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P. C. BHAKTA (*)

# Some results on the generation of quasi-uniformities. (**)

## 1. - Introduction.

Let $X$ be a non-void set. For non-void subsets $U$ and $V$ of $X\times X$, $U\circ V$ is a subset of $X\times X$ defined by

$$U\circ V=\{(x, y): (x, z)\in V \text{ and } (z, y)\in U \text{ for some } z\in X\}.$$

The set $\Delta$ is defined by $\Delta=\{(x, x): x\in X\}$.

A non-void family $\mathscr{U}$ of subsets of $X\times X$ is said to be « a quasi-uniformity on $X$ » if the following axioms are satisfied ([2], Ch. 11, p. 174):

(QU.1) $\Delta\subset U$ for every $U\in\mathscr{U}$.
(QU.2) If $V\subset X\times X$ and $U\subset V$ for some $U\in\mathscr{U}$, then $V\in\mathscr{U}$.
(QU.3) If $U, V\in\mathscr{U}$, then $U\cap V\in\mathscr{U}$.
(QU.4) For every $U\in\mathscr{U}$, there is a $V$ in $\mathscr{U}$ with $V\circ V\subset U$.

The pair $(X, \mathscr{U})$ is said to be « a quasi-uniform space ». A subfamily $\mathscr{B}$ of $\mathscr{U}$ is a base of the quasi-uniformity $\mathscr{U}$ if for any $U$ in $\mathscr{U}$, there is a member $V$ of $\mathscr{B}$ with $V\subset U$. A family $\mathscr{B}$ of subsets of $X\times X$ is a base for some quasi-uniformity on $X$ iff the following conditions hold:

(B.1) $\Delta\subset U$ for every $U\in\mathscr{B}$.
(B.2) For $U, V\in\mathscr{B}$ there is a $W$ in $\mathscr{B}$ with $W\subset U\cap V$.
(B.3) For every $U\in\mathscr{B}$, there is a $V$ in $\mathscr{B}$ with $V\circ V\subset U$.

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(*) Indirizzo: Department of Mathematics, Jadavpur University, Calcutta 32, India.

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A mapping $p: X \times X \to R$ ($R =$ field of real numbers) is «a quasi-metric on $X$» if the following axioms are satisfied ($x, y, z \in X$):

(QM.1) $p(x, y) \geqslant 0$ and $p(x, x) = 0$.
(QM.2) $p(x, y) \leqslant p(x, z) + p(z, y)$.

For any quasi-metric $p$ on $X$ and $r > 0$ we write $V_{(p,r)} = \{(x, y): x, y \in X$ and $p(x, y) < r\}$.

Let $\mathscr{P}$ be a family of quasi-metrics on $X$. Denote by $\mathscr{B}$ the family of all sets of the form $V = \bigcap_{i=1}^{n} V_{(p_i, r_i)}$ ($p_i \in \mathscr{P}$, $p_i > 0$ and $n = 1, 2, 3, \ldots$).

It is easy to verify that $\mathscr{B}$ is a base for some quasi-uniformity $\mathscr{U}$ on $X$. We say that the family $\mathscr{P}$ of quasi-metrics on $X$ generates the quasi-uniformity $\mathscr{U}$.

In the present paper we show (Th. 2.1) that every quasi-uniformity on a set $X$ can be generated by a family $\mathscr{P}$ of quasi-metrics on $X$. Next, we take a certain type of families $\mathscr{V}$ of subsets of $X \times X$ and define scalar multiple $\alpha V$ of an element $V$ of $\mathscr{V}$ by a positive number $\alpha$ satisfying some axioms; we show (Th. 2.2) that each such family $\mathscr{V}$ generates a quasi uniformity on $X$ and conversely, every quasi-uniformity on $X$ is generated by a family $\mathscr{V}$ of the above type (Th. 2.3). Lastly we show (Th. 2.4) that every topology on a set $X$ can be generated by a family of quasi-metrics on $X$.

