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# On certain classes of operators. (**)

## 1. - Introduction.

We shall call a bounded operator $T$ on a complex Hilbert space $H$ to be of class $(H;\,k)$, $k\geqslant 2$, if it satisfies the inequality

$$\|T^k x\| \geqslant \|T^* x\|^k ,$$

for all unit vectors $x$ in $H$. An operator $T$ is said to be *hyponormal* if $\|Tx\|\geqslant \|T^* x\|$ for all $x$ in $H$; *normaloid* if $\|T\|^n=\|T^n\|$ for all $n\geqslant 1$ and an operator of class $(N;\,k)$ if $\|T^k x\|\geqslant\|Tx\|^k$ for all unit vectors $x$, where $k\geqslant 2$ is a positive integer.

The object of this paper is to establish certain interesting properties of the class $(H;\,k)$ including its relations with the above defined classes of operators. Our first observation is that each hyponormal operator $T$ is in the class $(H;\,k)$ [4]; we shall prove below that $(H;\,k)\subseteq(N;\,k+1)$ for each $k\geqslant 2$. Since each operator of class $(N;\,k)$ is normaloid for all $k\geqslant 2$ [**5**]$_2$, we have the following inclusion relations:

$$\left\{\begin{matrix}\text{hyponormal}\\ \text{operators}\end{matrix}\right\}\subseteq(H;\,k)\subseteq(N;\,k+1)\subseteq\left\{\begin{matrix}\text{normaloid}\\ \text{operators}\end{matrix}\right\}.$$

The class $(H;\,2)$ has been recently studied by S. M. Patel [**5**]$_2$. Our main concern in this paper will be the class $(H;\,k)$, $k\geqslant 3$.

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2. – Theorem 1. *Let $k \geqslant 3$ be any positive integer. If $T$ has the property*

$$T^{*k}T^{k} - 2z(TT^{*})^{k/2} + z^{2} \geqslant 0 \qquad (z > 0)\,, \tag{1}$$

*then $T$ is of class $(H; k)$, but the converse is not true.*

Proof. For $\|x\| = 1$,

$$0 \leqslant \left(\left(T^{*k}T^{k} - 2z(TT^{*})^{k/2} + z^{2}\right)x, x\right) \leqslant \|T^{k}x\|^{2} - 2z(TT^{*}x, x)^{k/2} + z^{2}$$
$$\leqslant \|T^{k}x\|^{2} - 2z\|T^{*}x\|^{k} + z^{2}\,.$$

Taking $z = \|T^{*}x\|^{k}$, we obtain $\|T^{k}x\| \geqslant \|T^{*}x\|^{k}$.

To see that the converse is not true, let $K$ be the direct sum of an infinite number of copies of $H$ and let $A$ and $B$ be any two bounded positive operators on $H$. Define $T = T_{A,B}$ on $K$ as follows:

$$T\langle x_1, x_2, x_3, \ldots, x_n, \ldots\rangle = \langle 0, Ax_1, Bx_2, Bx_3, \ldots\rangle\,.$$

Then $T$ is hyponormal iff $B^{2} \geqslant A^{2}$. Now a simple computation shows that (1) is equivalent to

$$B^{2k} - 2zA^{k} + z^{2} \geqslant 0 \qquad (z > 0)\,. \tag{2}$$

If we take $A = C^{\frac{1}{2}}$ and $B = D^{\frac{1}{2}}$, where

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

are positive operators on the two dimensional Hilbert space $H$, then $B^{2} \geqslant A^{2}$. Thus $T$ is hyponormal and hence is in the class $(H; k)$, $k \geqslant 3$. In particular, $T$ is in the class $(H; 4)$. Since for $k = 4$ and $z = 1$

$$B^{2k} - 2zA^{k} + z^{2} = \begin{bmatrix} 109 & 48 \\ 48 & 21 \end{bmatrix} \not\geqslant 0,$$

it follows that $T$ does not satisfy (1) when $k = 4$. This proves our assertion.

However, it has been shown in [**5**]$_2$ that for $k = 2$, the condition (1) is also necessary for $T$ to belong to the class $(H; 2)$.

An important relation between an operator of class $(H; k)$ and $(k+1)$-paranormal operator is given by

Theorem 2. *If $k \geqslant 2$, then every operator of class $(H; k)$ is of class $(N; k+1)$.*

Proof. Let $T$ be an operator of class $(H; k)$, $k \geqslant 2$. Then $\|T^k x\| \|x\|^{k-1} \geqslant \|T^* x\|^k$, for every $x$ in $H$. Replacing $x$ by $Tx$, we obtain $\|T^{k+1}x\| \|Tx\|^{k-1} \geqslant \|T^* Tx\|^k$.

