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The change of scale and translation pathology in Yeh-Wiener space. (**)

1. - Introduction.

Let $C[a, b]$ denote the Wiener space of functions of one variable, i.e. $C[a, b] = \{x(\cdot)\,|\,x(a) = 0$ and $x(s)$ is continuous on $[a, b]\}$. Let $R = \{(s, t)\,|\,|\,a \leqslant s \leqslant b,\ \alpha \leqslant t \leqslant \beta\}$ and let $C_2[R]$, called Yeh-Wiener space, denote the Wiener space of functions of two variables over R, i.e. $C_2[R] = \{x(\cdot, \cdot)\,|\,x(a, t) = x(s, \alpha) = 0,\ x(s, t)$ continuous for $a \leqslant s \leqslant b,\ \alpha \leqslant t \leqslant \beta\}$. Yeh-Wiener measure on the unit square was defined by Yeh $[\mathbf{8}]_1$ (see also the reference to Kitagawa in $[\mathbf{8}]_1$) and was extended by Kuelbs $[\mathbf{5}]_1$ to regions other than squares and to higher (even infinite) dimensional spaces.

In section **2** we exhibit a subset of measure one in Yeh-Wiener space $C_2[R]$ which for all real $\lambda \neq \pm 1$ is transformed into a set of measure zero by the change of scale transformation $y(\cdot, \cdot) = \lambda x(\cdot, \cdot)$. In particular Yeh-Wiener measurability is not invariant under change of scale. These are generalizations of results obtained by Cameron and Martin [**3**] for Wiener space $C[0, 1]$.

In [**2**] Cameron showed that almost no translations in Wiener space preserve measurability. In section **3** we obtain this result for translations in $C_2[R]$.

In section **5** we show that the results of sections **2** and **3** also hold in N-dimensional Yeh-Wiener space.

We will include for the sake of completeness a brief discussion of Yeh-Wiener measure on $C_2[R]$. For a more complete discussion see $[\mathbf{8}]_1$, $[\mathbf{5}]_1$ and either of two recent papers by Cameron and Storvick $[\mathbf{4}]_{1,2}$.

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Let $a = s_0 < s_1 < \dots < s_m = b$ and $\alpha = t_0 < t_1 < \dots < t_n = \beta$ be partitions of $[a, b]$ and $[\alpha, \beta]$ respectively. Let $-\infty \leqslant P_{j,k} \leqslant Q_{j,k} \leqslant +\infty$ be given for $j = 1, \dots, m$ and $k = 1, \dots, n$. Then

$$I = \{x \in C_2[R] \,|\, P_{j,k} < x(s_j, t_k) \leqslant Q_{j,k} \text{ for } i = 1, \dots, m \text{ and } k = 1, \dots, n\}$$

is called an interval in $C_2[R]$. The measure of the interval I is given by

$$m(I) = \pi^{-mn/2}[(s_1 - s_0) \dots (s_m - s_{m-1})]^{-n/2}[(t_1 - t_0) \dots (t_n - t_{n-1})]^{-m/2} \cdot$$

$$\cdot \int_{P_{m,n}}^{Q_{m,n}} \dots \int_{P_{1,1}}^{Q_{1,1}} \exp\left\{-\sum_{j=1}^{m} \sum_{k=1}^{n} \frac{[u_{j,k} - u_{j-1,k} - u_{j,k-1} + u_{j-1,k-1}]^2}{(s_j - s_{j-1})(t_k - t_{k-1})}\right\} du_{1,1} \dots du_{m,n} .$$

This measure is countably additive on the set of all such intervals in $C_2[R]$ and can be extended in the usual way to the σ-algebra of sets generated by the intervals (sets in this σ-algebra are said to be strictly Yeh-Wiener measurable) and then can be further extended so as to be a complete measure which we will also denote by m. Integration of a functional F with respect to this measure will be denoted by $\int_{C_2[R]} F(x)\, dx$.

2. - Change of scale pathology.

