{d(x, Ty) + d(y, Tx)}$$

where $0 \leqslant \alpha < 1$ and $x, y \in X$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$. Put: $x_n = T(x_{n-1})$ $(n = 1, 2, 3, \ldots)$; then we have

$$d(x_1, x_2) = d(Tx_0, Tx_1) \leqslant \alpha \frac{d(x_0, Tx_0)\, d(x_0, Tx_1) + d(x_1, Tx_1)\, d(x_1, Tx_0)}{d(x_0, Tx_1) + d(x_1, Tx_0)}$$

$$= \alpha \frac{d(x_0, x_1)\, d(x_0, x_2) + d(x_1, x_2)\, d(x_1, x_1)}{d(x_0, x_2) + d(x_1, x_1)} .$$

Hence: $d(x_1, x_2) \leqslant \alpha d(x_0, x_1)$.

(*) Indirizzo: Department of Mathematics and Statistics, Aligarh Muslin University, Aligarh - 202001, India.
(**) Ricevuto: 20-II-1975.

Similarly, we have:

$$d(x_2, x_3) = d(Tx_1, Tx_2) \leqslant \alpha \frac{d(x_1, x_2)\, d(x_1, x_3) + d(x_2, x_3)\, d(x_2, x_2)}{d(x_1, x_3) + d(x_2, x_2)} .$$

Therefore: $d(x_2, x_3) \leqslant \alpha d(x_1, x_2)$.
In general, we have: $d(x_n, x_{n+1}) \leqslant \alpha^n d(x_0, x_1)$.

This means that the sequence $\{x_n\}$ is a Cauchy sequence. Hence, by the completeness of X, $\{x_n\}$ converges to some point x in X. For the point x

$$\begin{aligned} d(x, Tx) &\leqslant d(x, x_{n+1}) + d(Tx_n, Tx) \\ &\leqslant d(x, x_{n+1}) + \alpha \frac{d(x_n, Tx_n)\, d(x_n, Tx) + d(x, Tx)\, d(x, Tx_n)}{d(x_n, Tx) + d(x, Tx_n)} \\ &\leqslant d(x, x_{n+1}) + \alpha \frac{d(x_n, x_{n+1})\, d(x_n, Tx) + d(x, Tx)\, d(x, x_{n+1})}{d(x_n, Tx) + d(x, x_{n+1})} . \end{aligned}$$

Letting $n \to \infty$, then we have: $d(x, Tx) \leqslant 0$. Therefore $d(x, Tx) = 0$; that is, the point x is a fixed point of T. For the uniqueness of x, let y be any other fixed point of T. Then

$$\begin{aligned} d(x, y) = d(tx, Ty) &\leqslant \alpha \frac{d(x, Tx)\, d(x, Ty) + d(y, Ty)\, d(y, Tx)}{d(x, Ty) + d(y, Tx)} \\ &\leqslant \alpha \frac{d(x, x)\, d(x, y) + d(y, y)\, d(y, x)}{d(x, y) + d(y, x)} . \end{aligned}$$

Hence $d(x, y) = 0$; that is, $x = y$. This completes the proof of Theorem 1.

2. – Theorem 2. *Let S and T be mappings of a complete metric space (X, d) into itself. Suppose that there exists a non-negative real number α such that $\alpha < 1$ and*

$$d(Tx, Sy) \leqslant \alpha \frac{d(x, Tx)\, d(x, Sy) + d(y, Sy)\, d(y, Tx)}{d(x, Sy) + d(y, Tx)}$$

for all x, y in X. Then S, T have a unique common fixed point.

Proof. Let $x_0 \in X$. Define

$$x_{2n+1} = S(x_{2n}) , \qquad x_{2n+2} = T(x_{2n+1}), \qquad (n = 0, 1, 2, 3, \ldots) ,$$

then we have

$$d(x_1, x_2) = d(Sx_0, Tx_1) \leqslant \alpha \frac{d(x_0, Sx_0)\, d(x_0, Tx_1) + d(x_1, Tx_1)\, d(x_1, Sx_0)}{d(x_1, Sx_0) + d(x_0, Tx_1)}$$

$$\leqslant \alpha \frac{d(x_0, x_1)\, d(x_0, x_2) + d(x_1, x_2)\, d(x_1, x_1)}{d(x_1, x_1) + d(x_0, x_2)} .$$

Hence: $d(x_1, x_2) \leqslant \alpha d(x_0, x_1)$.

Similarly we have

$$d(x_2, x_3) = d(Tx_1, Sx_2) \leqslant \alpha \frac{d(x_1, Tx_1)\, d(x_1, Sx_2) + d(x_2, Sx_2)\, d(x_2, Tx_1)}{d(x_2, Tx_1) + d(x_1, Sx_2)}$$

$$\leqslant \alpha \frac{d(x_1, x_2)\, d(x_1, x_3) + d(x_2, x_3)\, d(x_2, x_2)}{d(x_2, x_2) + d(x_1, x_3)} .$$

This gives: $d(x_2, x_3) \leqslant \alpha d(x_1, x_2)$. In general, we have: $d(x_n, x_{n+1}) \leqslant \alpha^n d(x_0, x_1)$.