Now

$$\|T^{k+1}x\| \|x\|^k = \|T^k(Tx)\| \|x\|^k = \left\| T^k \frac{Tx}{\|Tx\|} \right\| \|x\|^k \|Tx\|$$

$$\geqslant \left\| T^* \frac{Tx}{\|Tx\|} \right\|^k \|x\|^k \times \|Tx\| = \|T^* Tx\|^k \|x\|^k / \|Tx\|^{k-1}$$

$$\geqslant ((T^* T)^2 x, x)^{k/2} \|x\|^k / \|Tx\|^{k-1} \geqslant \|Tx\|^{2k} / \|Tx\|^{k-1} = \|Tx\|^{k+1} ,$$

where we have used $(B^r x, x) \geqslant (Bx, x)^r$, for a positive operator $B$ and $r \geqslant 1$. Hence the proof.

**3.** – Next, we characterize weighted shift operators of class $(H; k)$ in terms of their weights in the following

Theorem 3. *Let $T$ be a weighted shift operator with weights $\{\alpha_n\}$. Then the following statements are equivalent.*

(i) *$T$ is an operator of class $(H; k)$, $k \geqslant 2$.*

(ii) $|\alpha_{n-1}|^k \leqslant |\alpha_n| |\alpha_{n+1}| \dots |\alpha_{n+k-1}|$, *for all integers $n$.*

Proof. Obviously (i) implies (ii). We prove that (ii) implies (i). It is known that for positive numbers $b$ and $c$, $c - 2bz + z^2 \geqslant 0$, $z > 0$, if and only if $b^2 \leqslant c$. If we take $b = |\alpha_{n-1}|^k$ and $c = \big(|\alpha_n| |\alpha_{n+1}| \dots |\alpha_{n+k-1}|\big)^2$, then $b^2 \leqslant c$, and hence we have: $\big(|\alpha_n| |\alpha_{n+1}| \dots |\alpha_{n+k-1}|\big)^2 - 2z|\alpha_{n-1}|^k + z^2 \geqslant 0$.
This, in turn, implies
$\big((T^{*k} T^k - 2z(TT^*)^{k/2} + z^2) e_n, e_n\big) \geqslant 0$ or $T^{*k} T^k - 2z(TT^*)^{k/2} + z^2 \geqslant 0$ $(z > 0)$.
It follows by Theorem 1 that $T$ is an operator of class $(H; k)$.

Theorem 3 leads us to a number of interesting conclusions which we collect in the following

Theorem 4. *If $k \geqslant 3$ is a positive integer, then*

(i) *there exists an operator of class $(H; k)$ which is not of $(N; k)$;*

(ii) *there exists an operator of class $(N; k)$ which is not of $(H; k)$;*

(iii) *there is an operator of class $(H; k)$ which is not of $(H; 2)$;*

(iv) *there is an operator of class $(H; 2)$ which is not of $(H; k)$.*

Proof. (i) We consider the bilateral weighted shift operator $T$ with weights $\{\alpha_n\}$ such that $\alpha_n = \frac{1}{2}$ $(n < 0)$, $\alpha_0 = 1$, $\alpha_n = \frac{1}{2}$ $(k-2 \geqslant n \geqslant 1)$, $\alpha_{k-1} = 2^{k-3}$, $\alpha_n = 2^{k-1}$ $(n \geqslant k)$. Then by Theorem 3, $T$ is an operator of class $(H; k)$; but since $\alpha_0^{k-1} > \alpha_1 \alpha_2 \dots \alpha_{k-1}$, $T$ is not an operator of class $(N; k)$. [*A* weighted shift operator $T$ with weights $\{\alpha_n\}$ belongs to the class $(N; k)$ if and only if $|\alpha_n|^{k-1} \leqslant |\alpha_{n+1}|\,|\alpha_{n+2}| \dots |\alpha_{n+k-1}|$ for all integers $n$].

(ii) Let $K$ be the direct sum of an infinite number of copies of $H$ and let $A$ and $B$ be two bounded positive operators on $H$. Define an operator $T = T_{A,B}$ on $K$ as follows

$$T\langle x_1, x_2, \dots\rangle = \langle 0, Ax_1, Ax_2, \dots, Ax_n, Bx_{n+1}, Bx_{n+2}, \dots\rangle .$$

If we take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then one can easily verify that

$$A^{k-n} B^{2n} A^{k-n} \geqslant A^{2k} \qquad (n = 1, 2, \dots, k-1) .$$

Thus $T^{*k}T^k \geqslant (T^*T)^k$ and hence $T^{*k}T^k - 2z(T^*T)^{k/2} + z^2 \geqslant 0 \qquad (z > 0)$.