In this section we consider the change of scale transformation

$$y(\cdot, \cdot) = \lambda x(\cdot, \cdot) \tag{1}$$

for λ real and x in $C_2[R]$. We will show that there exists a subset A_1 of $C_2[R]$ such that $n(A_1) = 1$ and which, for all real $\lambda \neq \pm 1$ is mapped by (1) into a null subset of $C_2[R]$. In particular we will show that Yeh-Wiener measurability is not preserved by change of scale.

The following lemma, whose proof is given in section 4, plays a key role in the development.

Lemma 1. *For $n = 1, 2, 3, \dots$ let σ_n be the set of points*

$$\sigma_n = \{(s_j^{(n)}, t_k^{(n)}) = \left(a + j(b - a)/2^n, \alpha + k(\beta - \alpha)/2^n\right) | j, k = 0, 1, \dots, 2^n\} .$$

For each x *in* $C_2[R]$ *let*

$$(2)\quad S_{\sigma_n}(x) = \sum_{j=1}^{2^n} \sum_{k=1}^{2^n} [x(s_j^{(n)}, t_k^{(n)}) - x(s_{j-1}^{(n)}, t_k^{(n)}) - x(s_j^{(n)}, t_{k-1}^{(n)}) + x(s_{j-1}^{(n)}, t_{k-1}^{(n)})]^2 .$$

Then

$$(3)\quad \lim_{n\to\infty} S_{\sigma_n}(x) = (b-a)(\beta-\alpha)/2 \qquad \textit{for almost all } x \textit{ in } C_2[R] .$$

Notation. For $E \subseteq C_2[R]$ and $-\infty < \lambda < \infty$ let $\lambda E = \{\lambda x(\cdot, \cdot) \,|\, x \in E\}$. For $\lambda \geqslant 0$ let $A_\lambda = \{x \in C_2[R] \,|\, \lim_{n\to\infty} S_{\sigma_n}(x) = \lambda^2(b-a)(\beta-\alpha)/2\}$, and finally let $D = \{x \in C_2[R] \,|\, \lim_{n\to\infty} S_{\sigma_u}(x)$ doesn't exist$\}$.

The following theorem is an easy consequence of Lemma 1.

Theorem 1.

a) $m(A_\lambda) = 1$ *if and only if* $\lambda = 1$,
b) $vA_\lambda = A_{v\lambda}$ *for* $v > 0$ *and* $\lambda \geqslant 0$,
c) $m(\lambda^{-1} A_\lambda) = 1$ *for* $\lambda > 0$,
d) $m(v^{-1} A_\lambda) = 0$ *for* $v \neq \lambda$, $v > 0$ *and* $\lambda \geqslant 0$,
e) $A_\lambda \cap A_v = \emptyset$ *for* $\lambda \neq v$,
f) $m(D) = 0$,
g) $m(\bigcup_{0<v\neq\lambda} v^{-1}A_\lambda) = 0$ *for each fixed* $\lambda \geqslant 0$,
h) $C_2[R] = D + A_0 + A_1 + \sum_{0<\lambda\neq 1} A_\lambda$.

In $[4]_2$ Cameron and Storvick make the remark: « As in the case of Wiener measure, strict Yeh-Wiener measurability is invariant under change of scale, but there is no reason to suppose that this is true of Yeh-Wiener measurability ». In fact Theorem 1 above shows that Yeh-Wiener measurability is not invariant under change of scale. For let H be a non-measurable subset of $C_2[R]$. Then $H \cap A_1$ is also a non-measurable subset of $C_2[R]$. Now for $0 < \lambda \neq 1$, $K \equiv \lambda^{-1}(H \cap A_1)$ is a null subset of $C_2[R]$, hence measurable, while $\lambda H = H \cap A_1$ is non-measurable.

Theorem 2. *Let* $f(\lambda)$ *be a given function with domain* $(0, \infty)$ *and satisfying* $0 \leqslant f(\lambda) \leqslant 1$. *Then there is a subset* E *of* $C_2[R]$ *which is measurable under every change of scale* (1) *and satisfies* $m(\lambda E) = f(\lambda)$ *for all* $\lambda > 0$.

