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A Hellinger-Hahn decomposition theorem. (**)

Introduction.

Many problems related to countably additive measures have been extended to finitely additive set functions. In [6], for example, the Hewitt-Yosida decomposition was generalized to s-bounded finitely additive set functions by using a variation of the Caratheodory process. In [7] the author obtained generalizations of the Hewitt-Yosida decomposition and the Lebesgue decomposition to finitely additive vector measures satisfying some continuity condition.

The present work generalizes a Hellinger-Hahn decomposition of the domain X of a Borel function (see [5]) in the case where the measure μ on the Borel σ-algebra of X is only finitely additive.

The result in [5] can be stated as follows: Let f be a countable to one Borel function from a complete separable metric space X into another complete separable metric space Y. Both X and Y are equipped with their usual Borel σ-algebras and X is further equipped with a finite measure μ. Then X can be partitioned into pair-wise disjoint Borel sets N, C_1, C_2, C_3, ... (this sequence may be finite) such that

(1) $\mu(N) = 0$;
(2) $\mu(C_i) > 0$ for each i;
(3) $f|_{C_i}$ is injective for each i;
(4) the measure induced by $f|_{C_{i+1}}$ is absolutely continuous with respect to the one induced by $f|_{C_i}$.

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In this paper we obtain a generalized result by using a variation of the Lebesgue decomposition (see [7]).

Preliminaries.

Throughout this paper X and Y are both Banach spaces equipped with their usual Borel σ-algebras $\mathscr{B}(X)$ and $\mathscr{B}(Y)$, respectively. Let μ be a non-negative finitely additive measure on $\mathscr{B}(X)$ and let f be a Borel function from X into Y. The function f is said to be *countable to one* if the inverse image of every singleton is either finite or countable.

Theorem 1 (Lusin) (see [3]). *Let f be a countable to one Borel function from X into Y. Then X can be partitioned into disjoint sets $A_1, A_2, A_3, \ldots$ such that the restriction $f|A_K$ of f to A_K is injective for each K.*

The following is a variation of the Lebesgue decomposition obtained in [7].

Theorem 2. *Let m be a finite non-negative finitely additive measure on a σ-algebra Σ. If λ is a finitely additive non-negative measure on Σ, m is uniquely representable as the sum $m = m_s + m_a$, where m_s is λ-singular and m_a is λ-continuous.*

Note that two positive measures μ and ν are *singular* (to each other) if $\mu \wedge \nu = 0$. If μ and ν are finitely additive, then given $\varepsilon > 0$ there exists a set $E \in \Sigma$ such that $\mu(E) < \varepsilon$ and $\nu(E') < \varepsilon$. If μ and ν are both countably additive, then $\mu(E)$ and $\nu(E')$ can both be made zero (see [4]). To distinguish these two cases, as far as the decomposition of the original set into E and E' is concerned, in the former case we say that μ and ν are *topologically singular* (see [6]).

Decomposition of the domain.

Definition 1. Let λ and μ be two finitely additive measures on a σ-algebra Σ. For $\varepsilon > 0$, λ is said to be *ε-approximately equal to μ* if

$$\sup_{B \in \Sigma} |\lambda(B) - \mu(B)| < \varepsilon.$$

Definition 2. Let λ_1 and λ_2 be two finitely additive measures on a σ-algebra Σ. For $\varepsilon > 0$, λ_1 is said to be *ε-approximately absolutely continuous to λ_2* if there exists finitely additive measures μ_1 and μ_2 which are ε-approximately equal to λ_1 and λ_2, respectively, and such that $\mu_1 \ll \mu_2$.

In the following we will indicate by the notation $A_{ijj\ldots j;r}$ that the subscript contains K elements up to the colon «:». If no colon appears then there are a total of K elements in the subscript.

Theorem 3. *Let f be a countable to one Borel function from X into Y. Then for any $\varepsilon > 0$ there exists a sequence $\{D_i\}$ of disjoint Borel sets in X such that each D_i is the limit of a sequence $\{B_i^n\}_{n=1}^{\infty}$ of Borel sets satisfying the following properties:*

(1) $B_i^n \subset B_i^{n+1}$.

