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On entire functions of bounded index defined by Dirichlet expansions. (**)

The definition of function of bounded index, which is due to B. Lepson, (see [1], p. 304), can be restated in the following form:

Definition. *An entire function $f(s)$ is of bounded index if and only if there is an integer ν such that for all s*

$$\max\{|f^{(j)}(s)|/j!\,|\,c \leqq j \leqq \nu\} \geqq \sup\{|f^{(j)}(s)|/j!\,|_{j=0,1,2,\dots}\},$$

where $f^{(0)}(s)$ stands for $f(s)$. The smallest of all the integers with such property is called the index of $f(s)$.

In this paper we consider an entire function $f(s)$ which admits a Dirichlet expansion of the form

$$(1) \qquad f(s) = \sum_{n=0}^{\infty} a_n \exp(\lambda_n s) \qquad (\lambda_0 \geqq 0,\ \lambda_{n+1} > \lambda_n),$$

absolutely convergent everywhere and such that $\liminf_{n=\infty}(\lambda_{n+1} - \lambda_n) > 0$. For this type of function we prove the following

Theorem. *If $f(s)$ is of bounded index N, then it reduces to an exponential polynomial (that is, $a_n = 0$ from some n on).*

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Proof. Let us assume that the expansion in (1) is infinite. Then if we write $s = \sigma + it$ and define the maximum term in the usual form

$$m(\sigma) = \max\{|a_n|\exp(\lambda_n\sigma)\,|\,n \geqq 0\},$$

it is known (see [**2**], p. 717), that the index function (central index)

$$n(\sigma) = \max\{n\,|\,m(\sigma) = |a_n|\exp(\lambda_n\sigma)\}$$

is monotonically divergent to $+\infty$ with σ and that the same is true for the function $\lambda(\sigma) = \lambda_{n(\sigma)}$, that is

$$\lim_{\sigma=+\infty}\lambda(\sigma) = +\infty. \tag{2}$$

On the other hand, if we define

$$M_j(\sigma) = \max\{|f^{(j)}(\sigma + it)|\,|-\infty < t < +\infty\} \qquad (j = 0, 1, 2, \ldots),$$

it is also known (see [**3**], theorem A and theorem 2), that for any sequence of values of σ divergent to $+\infty$ in the complement S of the union of the exceptional sets of the functions $f^{(0)}, f^{(1)}, \ldots, f^{(N+1)}$, we have

$$\lim_{j=\infty}[M_j(\sigma)/M_k(\sigma)][\lambda(\sigma)]^{k-j} = 1, \qquad 0 \leqq j,\ k \leqq N+1.$$

From this and the definition of bounded index it follows that for any given $\varepsilon > 0$ and large $\sigma \in S$

$$\begin{aligned}[\lambda(\sigma) - \varepsilon]^{N+1} &\leqq M_{N+1}(\sigma)/M_c(\sigma) \leqq \max\{(N+1)!\,M_k(\sigma)/[k!\,M_0(\sigma)]\,|\,0 \leqq k \leqq N\}\\ &\leqq (N+1)\,|\max\{(1/k!)[M_k(\sigma)/M_{k-1}(\sigma)][M_{k-1}(\sigma)/M_{k-2}(\sigma)]\ldots\\ &\qquad\qquad \ldots[M_1(\sigma)/M_0(\sigma)]\,|\,c \leqq k \leqq N\}\\ &\leqq (N+1)!\,[\lambda(\sigma) + \varepsilon]^N,\end{aligned}$$

therefore

$$([\lambda(\sigma) - \varepsilon]/[\lambda(\sigma) + \varepsilon])^N[\lambda(\sigma) - \varepsilon] \leqq (N+1)! \tag{3}$$

which clearly contradicts (2) since the left side of (3) has limit $+\infty$ as $\sigma \to +\infty$. Hence, the expansion (1) must be finite and the theorem is proved.

A parallel result for the case of ordinary Taylor series is given by S. M. Shah (see [**4**], theorem 1).

References.

[1] B. Lepson, *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index*, Proceedings of Symposia in Pure Mathematics, Vol. II, Entire Functions and Related Parts Analysis, and Lecture Notes, 1966, Summer Institute in Entire Functions, University of California, La Jolla.

[2] A. G. Azpeitia, *On the maximum modulus and the maximum term of an entire Dirichlet series*, Proc. Amer. Math. Soc. **12** (1961), 717-721.

[3] F. Sunyer y Balaguer, *Generalizacion del metodo de Wiman-Valiron a una clase de series de Dirichlet*, Rev. Acad. Ci. Zaragoza (2) **16** (1961), 9-13.

[4] S. M. Shah, *Entire functions of bounded index*, Proc. Amer. Math. Soc. **19** (1968), 1017-1022.

Resumen.

Si la function $f(s)$ *es de indice acotado y admite un desarrollo en series de Dirichlet de la forma* $f(s)=\sum_{n=0}^{\infty} a_n \exp[\lambda_n s]$, $(\lambda_0 \geqslant 0,\ \lambda_{n+1} > \lambda_n)$ *absolutamente convergente en todo el plano y tal que* $\liminf_{n=\infty}(\lambda_{n+1}-\lambda_n) > 0$, *entonces, la serie se reduce a un polinomio exponencial* (*es decir* $a_n = 0$ *desde un valor de* n *en adelante*).

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