Proof. For each $\lambda > 0$ let E_λ be a subset of $C_2[R]$ such that: $m(E_\lambda) = f(\lambda)$. (If $0 < f(\lambda) < 1$ we can simply choose $E_\lambda = \{x \in C_2[R] \,|\, -\infty < x(b, \beta) \leqslant \gamma\}$ where γ is chosen to satisfy $f(\lambda) = [\pi(b-a)(\beta-\alpha)]^{-\frac{1}{2}}\int_{-\infty}^{\gamma}\exp\{-u^2/[(b-a)\cdot(\beta-\alpha)]\}\,du$.) Then $E = \sum_{\lambda>0}\lambda^{-1}(A_1 \cap E_\lambda)$ is a set with the desired property.

The following well known fact follows easily as a corollary to Lemma 1.

Corollary. *The set of functions x in $C_2[R]$ which are of bounded variation on R have Yeh-Wiener measure zero.*

Proof. Let $\mathrm{Var}[x, R]$ denote the total variation of $x(s, t)$ on R. Then for each positive integer n,

$$\mathrm{Var}\,[x, R] \geqslant \sum_{j=1}^{2^n}\sum_{k=1}^{2^n}\left|x(s_j^{(n)}, t_k^{(n)}) - x(s_{j-1}^{(n)}, t_k^{(n)}) - x(s_j^{(n)}, t_{k-1}^{(n)}) + x(s_{j-1}^{(n)}, t_{k-1}^{(n)})\right|$$

$$\geqslant \frac{S_{\sigma_n}(x)}{\max\limits_{\substack{1\leqslant j\leqslant 2^n\\ 1\leqslant k\leqslant 2^n}}\left|x(s_j^{(n)}, t_k^{(n)}) - x(s_{j-1}^{(n)}, t_k^{(n)}) - x(s_j^{(n)}, t_{k-1}^{(n)}) + x(s_{j-1}^{(n)}, t_{k-1}^{(n)})\right|}\,.$$

But for almost all x in $C_2[R]$, $\lim\limits_{n\to\infty} S_{\sigma_n}(x) = (b-a)(\beta-\alpha)/2$ so that $\mathrm{Var}[x, R] = \infty$ for almost all x in $C_2[R]$.

3. - Translation pathology.

For z in $C_2[R]$ let T_z denote the translation $T_z x = x + z$, which maps $C_2[R]$ onto $C_2[R]$ and is $1-1$. Translations in Yeh-Wiener space have been studied by Yeh $[\mathbf{8}]_{2,3}$, Kuelbs $[\mathbf{5}]_2$ and others. It is known that measurability is preserved under translations by sufficiently smooth functions. This set of smooth functions is however of measure zero in $C_2[R]$. We will show that this must be so for any translation theorem which preserves Yeh-Wiener measurability.

A translation theorem for Yeh-Wiener space was established by Yeh $[\mathbf{8}]_{2,3}$ and was later generalized by Kuelbs $[\mathbf{5}]_2$. For completeness we will state Kuelb's theorem as it applies to $C_2[R]$.

Translation Theorem (Kuelbs). Let p be in $L_2(R)$. Let $z(s, t) = \int_a^s \int_\alpha^t p(u, v)\, dv\, du$. Let F be a measurable functional on $C_2[R]$. Then

$$\int_{C_2[R]} F(y)\, dy = \exp\left[-\int_R p^2\right] \int_{C_2[R]} F(x+z) \exp\left[-2 \int_R p\, d\tilde{x}\right] dx$$

in the sense that if either integral exists, both exist and are equal. (If $p(s, t)$ is not of bounded variation on R the integral $\int_R p\, d\tilde{x}$ is interpreted as a generalized Paley-Wiener-Zygmund integral [7], $[5]_2$.)

