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On a particular nuclear reactor with a spherical geometry. (**)

1. - Introduction.

The neutron flux in a nuclear reactor with no reflector, as it is well known, is maximum in the center of the core and vanishes at the extrapolated boundary. Moreover the evaluation of the average flux gives values which depend on the particular geometry of the reactor and are in any case considerably less than the maximum value. Apart from the use of technical expedients that however present disadvantages, it is possible to flatten the flux, i.e., to increase the ratio γ of maximum to average flux with the use of a peripherical reflector, in the presence of which the flux is zero at the extrapolated boundary of the reflector and then its average value increases.

If we refer to unreflected thermal reactors, the value of γ obtained for an infinite plane slab ($\gamma = 0{,}637$) is much greater than the values calculated for the conventional geometrical shapes (only in the case of the sphere we have $\gamma = 0{,}304$). Therefore the idea of a hollow spherical reactor rises, in order to realize a critical system, which, when the radius R_0 of the inner cavity increases, may approximate more and more the physical behaviour of the infinite plane slab.

In this work we relate only the nuclear calculation that has quite shown the exactness of such a physical intuition.

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We want only to hint at a possible solution for the structure of the reactor: the reactor is graphite-moderated and gas-cooled; the fuel elements (such as the control rods) have the shape of a frustum of cone convergent towards the center of the reactor and carry their own portion of peripherical reflector; the gas-coolant enters in radial direction crossing the reflector and the core and it flows into the inner cavity, from which it goes backwards through different pipes and gathers in a peripherical chamber. The reactor is anchored to a structure that is outside, has a spherical shape and is connected with the vertices of an icosahedron to keep the spherical symmetry.

This solution allows us to distribute the fuel elements uniformly, therefore all the elements are in the same condition of working, whence the possibility of discharging simultaneously the elements, which are consumed enough uniformly also in their length by virtue of the high values of γ. To change the fuel elements we might consider, for example, these two solutions: to wheell the reactor with two couples of jacks to be pitched in holes which are made in the carrying structure where the vertices of the icosahedron are, or to wheel the fuelling machine around the core. But now we do not dwell on such subjects any longer.

2. - Neutron flux distribution.

The reactor, that we take into consideration, has the shape of a hollow sphere surrounded by a reflector, therefore, in order to calculate the neutron flux, we must consider three different regions: the inner central cavity, the active zone of the reactor (core) having the shape of a spherical bark and, at last, the peripherical reflector with the same shape. Therefore the characteristic sizes will be the radii R_0 and R of the two spheres that delimitate the cavity and the core of the reactor respectively, besides the thickness T of the reflector; we shall suppose that T includes the extrapolated distance as well as R when $T = 0$.

For the nuclear calculation we suppose that the reactor is homogeneous and thermal, moreover we assume that both the reflector and the moderator are made of the same material.

Denoting by Φ_c the neutron flux in the core and by B^2 the buckling of the critical reactor, the diffusion equation in the steady-state is

$$\Delta^2\Phi_c + B^2\Phi_c = 0 . \tag{2.1}$$

This differential equation, using a system of polar coordinates with the origine in the center of the reactor and taking into account the particular spherical symmetry, assumes the following form

$$\frac{d^2\Phi_c(r)}{dr^2} + \frac{2}{r}\frac{d\Phi_c(r)}{dr} + B^2\,\Phi_c(r) = 0\,, \tag{2.2}$$

where r is the radial distance from the center.

Putting

$$\Phi_c(r) = \frac{u(r)}{r}\,, \tag{2.3}$$

the second order differential equation (2.2), whose coefficients are not constant, becomes simply

$$\frac{d^2u(r)}{dr^2} + B^2u(r) = 0\,, \tag{2.4}$$

from which it follows that

$$u(r) = A_1 \sin(Br) + A_2 \cos(Br) \tag{2.5}$$

whence, using (2.3), we have

$$\Phi_c(r) = \frac{1}{r}[A_1 \sin(Br) + A_2 \cos(Br)]\,. \tag{2.6}$$

The neutron flux Φ_r in the reflector may be deduced again by the diffusion equation, which now may be written in the form

$$\Delta^2\Phi_r - \frac{1}{L_r^2}\Phi_r = 0, \tag{2.7}$$

where L_r is the thermal diffusion length in the reflector. The general solution of this equation may be deduced in the same way used for (2.1); we obtain

$$\Phi_r(r) = \frac{1}{r}(C_1 \exp[kr] + C_2 \exp[-kr])\,, \tag{2.8}$$

where we have put

$$k^2 = \frac{1}{L_r^2}\,. \tag{2.9}$$

The two solutions (2.6) and (2.8) must satisfy the boundary conditions, which impose the continuity of the flux and of the neutron current density at the spherical surfaces that delimitate the inner zones of the reactor, besides the fact that the flux be zero at the extrapolated distance of the reflector. Therefore, besides