## 2. - On the generation of quasi-uniformities.

Theorem 2.1. *Every quasi-uniformity on a set $X$ can be generated by a family of quasi-metrics on $X$.*

Proof. Let $\mathscr{U}$ be a quasi-uniformity on the set $X$. Let $\mathscr{B}$ be a base for $\mathscr{U}$ such that no member of $\mathscr{B}$ is equal to $X \times X$. For each $V$ in $\mathscr{B}$ we can choose, proceeding as in Theorem 11.1.1 ([2], Ch. 11, p. 175), a sequence $\{U_n^{(V)}\}_{n=0}^{\infty}$ of sets in $\mathscr{U}$ with $U_{n+1}^{(V)} \circ U_{n+1}^{(V)} \circ U_{n+1}^{(V)} \subset U_n^{(V)}$, where $U_0^{(V)} = X \times X$ and $U_1^{(V)} = V$. By metrization lemma ([1], Ch. 6,12, p. 185) there is a quasi-metric $p_V$ on $X$ such that

$$U_n^{(V)} \subset \{(x, y): p_V(x, y) < 2^{-n+2}\} \subset U_{n-1}^{(V)} \qquad (n = 1, 2, 3, \ldots). \tag{1}$$

Let $\mathscr{P} = \{p_V: \mathscr{V} \in \mathscr{B}\}$. Denote by $\mathscr{V}$ the quasi-uniformity on $X$ generated by the family $\mathscr{P}$ of quasi-metrics on $X$. Let $U$ be any member of $\mathscr{U}$. There is a set $V$ in $\mathscr{B}$ with $V \subset U$. From (1) we have $V_{(p_V, 1)} \subset U_1^{(V)} = V$. This gives that $U \in V$ and so $\mathscr{U} \subset V$.

Next, let $W \in \mathscr{V}$. Then there are finite number of members $V_1, V_2, \ldots, V_m$ in $\mathscr{B}$ and $r_i > 0$ ($i = 1, 2, \ldots, m$) such that $\bigcap_{i=1}^{m} V_{(p_{V_i}, r_i)} \subset W$. Choose positive integers $n_1, n_2, \ldots, n_m$ with $2^{-n_i+2} < r_i$ ($i = 1, 2, \ldots, m$).

From (1) we have

$$U_{n_i}^{(V_i)} \subset \{(x, y): p_{V_i}(x, y) < 2^{-n_i+2}\} \subset V_{(p_{V_i}, r_i)} \qquad (i = 1, 2, \dots, m)\,.$$

Write $U = \bigcap_{i=1}^{m} U_{n_i}^{(V_i)}$. Then $U \in \mathscr{U}$ and $U \subset \bigcap_{i=1}^{m} V_{(p_{V_i}, r_i)} \subset W$.

This gives that $W \in \mathscr{U}$ and so $\mathscr{V} \subset \mathscr{U}$. Therefore we have $\mathscr{V} = \mathscr{U}$ and the proof is complete.

Now let $X$ be a non-void set and $\mathscr{V}$ be a non-void family of subsets of $X \times X$ having the following properties:

(i) $\Delta \subset V$ for every $V \in \mathscr{V}$.
(ii) $V_1 \cap V_2 \in \mathscr{V}$ for all $V_1, V_2 \in \mathscr{V}$.

Clearly $\mathscr{V}$ does not satisfy all the conditions for being a base for some quasi-uniformity on $X$. We denote by $R_+$ the set of all positive real numbers. Let $f$ be a mapping of $R_+ \times \mathscr{V}$ into $\mathscr{V}$. For simplicity we write $\alpha \cdot V$ or $\alpha V$ for $f(\alpha, V)$, where $\alpha \in R_+$ and $V \in \mathscr{V}$ and call $\alpha V$ as scalar multiple of $V$. We suppose that this multiplication satisfies the following axioms. For $\alpha, \beta \in R_+$ and $V \in \mathscr{V}$

(iii) $1 \cdot V = V$,
(iv) $\alpha(\beta V) = (\alpha\beta) V$,
(v) $\alpha V \subset \beta V$ if $\alpha < \beta$,
(vi) $\alpha V \circ \beta V \subset (\alpha + \beta) V$,
(vii) if $(x, y) \in X \times X$ and $V \in \mathscr{V}$ there is a $\lambda > 0$ such that $(x, y) \in \lambda V$.