In particular the translation theorems asserts that if z is of the above form then the translation T_z takes each Yeh-Wiener measurable set Γ into a Yeh-Wiener measurable set $T_z\Gamma$ whose measure is given by

$$m(T_z\Gamma) = \exp\left[-\int_R p^2\right] \int_\Gamma \exp\left[-2 \int_R p\, d\tilde{x}\right] dx\,.$$

Theorem (Bearman). Let $F(x, y)$ be an integrable functional on $C_2[R] \times C_2[R]$. Then for all real θ,

$$\int_{C_2[R]} \int_{C_2[R]} F(x, y)\, dx\, dy = \int_{C_2[R]} \int_{C_2[R]} F(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)\, dx\, dy\,.$$

Bearman [1] proved this result about rotations in Wiener space $C[0, 1]$. The extension to Yeh-Wiener space $C_2[R]$ is straight forward.

We will use the following modification of Bearman's theorem in the proofs of Theorems 3 and 4 below.

Lemma 2. *Let $F[(p^2+q^2)^{\frac{1}{2}}x]$ be Yeh-Wiener integrable on $C_2[R]$, where p and q are real numbers. Then $F[py+qz]$ is integrable on $C_2[R] \times C_2[R]$ and*

$$\int_{C_2[R]} \int_{C_2[R]} F[py+qz]\, dy\, dz = \int_{C_2[R]} F[(p^2+q^2)^{\frac{1}{2}} x]\, dx\,.$$

Theorem 3. *Almost no translations in Yeh-Wiener space preserve measurability.*

Proof. Let E be any Yeh-Wiener measurable subset of $C_2[R]$. Then using Lemma 2 we obtain

$$(4) \qquad \int_{C_2[R]} m(T_zE)\, dz = \int_{C_2[R]} \int_{C_2[R]} \chi_{T_zE}(x)\, dx\, dz = \int_{C_2[B]} \int_{C_2[R[} \chi_E(x-z)\, dx\, dz$$
$$= \int_{C_2[R]} \chi_E(2^{\frac{1}{2}}y)\, dy = \int_{C_2[R]} \chi_{2^{-\frac{1}{2}}E}(y)\, dy = m(2^{-\frac{1}{2}}(E)\,.$$

Hence letting $E = A_{2^{\frac{1}{2}}}$ we see that $m(T_z A_{2^{\frac{1}{2}}}) = 1$ for almost all z in $C_2[R]$. Hence almost all translations T_z take $A_{2^{\frac{1}{2}}}$, a set of measure zero, into a set of measure one. Again let H be a non-measurable subset of $C_2[R]$. Then $H \cap (T_z A_{2^{\frac{1}{2}}})$ is also non-measurable for almost all z in $C_2[R]$. But $T_z^{-1}[H \cap \cap(T_z A_{2^{\frac{1}{2}}})]$ is a subset of $A_{2^{\frac{1}{2}}}$, hence null, and so T_z is almost never measurability preserving.

A subset E of $C_2[R]$ is said to be an invariant null set if for all $\lambda > 0$, $m(\lambda E) = 0$. Our next theorem shows that almost all translations restricted to invariant null sets preserve measurability.

Theorem 4. *If E is an invariant null set, then $m(T_z E) = 0$ for almost all z in $C_2[R]$.*

Proof. Let E be an invariant null set. Then $m(2^{-\frac{1}{2}}E) = 0$. Hence by (4), $\int\limits_{C_2[R]} m(T_z E)\, dz = m(2^{-\frac{1}{2}}E) = 0$. Thus $m(T_z E) = 0$ for almost all z in $C_2[R]$.

Remark. (i) The converse of Theorem 4 is not true since for $\lambda > 0$, A_λ is not an invariant null set while for $\lambda \neq 2^{\frac{1}{2}}$, $\int\limits_{C_2[R]} m(T_z A_\lambda)\, dz = m(2^{-\frac{1}{2}} A_\lambda) = 0$.

(ii) Proceeding as above one readily sees that the result in Wiener space $C[a, b]$ corresponding to Theorem 4 is also true. That is to say, if E is an invariant null set in Wiener space $C[a, b]$ then the set $T_z E$ has Wiener measure zero for almost all z in $C[a, b]$.