$$\Phi_r(R+T)=0\,, \tag{2.10}$$

we must have

$$\Phi_c(R)=\Phi_r(R) \tag{2.11}$$

and

$$D_c\left[\frac{\mathrm{d}\Phi_c}{\mathrm{d}r}\right]_{r=R}=D_r\left[\frac{\mathrm{d}\Phi_r}{\mathrm{d}r}\right]_{r=R}, \tag{2.12}$$

where D_c and D_r are the diffusion coefficients of the core and of the reflector respectively; at last, it is enough to observe that we have neither production nor absorption of neutrons in the cavity to may state that

$$\Phi(r)=\Phi_c(R_0) \qquad \forall\, r<R_0 \tag{2.13}$$

and

$$\left[\frac{\mathrm{d}\Phi_c}{\mathrm{d}r}\right]_{r=R_0}=0\,. \tag{2.14}$$

From the condition (2.10) we have at once

$$C_2=-\,C_1\exp\,[2k(R+T)] \tag{2.15}$$

and, if we put

$$C\;=-\,2C_1\exp\,[k(R+T)]\,, \tag{2.16}$$

the solution (2.8) assumes the following form

$$\Phi_r(r)=\frac{C}{r}\sinh\,[k(R+T-r)]\,. \tag{2.17}$$

The other three conditions (2.11), (2.12) and (2.14), taking into account the fact that the reflector and the moderator are made of the same material (therefore the diffusion coefficients are equal, see on page 125 of [**3**]), give the following system

$$\left\{\begin{aligned}&A_1 \sin(BR) + A_2 \cos(BR) = C \sinh(kT)\\&A_1 B \cos(BR) - A_2 B \sin(BR) = - Ck \cosh(kT)\\&A_1 [BR_0 \cos(BR_0) - \sin(BR_0)] - A_2 [BR_0 \sin(BR_0) + \cos(BR_0)] = 0\end{aligned}\right. \tag{2.18}$$

linear and homogeneous, which has nontrivial solutions for A_1, A_2 and C if and only if the determinant of the matrix of the coefficients is zero. From this condition we derive, with some calculus, the critical equation of the reactor

$$\begin{aligned}&[B^2 R_0 \tanh(kT) - k] \sin[B(R - R_0)] =\\&\qquad = B[\tanh(kT) + kR_0] \cos[B(R - R_0)] .\end{aligned} \tag{2.19}$$

We observe that in this trascendental equation we have certainly

$$\sin[B(R - R_0)] \neq 0 , \tag{2.20}$$

because, otherwise, also the right-hand side ought to vanish, but that is not possible for

$$\tanh(kT) + kR_0 > 0 . \tag{2.21}$$

Then, the equation (2.19) reduces to

$$\cot[B(R - R_0)] = \frac{1}{B} \frac{B^2 R_0 \tanh(kT) - k}{\tanh(kT) + kR_0} , \tag{2.22}$$

from which we may obtain for R the following relation

$$R = R_0 + \frac{1}{B}\left[\cot^{-1} \frac{BR_0 \tanh(kT) - k/B}{\tanh(kT) + kR_0} + n\pi\right] , \tag{2.23}$$

where n is zero or an integer such that the neutron flux is positive for any $r \in [R_0, R]$.

On the other hand, writing (2.6) in terms of $r - R_0$ and taking account of $(2.18)_3$, we have

$$\Phi_c(r) = \Phi_c(R_0)\,\frac{\sin[B(r-R_0)] + BR_0\cos[B(r-R_0)]}{Br}\,, \tag{2.24}$$

from which it follows that Φ_c vanishes for the values $\bar{r} > R_0$ such that

$$\cot[B(\bar{r}-R_0)] = -\frac{1}{BR_0} \tag{2.25}$$

i.e., when

$$\bar{r} = R_0 + \frac{1}{B}\left[\cot^{-1}\left(-\frac{1}{BR_0}\right) + m\pi\right], \tag{2.26}$$

where m is integer.

We observe that (2.23), when $T = 0$, coincides with (2.26), hence, whatever the values of B and R_0 may be, the first value of $\bar{r}$ ($m = 0$) is equal to $R = \bar{R}$ calculated for $T = 0$ and $n = 0$, assuming that $\bar{R}$ includes the extrapolated distance of the bare reactor; moreover, when $T \neq 0$ the critical radius R is surely less than $\bar{R}$ (in fact the argument of $\cot^{-1}$ in (2.23) is an increasing function of T), therefore the flux does not vanish within the core, whatever the values of B and R_0 may be, provided that we assume only $n = 0$ in (2.23); for any $n > 0$ we have always R greater than the first value of $\bar{r}$.