It follows that $T$ is $k$-paranormal [**3**]. Next, we observe that $N(A) \neq \{0\} \neq N(B)$ and $N(A) \cap N(B) = \{0\}$. Taking $x$ as a non zero vector in $N(B)$ and $y = \langle 0, 0, 0, \dots, (n+1)/x, 0, 0, \dots\rangle$, one can see that $Ty = 0$ but $T^* y \neq 0$. Hence $T$ is not an operator of class $(H; k)$.

(iii) We consider the bilateral weighted shift operator with weights $\alpha_n = (1/\sqrt{5})$ $(n \leqslant -1)$, $\alpha_0 = 3/5$, $\alpha_n = n/n+1$ $(n \geqslant 1)$. It is an operator of class $(H; k)$ for $k \geqslant 3$ but since the inequality $|\alpha_0|^2 \leqslant |\alpha_1| \cdot |\alpha_2|$ is not satisfied, it is not an operator of class $(H; 2)$.

(iv) Let $T$ be the bilateral weighted shift operator with weights $\{\alpha_n\}$ such that

$$\alpha_n = \frac{1}{2}\ (n < 0)\,, \quad \alpha_0 = 1\,, \quad \alpha_n = \frac{1}{2^{k-1}}\ (k-1 \geqslant n \geqslant 1)\,, \quad \alpha_k = 2^{k-1}\,,$$

$$\alpha_n = 2^{k-1}\ (n \geqslant k+1)\,.$$

Then $T$ is an operator of class $(H;2)$. However as the inequality $\alpha_0^k \leqslant \alpha_1\alpha_2 \ldots \alpha_k$ is not satisfied for $k \geqslant 3$, it is not an operator of class $(H;k)$, $k \geqslant 3$.

In $[\mathbf{5}]_1$ it is proved that for $k > 2$, the inverse of a non-singular $k$-paranormal operator may not be $k$-paranormal. We prove a corresponding result for the class $(H;k)$, $k \geqslant 3$.

Theorem 5. *For $k \geqslant 3$, there exists a non-singular operator of class $(H;k)$ whose inverse is not an operator of class $(H;k)$.*

Proof. One can easily verify that the inverse of a non-singular weighted shift operator $T$ with weights $\{\alpha_n\}$ is an operator of class $(H;k)$ if and only if $|\alpha_n|^k \geqslant |\alpha_{n-1}|\,|\alpha_{n-2}| \ldots |\alpha_{n-k}|$, for all integers $n$.

We consider the non-singular operator of class $(H;k)$ defined in Theorem 4 (i). For $n = 1$, we have

$$|\alpha_1|^k \geqslant |\alpha_0|\,|\alpha-1|\,|\alpha-2| \ldots |\alpha-(k-1)|\,,$$

so that $(\frac{1}{2})^k \geqslant (\frac{1}{2})^{k-1}$ or $1 \geqslant 2$ which is absurd. Hence $T^{-1}$ is not an operator of class $(H;k)$.

Recently, T. Andô [**1**] has proved that the sum of a scalar and a paranormal operator may not be paranormal. In $[\mathbf{5}]_2$, it is shown that the sum of an operator of class $(H;2)$ and a scalar is not necessarily an operator of class $(H;2)$. Since $(H;2)$ neither includes nor is included by $(H;k)$, $k \geqslant 3$, it is interesting to know whether a similar result holds for operators of class $(H;k)$ for $k \geqslant 3$.

Theorem 6. *The sum of an operator of class $(H;3)$ and a scalar may not be of class $(H;3)$.*

Proof. Let $T$ be the unilateral weighted shift with weights $\{\alpha_n\}$ such that $\alpha_0 = (1/\sqrt{2})$, $\alpha_1 = 1/6$ and $\alpha_n = n$ $(n \geqslant 2)$. Then by Theorem 3, $T$ turns out to be an operator of class $(H;3)$. Let $z$ be any complex number. If $T + zI$ is also in $(H;3)$, then

$$\|(T+zI)^3 e_n\| \geqslant \|(T^* + \bar{z}I)e_n\|^3 \qquad \text{for } n = 0, 1, 2, \ldots\,.$$

This gives

$$(\alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-1}^6) + 3|z|^2(3\alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^4) + 3|z|^4(3\alpha_n^2 - \alpha_{n-1}^2) \geq 0 ,$$

for $n = 0, 1, 2, \ldots$. But for $n = 1$, this is not satisfied for a non zero $z$. Hence $T + zI$ is not an operator of class $(H; 3)$.

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