We denote by $\mathscr{F}$ the class of all families $\mathscr{V}$ having the properties (i) and (ii) with a (scalar) multiplication satisfying the axioms (iii)-(vii).

Theorem 2.2. *For each $\mathscr{V}$ in $\mathscr{F}$ there is a quasi-uniformity $\mathscr{U}$ on $X$ such that $\mathscr{V}$ is a base for $\mathscr{U}$; and for each $V$ in $\mathscr{V}$ there is a quasi-metric $p_V$ on $X$ such that*

$$V_{(p_V, r)} \subset rV \subset V_{(p_V, \varrho)} \qquad (0 < r < \varrho)$$

*and the family $\{p_V: V \in \mathscr{V}\}$ of quasi-metrics on $X$ generates the quasi-uniformity $\mathscr{U}$.*

Proof. We prove the theorem by the following steps.

(I) Let $V \in \mathscr{V}$. Take $U = \alpha V$, where $0 < \alpha < \frac{1}{2}$. Then $U \in \mathscr{V}$. By (vi) and (v) we have $U \circ U = \alpha V \circ \alpha V \subset 2\alpha V \subset V$. Thus $\mathscr{V}$ satisfies the conditions (B.1), (B.2) and (B.3). Hence $\mathscr{V}$ is a base for some quasi-uniformity $\mathscr{U}$ on $X$.

(II) Let $V \in \mathscr{V}$. Take any $x$, $y$ in $X$. Then by (vii) there is a $\lambda > 0$ such that $(x, y) \in \lambda V$. Let $A_{(x,y)} = \{\lambda: \lambda > 0 \text{ and } (x, y) \in \lambda V\}$. Define $p_V(x, y)$ by $p_V(x, y) = \inf\{\lambda: \lambda \in A_{(x,y)}\}$. From definition it is clear that $p_V(x, y) \geqslant 0$. Since $(x, x) \in \lambda V$ for every $\lambda > 0$, we have $A_{(x,x)} = \{\lambda: \lambda > 0\}$ and so $p_V(x, x) = 0$.

Now let $x$, $y$, $z$ be any three elements in $X$. Write $p_V(x, z) = r_1$ and $p_V(z, y) = r_2$. Choose $\varepsilon > 0$ arbitrarily. Then $(x, z) \in (r_1 + \varepsilon) V$ and $(z, y) \in (r_2 + \varepsilon) V$. By (vi) we have $(x, y) \in (r_1 + \varepsilon) V_0 (r_2 + \varepsilon) V \subset (r_1 + r_2 + 2\varepsilon) V$, which gives that $r_1 + r_2 + 2\varepsilon \in A_{(x,y)}$. Hence $p_V(x, y) \leqslant r_1 + r_2 + 2\varepsilon = p_V(x, z) + p_V(z, y) + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary we obtain: $p_V(x, y) \leqslant p_V(x, z) + p_V(z, y)$. Thus $p_V$ satisfies the axioms (QM.1) and (QM.2). So $p_V$ is a quasi-metric on $X$.

Let $r$ and $\varrho$ be any two positive numbers with $r < \varrho$. If $(x, y) \in V_{(p_V, r)}$, then $p_V(x, y) < r$ which gives that $(x, y) \in rV$ and so $V_{(p_V,r)} \subset rV$. If $(x, y) \in rV$, then $p_V(x, y) \leqslant r < \varrho$ which gives that $(x, y) \in V_{(p_V,\varrho)}$; so $rV \subset V_{(p_V,\varrho)}$. Hence $V_{(p_V,r)} \subset rV \subset V_{(p_V,\varrho)}$ for $0 < r < \varrho$.