4. - Proof of Lemma 1.

We will first show that $\int\limits_{C_2[R]} S_{\sigma_n}(x)\, dx = (b-a)(\beta-\alpha)/2$ for all n. So let n be a fixed positive integer and let

$$\varphi_{j,k}(s, t) = \begin{cases} [(s_j^{(n)} - s_{j-1}^{(n)})(t_k^{(n)} - t_{k-1}^{(n)})]^{-\frac{1}{2}}, & s_{j-1}^{(n)} < s \leqslant s_j^{(n)} \text{ and } t_{k-1}^{(n)} < t \leqslant t_k^{(n)} \\ 0, & \text{elsewhere} \end{cases}$$

for $j, k = 1, 2, \ldots, 2^n$. Then $\{\varphi_{j,k}\}_{j,k=1}^{\infty}$ is an orthonormal set of functions each of bounded variation on R such that

$$(s_j^{(n)} - s_{j-1}^{(n)})^{\frac{1}{2}}(t_k^{(n)} - t_{k-1}^{(n)})^{\frac{1}{2}} \int_a^b \int_\alpha^\beta \varphi_{j,k}(s, t)\, dx(s, t) =$$

$$= x(s_j^{(n)}, t_k^{(n)}) - x(s_{j-1}^{(n)}, t_k^{(n)}) - x(s_j^{(n)}, t_{k-1}^{(n)}) + x(s_{j-1}^{(n)}, t_{k-1}^{(n)}) .$$

Hence by the Paley-Wiener-Zygmund formula ([**8**]$_2$, theorem II)

$$(5)\quad \int_{C_2[R]} S_{\sigma_n}(x)\,dx = \sum_{j=1}^{2^n}\sum_{k=1}^{2^n}(s_j^{(n)}-s_{j-1}^{(n)})(t_k^{(n)}-t_{k-1}^{(n)})\int_{C_2[R]}\Big[\int_a^b\int_\alpha^\beta \varphi_{j,k}(s,t)\,dx(s,t)\Big]^2 dx =$$

$$= \sum_{j=1}^{2^n}\sum_{k=1}^{2^n}(s_j^{(n)}-s_{j-1}^{(n)})(t_k^{(n)}-t_{k-1}^{(n)})\Big[\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}u^2\exp(-u^2)\,du\Big] = (b-a)(\beta-\alpha)/2\,.$$

In a completely analogous manner we obtain

$$(6)\quad \int_{C_2[R]}[S_{\sigma_n}(x)]^2\,dx = [(b-a)(\beta-\alpha)/2]^2 + \frac{1}{2}\sum_{j=1}^{2^n}\sum_{k=1}^{2^n}(s_j^{(n)}-s_{j-1}^{(n)})^2(t_k^{(n)}-t_{k-1}^{(n)})^2\,.$$

Now using equations (5) and (6) we easily obtain

$$(7)\qquad \int_{C_2[R]}[S_{\sigma_n}(x)-(b-a)(\beta-\alpha)/2]^2\,dx = (b-a)^2(\beta-\alpha)^2\,2^{-(1+2n)}\,.$$

Hence $S_{\sigma_n}(x)$ converges in mean-square to $(b-a)(\beta-\alpha)/2$. Now for $n=1,2,\dots$ let

$$E_n = \{x\in C_2[R]\,|\,|S_{\sigma_n}(x)-(b-a)(\beta-\alpha)/2|\geqslant(b-a)(\beta-\alpha)2^{-(1+2n/3)}\}\,,$$

$F_n = \bigcup_{k=n}^{\infty}E_k$ and $F=\bigcap_{n=1}^{\infty}F_n$. Then for each fixed n

$$m(F)\leqslant m(F_n)\leqslant\sum_{k=0}^{\infty}m(E_{n+k})\leqslant\sum_{k=0}^{\infty}2^{1-2(n+k)/3}\to 0 \qquad \text{as } n\to\infty\,.$$

Hence $m(F)=0$. But for x not in F, $\lim_{n\to\infty}S_{\sigma_n}(x)=(b-a)(\beta-\alpha)/2$ which establishes (3) and completes the proof of Lemma 1.