The critical radius of the reactor, therefore, is given by the following expression

$$R = R_0 + \frac{1}{B}\cot^{-1}\frac{BR_0\tanh(kT) - k/B}{\tanh(kT) + kR_0}\,. \tag{2.27}$$

We note at once that (2.19) when $R_0 = 0$ becomes the critical equation of a full spherical reactor with the reflector (see (5.80) of [**3**])

$$k\sin(BR) + B\tanh(kT)\cos(BR) = 0\,; \tag{2.28}$$

then from (2.27), when R_0 increases without bound, we have the following limit

$$\lim_{R_0\to+\infty}(R - R_0) = \frac{1}{B}\cot^{-1}\left[\frac{B}{k}\tanh(kT)\right], \tag{2.29}$$

that gives the half-thickness of a critical infinite plane slab.

This second result derives just from the fact that when $R_0 \to +\infty$ the hollow spherical reactor becomes an infinite plane slab; moreover the condition that the derivative of the flux is zero at $r = R_0$ is true also in the middle of the infinite plane slab, whose thickness is twice the limit expressed by (2.29).

We also observe that, as T tends to infinite, we have

$$\lim_{T\to+\infty} R = R_0 + \frac{1}{B} \cot^{-1} \frac{BR_0 - k/B}{1 + kR_0}, \tag{2.30}$$

from which we may calculate the greatest savings we can obtain with the reflector for any value of B and R_0, of course to be referred to the case with $T = 0$. Such savings can be obtained in practice by the values of T such that $\tanh(kT) \simeq 1$ and hence when $T = 2L_r \div 3L_r$ (being $\tanh 2 = 0{,}964$).

At last, we note that also the constant C of (2.17) depends upon $\Phi_c(R_0)$, being for $T \neq 0$ (when $T = 0$ we have $C = 0$ too), on the ground of (2.11) and $(2.18)_1$,

$$C = \frac{R}{\sinh(kT)} \Phi_c(R), \tag{2.31}$$

where $\Phi_c(R)$ is given by (2.24).

3. - Neutron flux flattening.

As it is well known, a particularly important index of a good working of a reactor is the ratio

$$\gamma = \frac{\bar{\Phi}_c}{(\Phi_c)_{\max}}, \tag{3.1}$$

where

$$\bar{\Phi}_c = \frac{1}{V} \int_V \Phi_c \, \mathrm{d}V \tag{3.2}$$

is the average value of the flux in all the volume V of the core and $(\Phi_c)_{\max}$ is the maximum value that the flux assumes in the core.

As regards the maximum we have

$$(\Phi_c)_{\max} = \Phi_c(R_0). \tag{3.3}$$

To prove this we begin by observing that

$$\Phi_c(r) > 0 \qquad \forall\, r \in [R_0, R)\,, \tag{3.4}$$

hence and from (2.2) it follows that in such an interval we have also

$$\frac{d^2\Phi_c}{dr^2} + \frac{2}{r}\frac{d\Phi_c}{dr} = -B^2\Phi_c < 0\,, \tag{3.5}$$

whence

$$\frac{d^2\Phi_c}{dr^2} < -\frac{2}{r}\frac{d\Phi_c}{dr}\,. \tag{3.6}$$

If we suppose that in a right-neighbourhood of R_0 ($r > R_0$) we have $d\Phi_c/dr > 0$, from (3.6) we ought to have in such a neighbourhood $d^2\Phi_c/dr^2 < 0$ and hence the function $\Phi_c(r)$ would be there increasing and concave downward; but this is not possible for the neutron current is continuous and (2.14) holds. Therefore we can state that the neutron flux is not an increasing function of r in $[R_0, R)$ and hence (3.3) follows.

The value of $\Phi_c(R_0)$ must be chosen on the ground of the power of the reactor.

To obtain $\bar{\Phi}_c$ we must calculate the integral of (3.2) over the volume

$$V = \frac{4}{3}\pi(R^3 - R_0^3) \tag{3.7}$$

of the core, i.e., with r which ranges from R_0 to R. Thus we have

$$\gamma = \frac{3}{B^3(R^3 - R_0^3)}\{(1 + B^2R_0R)\sin[B(R-R_0)] - B(R-R_0)\cos[B(R-R_0)]\}\,. \tag{3.8}$$

4. - Numerical calculus.

Because of the particularly complicated form of the deduced relations, we have preferred to study them in function of the various parameters, also to have a complete idea of the nuclear behaviour of the reactor. Thus, we have fixed B, R_0 and T and we have calculated the corresponding critical sizes (the radius R and the volume V) and the flux flattening γ with the computer.