(III) Let $\mathscr{U}_0$ denote the quasi-uniformity on $X$ generated by the family $\{p_V: V \in \mathscr{V}\}$ of quasi-metrics on $X$. Let $U \in \mathscr{U}$. Since $\mathscr{V}$ is a base for $\mathscr{U}$, there is a member $V \in \mathscr{V}$ with $V \subset U$. Again, since $V_{(p_V,1)} \subset V$, it follows that $U \in \mathscr{U}_0$ and so $\mathscr{U} \subset \mathscr{U}_0$.

Next, let $U \in \mathscr{U}_0$. Then there is a set $W$ of the form $W = \bigcap_{i=1}^{n} V_{(p_{V_i}, r_i)}$ $(V_i \in \mathscr{V})$ such that $W \subset U$. Let $\alpha$ be a positive number with $0 < \alpha < 1$. Write $W_0 = \bigcap_{i=1}^{n} (\alpha r_i) V_i$. Then $W_0 \in \mathscr{U}$. Since $(\alpha r_i) V_i \subset V_{(p_{V_i}, r_i)}$ $(i = 1, 2, \ldots, n)$, we have $W_0 \subset W \subset U$ which gives that $U \in \mathscr{U}$; so $\mathscr{U}_0 \subset \mathscr{U}$. Therefore $\mathscr{U}_0 = \mathscr{U}$ and the proof of the theorem is complete.

Theorem 2.3. *If $\mathscr{U}$ is a quasi-uniformity on $X$, then there is a family $\mathscr{V}$ in $\mathscr{F}$ such that $\mathscr{V}$ generates $\mathscr{U}$.*

Proof. Let $\mathscr{U}$ be a quasi-uniformity on the set $X$. Then by Theorem 2.1 there is a family $\mathscr{P}$ of quasi-metrics on $X$ such that $\mathscr{P}$ generates the quasi-uniformity $\mathscr{U}$. Denote by $\mathscr{V}$ the family of all sets of the form $\bigcap_{i=1}^{n} V_{(p_i, r_i)}$ $(p_i \in \mathscr{P},\ r_i > 0 \text{ and } n = 1, 2, 3, \ldots)$.

Then $\mathscr{V}$ is a base for the quasi-uniformity $\mathscr{U}$. Clearly $\mathscr{V}$ possesses the properties (i) and (ii).

Let $V \in \mathscr{V}$. Then we have

$$V = \bigcap_{i=1}^{n} V_{(p_i, r_i)} \qquad (p_i \in \mathscr{P} \text{ and } r_i > 0)$$

If $\alpha \in R_+$, then $\bigcap_{i=1}^{n} V_{(p_i, \alpha r_i)}$ is also are element of $\mathscr{V}$. We define $\alpha V = \bigcap_{i=1}^{n} V_{(p_i, \alpha r_i)}$. It is obvious that the axioms (iii) and (iv) are satisfied.

Let $\alpha, \beta \in R_+$ and $\alpha < \beta$. If $(x, y) \in \alpha V$, then $(x, y) \in V_{(p_i, \alpha r_i)}$ $(i = 1, 2, \ldots, n)$. So $p_i(x, y) < \alpha r_i < \beta r_i$ $(i = 1, 2, \ldots, n)$ which gives that $(x, y) \in V_{(p_i, \beta r_i)}$ $(i = 1, 2, \ldots, n)$. Thus $(x, y) \in \beta V$ and so $\alpha V \subset \beta V$ which proves (v).