Thus $\{x_n\}$ is a Cauchy sequence. Hence, by the completeness of X, $\{x_n\}$ converges to some point x in X. For the point x

$$d(x, Tx) \leqslant d(x, x_{n+1}) + d(Tx_n, Tx)$$

$$\leqslant d(x, x_{n+1}) + \alpha \frac{d(x_n, Tx_n)\, d(x_n, Tx) + d(x, Tx)\, d(x, Tx_n)}{d(x_n, Tx) + d(x, Tx_n)}$$

$$\leqslant d(x, x_{n+1}) + \alpha \frac{d(x_n, x_{n+1})\, d(x_n, Tx) + d(x, Tx)\, d(x, x_{n+1})}{d(x_n, Tx) + d(x, x_{n+1})} .$$

As $n \to \infty$; we find that $d(x, Tx) = 0$, that is, x is a fixed point of T. Similarly x is a fixed point of S. To show that x is a unique common fixed point of S and T, let y be a fixed point of T. Then

$$d(x, y) = d(Sx, Ty) \leqslant \alpha \frac{d(x, Sx)\, d(x, Ty) + d(y, Ty)\, d(y, Sx)}{d(x, Ty) + d(y, Sx)}$$

$$\leqslant \alpha \frac{d(x, x)\, d(x, y) + d(y, y)\, d(y, x)}{d(x, y) + d(y, x)} .$$

This shows that: $d(x, y) = 0$ or $x = y$. So T has a unique fixed point.

Similarly, S has a unique fixed point.

Remark. By replacing S and T by S^p and T^q respectively, for some positive integers p, q, one can prove that in this case also S and T have a unique common fixed point.

3. – Theorem 3. *Let $\{T_n\}$ be a sequence of mappings of a complete metric space (X, d) into itself. Let x_n be a fixed point of T_n $(n = 1, 2, \ldots)$ and suppose T_n converges uniformly to T_0. If T_0 satisfies the condition*

$$(*) \qquad d(T_0 x, T_0 y) \leqslant \alpha \frac{d(x, T_0 x)\, d(x, T_0 y) + d(y, T_0 y)\, d(y, T_0 x)}{d(x, T_0 y) + d(y, T_0 x)},$$

where $0 \leqslant \alpha < 1$, then $\{x_n\}$ converges to the fixed point x_0 of T_0.

Proof. Under the condition $(*)$, T_0 has a unique fixed point by the Theorem 1.

Let $\varepsilon > 0$ be given, then there is a natural number N such that: $d(T_n x, T_0 x) < \varepsilon(1 - \alpha)$ for all $x \in X$ and $n \geqslant N$. Hence

$$d(x_n, x_0) = d(T_n x_n, T_0 x_0) \leqslant d(T_n x_n, T_0 x_n) + d(T_0 x_n, T_0 x_0)$$

$$\leqslant d(T_n x_n, T_0 x_n) + \alpha \frac{d(x_n, T_0 x_n)\, d(x_n, T_0 x_0) + d(x_0, T_0 x_0)\, d(x_0, T_0 x_n)}{d(x_n, T_0 x_0) + d(x_0, T_0 x_n)}$$

$$\leqslant d(T_n, x_n, T_0 x_0) + \alpha \frac{d(x_n, T_0 x_n)\, d(x_n, x_0) + d(x_0, x_0)\, d(x_0, T_0 x_n)}{d(x_n, x_0) + d(x_0, T_0 x_n)}$$

$$\leqslant d(T_n x_n, T_0 x_n) + \alpha \frac{d(x_n, T_0 x_n)\, d(x_n, x_0)}{d(x_n, x_0) + d(x_0, T_0 x_n)}$$

$$\leqslant d(T_n x_n, T_0 x_n) + \alpha \frac{d(x_n, x_0)\big(d(x_n, x_0) + d(x_0, T_0 x_n)\big)}{\big(d(x_n, x_0) + d(x_0, T_0 x_n)\big)}.$$

Therefore, $d(x_n, x_0) \leqslant \big(1/(1 - \alpha)\big)\, d(T_n x_n, T_0 x_n) \leqslant \varepsilon$, for $n > N$. Which shows that $\{x_n\}$ converges to x_0.

References.

[1] M. Edelstein: $[\bullet]_1$ *On fixed and periodic points under contraction mapping*, J. London Math. Soc. **37** (1962), 74-79; $[\bullet]_2$ *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc. **12** (1961), 7-10.

[2] R. Kannan, *Some results on fixed points* II, Amer. Math. Monthly **76** (1969), 405-408.

Summary.

The object of this paper is to prove a fixed point theorem using symmetric rational-expression and to study related results.

* * *