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Derivations of Lie algebras into bimodules. (**)

1. – For a Lie algebra L over a commutative ring K we shall denote by $U(L)$ the universal enveloping algebra of L over K with canonical map $i: L \to U(L)$. If M is an L-bimodule (that is, $U(L)$-bimodule) we shall write $[x, m] = x \cdot m - m \cdot x = -[m, x]$ for any x in L and m in M. By a derivation of L in M we shall mean a K-linear function $f: L \to M$ such that $f([x, y]) = [f(x), y] + [x, f(y)]$. A K-linear map $g: U(L) \to M$ is a derivation extending f if $g(uv) = g(u) \cdot v + u \cdot g(v)$ and $f = gi$.

Theorem. *Every derivation of L into a bimodule M has a unique extension to a derivation of $U(L)$ into M.*

Proof. Since M is an L-bimodule, if we consider M to be a commutative Lie algebra, then $H = L \oplus M$ becomes a Lie algebra with $[x, m] = xm - mx$ for all x in L and m in M. It is the semidirect product of L by M corresponding to the homomorphism $x \to [x, \cdot\,]$ of L into $\mathrm{Der}(M)$. Since there is a retraction of H onto the subalgebra L, the injection of L into H induces an injection of $U(L)$ into $U(H)$. Extend f by setting $f(x + m) = f(x)$ for all x in L, m in M, then f is a derivation of H into H. It is well known ([**1**], § 2, n. 8, Proposition 7) that then f has a unique extension to a derivation $D: U(H) \to U(H)$. Now M is a $U(L)$ bimodule so $U(L) \oplus M$ is naturally an associative algebra containing $U(L)$ as subalgebra and M as an ideal. The canonical map $L \oplus M \to U(L) \oplus M$ then induces an algebra homomorphism

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$s: U(H) \to U(L) \oplus M$. Let g be the composite of the injection $U(L) \to U(H)$ followed by sD. Then g is a derivation of $U(L)$ into $U(L) \oplus M$ and since $g(L)$ is contained in the ideal M, $g(U(L))$ is contained in M. Clearly g is the unique extension of f to a derivation of $U(L)$ into M.

Corollary. *Suppose f is an inner derivation, that is, there is an element m in M such that $f(x) = xm - mx$ for all x in L. Then its unique extension g is also an inner derivation.*

2. – The Hochschild cohomology of L or $U(L)$ with values in the bimodule M can be constructed from the bimodule bar resolution ([**3**], chap. X, p. 282). When this is done the derivations (crossed homomorphisms) of $U(L)$ into M are the 1-cocycles and the inner derivations (principal crossed homomorphisms) are the 1-coboundaries. Thus $H^1(U(L), M)$ is the K-module of outer derivations of L or $U(L)$ into M. Also $H^0(U(L), M)$ is the K-module of invariants, that is, those m in M for which $xm = mx$ for all x in L or for all x in $U(L)$. In the more classical situation ([**2**], p. 93) where M is initially just a left L-module, let $mx = 0$ for all m in M and x in L, then M is an L-bimodule. Then an element m is invariant iff $xm = 0$ for all x in L, or iff $um = me(u)$ for all u in $U(L)$, where $e: U(L) \to K$ is the augmentation. Moreover $f: L \to M$ is then a crossed homomorphism iff $f([x, y]) = xf(y) - yf(x)$. In this form it was not so obvious that 1-cocycles were indeed derivations of some kind.

References.

[**1**] N. Bourbaki, *Groupes et Algèbres de Lie*, Chap. I, Algèbres de Lie, Hermann, Paris 1960.

[**2**] N. Jacobson, *Lie Algebras*, Interscience, New York 1962.

[**3**] S. MacLane, *Homology*, Springer, Berlin 1963.

Abstract.

It is well-known that a derivation of a Lie algebra into itself extends uniquely to a derivation of its enveloping algebra into itself. We prove that this remains true for derivations into bimodules.

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