To do this, in particular, we have modified the equation (2.27), which, on the ground of the well known relation

$$\cot^{-1} y = \frac{\pi}{2} - \tan^{-1} y \,, \tag{4.1}$$

may be written in the form

$$R = R_0 + \frac{1}{B}\left[\frac{\pi}{2} - \tan^{-1}\frac{BR_0 \tanh(kT) - k/B}{\tanh(kT) + kR_0}\right]. \tag{4.2}$$

The programme has been made using the FORTRAN IV language, here related.

```
      READ(2,1)AL,P,PG
1     FORMAT(2F12.4,F12.7)
      AK=1./AL
      DO 2 I=1,7
      T=20*I-20
      TH=TANH(AK*T)
      DO 2 J=1,12
      RO=25*J
      RO3=RO**3
      DEN=TH+AK*RO
      DO 2 K=1,20
      B=P*FLOAT(K)
      BO=B*RO
      X=(BG*TH-AK/B)/DEN
      Y=PG/2.-ATAN(X)
      R=RO+Y/B
      A=R**3-RO3
      V=4.*PG*A/3.
      APP=3.*((1.+BO*B*R)*SIN(Y)-Y*COS(Y))/(B**3*A )
      WRITE(3,3)T,RO,B,R,V,APP
2     CONTINUE
3     FORMAT(1X,2(F5.0,2X),F7.4,2X,3(E14.4,2X))
      STOP
      END
```

In it the values of the thickness T of the reflector vary from 0 cm to 120 cm with variations of 20 cm. When $T = 0$ cm we have the case of the bare reactor, useful to calculate the savings corresponding to the various thicknesses of the reflector; the greatest savings may be derived from the case with $T = 120$ cm which is a thickness greater than $2L_r$ (we have assumed for the graphite $AL = L_r = 52$ cm).

The values of R_0 range from 25 cm to 300 cm with variations of 25 cm; thus we have obtained a detailed study of the influence of such a parameter on the nuclear calculation of the reactor.

At last, for any value of T and R_0 we have varied B from 0,005 cm^{-1} to 0,1 cm^{-1} increasing it with variations of 0,005 cm^{-1}. Such values comprehend widely the set of the values of B's of the extant reactors and this only to have a complete study of the peculiarities of the spherical reactor in examination.

5. - Results.

Also for $T = 0$ cm (the bare reactor) we have considerable improvements in γ's of the reactor in examination for any value of B and R_0 in comparison with the other conventional geometries. Such an advantage is accentuated as R_0 increases for any fixed value of B; an analogous result is obtained if we fix R_0 and we increase B.

The values of γ, obtained when the thickness of the reflector is little greater than $2L_r$ (for $T = 120$ cm), are excellent; for example if $B = 0{,}02$ cm^{-1} γ varies from 0,596, when $R_0 = 25$ cm, to 0,859, when $R_0 = 300$ cm. At last, we observe that even with a small thickness of the reflector (for example when $T = 40$ cm) we can obtain considerable improvements in comparison with the case $T = 0$ cm.

The following table lists some values of R and γ corresponding to some values of B, T and R_0; in it we express B in cm^{-1} and R, R_0, T in cm.

Table I.

	$B = 0.01$				$B = 0.02$				$B = 0.03$			
	$T = 0$		$T = 120$		$T = 0$		$T = 120$		$T = 0$		$T = 120$	
R_0	R	γ	R	γ	R	γ	R	γ	R	γ	R	γ
25	314.6	0.312	267.5	0.459	158.8	0.333	119.1	0.596	108.2	0.360	75.2	0.702
50	317.7	0.333	270.6	0.488	167.8	0.387	128.0	0.673	121.9	0.434	88.9	0.796
75	324.8	0.360	277.6	0.523	182.9	0.434	143.1	0.731	141.3	0.483	108.2	0.845
100	335.6	0.387	288.4	0.558	201.7	0.469	161.9	0.770	163.0	0.515	130.0	0.871
125	349.2	0.412	302.4	0.588	222.5	0.495	182.8	0.795	186.0	0.536	152.9	0.887
150	365.8	0.434	318.7	0.614	244.6	0.515	204.8	0.813	209.6	0.552	176.5	0.898

In any case R decreases if B increases, therefore the thicknesses $R - R_0$ of the core become very small while the corresponding values of γ become very high.

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Sommario.

Si considera un particolare reattore nucleare avente la forma di una sfera cava con o senza riflettore periferico, mettendone in risalto il maggiore appiattimento del flusso neutronico rispetto alle altre geometrie convenzionali.

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