Remark. Equations (5) and (6) could also have been obtained using well known facts about Gaussian functionals [**6**].

5. - N-dimensional Yeh-Wiener space.

Let N be a positive integer. Let Y_N denote the product space $\prod_{k=1}^{N}[a_k,b_k]$, where $-\infty<a_k<b_k<+\infty$ for all k. We will denote the points of Y_N by $=(s_1,\dots,s_N)$. The N-dimensional Yeh-Wiener space $C_N(Y_N)$ is the set of all real-valued continuous functions $x(s)$ on Y_N such that $x(s) = x(s_1, \dots, s_N) = 0$ if $s_j = a_j$ for some $1 \leq j \leq N$. See **[5]**$_1$ for a discussion of the measure of this space.

It turns out that all the results of sections **2** and **3** above also hold in Yeh-Wiener space $C_N(Y_N)$. The main fact that needs to be established is an N-dimensional version of Lemma 1. For then the results about « change of scale » in $C_N(Y_N)$ follow readily while using the N-dimensional version of the Translation Theorem and the N-dimensional version of Lemma 2 it is easy to see that almost no translations in $C_N(Y_N)$ preserve measurability.

Lemma 3. *For $n = 1, 2, \dots$ let σ_n be the set of points*

$$\sigma_n = \{(s^{(n)}_{1,j_1}, \dots, s^{(n)}_{N,j_N}) \,|\, j_1, \dots, j_N = 0, 1, \dots, 2^n,$$

$$s^{(n)}_{k,j_k} = a_k + j_k(b_k - a_k)/2^n \text{ for } k = 1, \dots, N\}.$$

For each x in $C_N(Y_N)$ let

$$S_{\sigma_n}(x) = \sum_{j_1=1}^{2^n} \cdots \sum_{j_N=1}^{2^n} [\Delta_1 \Delta_2, \dots, \Delta_N x(s^{(n)}_{1,j_1}, \dots, s^{(n)}_{N,j_N})]^2,$$

where for $k = 1, 2, \dots, N$,

$$\Delta_k x(s^{(n)}_{1,j_1}, \dots, s^{(n)}_{N,j_N}) =$$

$$= x(s^{(n)}_{1,j_1}, \dots, s^{(n)}_{N,j_N}) - x(s^{(n)}_{1,j_1}, \dots, s^{(n)}_{k-1,j_{k-1}}, s^{(n)}_{k,j_k-1}, s^{(n)}_{k+1,j_{k+1}}, \dots, s^{(n)}_{N,j_N}).$$

Then $\lim_{n \to \infty} S_{\sigma_n}(x) = \frac{1}{2} \prod_{k=1}^{N} (b_k - a_k)$ for almost all x in $C_N(Y_N)$.

Proof (outline). Proceeding as in section **4** above, using the N-dimensional Paley-Wiener-Zygmund formula we obtain

$$\int_{C_N(Y_N)} S_{\sigma_n}(x)\, dx = \frac{1}{2} \prod_{k=1}^{N} (b_k - a_k),$$

$$\int_{C_N(Y_N)} [S_{\sigma_n}(x)]^2\, dx = \left[\frac{1}{2} \prod_{k=1}^{N} (b_k - a_k)\right]^2 + 2^{-(1+Nn)} \prod_{k=1}^{N} (b_k - a_k)^2.$$

Hence

$$\int_{C_N(Y_N)} [S_{\sigma_n}(x) - \frac{1}{2} \prod_{k=1}^{N} (b_k - a_k)]^2 \, dx = 2^{-(1+Nn)} \prod_{k=1}^{N} (b_k - a_k)^2 ,$$

from which the desired convergence follows as in the proof of Lemma 1.

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Summary.

In this paper we show that Yeh-Wiener measurability is not invariant under change of scale. In addition we show that almost no translations in Yeh-Wiener space preserve measurability.

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