Let $\alpha, \beta$ be any two elements of $R_+$. If $(x, y) \in \alpha V_0 \beta V$, there is an element $z$ in $X$ such that $(x, z) \in \beta V$ and $(z, y) \in \alpha V$. So $p_i(x, z) < \beta r_i$ and $p_i(z, y) < \alpha r_i$ $(i = 1, 2, \ldots, n)$. Now $p_i(x, y) \leqslant p_i(x, z) + p_i(z, y) < (\alpha + \beta) r_i$ $(i = 1, 2, \ldots, n)$ which gives $(x, y) \in (\alpha + \beta) V$. So $\alpha V_0 \beta V \subset (\alpha + \beta) V$. Next, let $(x, y) \in X \times X$. Write $\alpha_i = p_i(x, y)$ $(i = 1, 2, \ldots, n)$ and let $\theta = \max\{\alpha_i / r_i : i = 1, 2, \ldots, n\}$. Choose $\lambda > \theta$. We have $p_i(x, y) = \alpha_i \leqslant \theta r_i < \lambda r_i$ $(i = 1, 2, \ldots, n)$, which gives that $(x, y) \in \lambda V$. Thus the (scalar) multiplication $\alpha V$ satisfies all the conditions (iii)-(vii). Therefore $\mathscr{V} \in \mathscr{F}$. By Theorem 2.2, $\mathscr{V}$ generates a quasi-uniformity $\mathscr{U}_0$ (say) on $X$ for which $\mathscr{V}$ is a base. Since $\mathscr{V}$ is a base of each of the quasi-uniformities $\mathscr{U}$ and $\mathscr{U}_0$ we have $\mathscr{U} = \mathscr{U}_0$. This completes the proof of the theorem.

Let $\mathscr{U}$ be a quasi-uniformity on $X$. For each $x \in X$ and $U \in \mathscr{U}$, let $U[x] = \{y : y \in X \text{ and } (x, y) \in U\}$.

Denote by $\tau$ the family of all subsets $G$ of $X$ such that for each $x \in G$ there is a member $U \in \mathscr{U}$ with $U[x] \subset G$. Then $\tau$ is a topology on $X$. We say that $\mathscr{U}$ induces the topology $\tau$. By Theorem 11.1.2 ([2], Ch. 11, p. 177) we see that if $\tau$ is a topology on $X$, there is a quasi-uniformity $\mathscr{U}$ on $X$ such that the topology induced by $\mathscr{U}$ on $X$ is identical with $\tau$.

Let $\mathscr{P}$ be a family of quasi-metrics on $X$. For any $p \in \mathscr{P}$, $r > 0$ and $x \in X$, let $S_p(x; r) = \{y : y \in X \text{ and } p(x, y) < r\}$.

Denote by $\mathscr{C}$ the family of all subsets of $X$ of the form

$$\bigcap_{i=1}^{n} S_{p_i}(x; p_i) \qquad (p_i \in \mathscr{P},\ r_i > 0 \text{ and } n = 1, 2, \ldots).$$

Then $\mathscr{C}$ is a base for some topology $\tau$ on $X$. We say that the topology $\tau$ is generated by the family $\mathscr{P}$ of quasi-metrics. Since

$$\bigcap_{i=1}^{n} S_{p_i}(x; r_i) = \bigcap_{i=1}^{n} V_{(p_i, r_i)}[x],$$

it follows that the topology induced by $\mathscr{U}$ on $X$ is identical with that generated by the family $\mathscr{P}$ of quasi-metrics.

**Theorem 2.4.** *Every topology on $X$ can be generated by a family of quasi-metrics on $X$.*

**Proof.** Let $\tau$ be a topology on $X$. By Theorem 11.1.2 ([2], Ch. 11, p. 177) there is a quasi-uniformity $\mathscr{U}$ on $X$ such that $\mathscr{U}$ induces the topology $\tau$. By Theorem 2.1, $\mathscr{U}$ can be generated by a family $\mathscr{P}$ of quasi-metrics on $X$. From above it follows that $\mathscr{P}$ generates $\tau$.

## References.

[1] J. L. KELLY, *General topology*, D. Van Nostrand Company, Princeton 1955.

[2] W. J. PERVIN, *Foundations of general topology*, Academic Press, N. Y. 1964.

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