(2) $\mu f|_{B_{i+1}^n}^{-1}$ *is ε-approximately absolutely continuous with respect to* $\mu f|_{B_i^n}^{-1}$.

(3) *Each B_i^n is a finite disjoint union of Borel sets of the form*

$$B_i^n = A_{i22\ldots2} \cup A_{i+1,22\ldots2:1} \cup \ldots \cup A_{i+K-1,22\ldots2:1} .$$

Moreover the sets A's satisfy the following properties:

(a) *The function f is one-one on each A.*

(b) $A_{n+K,22\ldots2} = A_{n+K,22\ldots2:1} \cup A_{n+K,22\ldots2:2}$.

(c) $\mu f|_{A_{n+K,22\ldots2:i}}^{-1}$ $(i = 1, 2)$ *is ε-approximately equal to $m_{n+K,22\ldots2:i}$ and $m_{n+K,22\ldots2} = m_{n+K,22\ldots2:1} + m_{n+K,22\ldots2:2}$, where $m_{n+K,22\ldots2:1}$ is topologically singular and $m_{n+K,22\ldots2:2}$ is absolutely continuous with respect to* $(m_{K,22\ldots2} + m_{K+1,22\ldots2:1} + \ldots + m_{n+K-1,22\ldots2:1})$.

(d) $\mu f|_{B_K^n}^{-1}$ *is ε-approximately equal to* $(m_{K,22\ldots2} + m_{K+1,22\ldots2:1} + \ldots + m_{n+K-1,22\ldots2:1})$.

(e) $(m_{K+1,22\ldots2:2} + m_{K+2,22\ldots2:21} + \ldots + m_{n+K,22\ldots2:21})$ *is absolutely continuous with respect to* $(m_{K,22\ldots2} + m_{K+1,22\ldots2:1} + \ldots + m_{n+K-1,22\ldots2:1})$.

Proof. Consider the partition of X into disjoint Borel sets $A_1, A_2, A_3, \ldots$ by Lusin's theorem.

Let $\mu f|_{A_i}^{-1}$ be defined by $\mu f|_{A_i}^{-1}(B) = \mu(f|_{A_i}^{-1}(B))$ where $B \in \mathscr{B}(Y)$. Now $\mu f|_{A_i}^{-1}$ is a non-negative finitely additive measure defined on $\mathscr{B}(Y)$. Given $\varepsilon > 0$, let ε_j^i be a double sequence of positive numbers such that $\sum_{i,j} \varepsilon_j^i < \varepsilon$. We set $m_i = \mu f|_{A_i}^{-1}$. By Theorem 2, m_2 can be written as $m_2 = m_{21} + m_{22}$ where m_{21} is topologically singular and m_{22} is absolutely continuous with respect to m_1. Then there exists $B_0 \in \mathscr{B}(Y)$ such that $m_1(B_0) < \varepsilon_1^1/2$, $m_{22}(B_0) < \varepsilon_1^1/2$, $m_{21}(B_0') < \varepsilon_1^1/2$. The Borel sets $A_{21} = f|_{A_2}^{-1}(B_0) = f|_{A_{21}}^{-1}$ and $A_{22} = f|_{A_2}^{-1}(B_0') = f|_{A_{22}}^{-1}(B_0')$ form a partition of A_2. For $B \in \mathscr{B}(Y)$ we have $f|_{A_{21}}^{-1}(B) = f|_{A_2}^{-1}(B \cap B_0)$ and $f|_{A_{22}}^{-1}(B) = f|_{A_2}^{-1}(B \cap B_0')$. This gives $\mu f|_{A_{21}}^{-1}(B) = m_2(B \cap B_0) = m_{21}(B) - m_{21}(B \cap B_0') + m_{22}(B \cap B_0)$ and $\mu f|_{A_{22}}^{-1}(B) = m_{21}(B \cap B_0') + m_{22}(B) - m_{22}(B \cap B_0)$. Therefore,

$$|\mu f|_{A_{21}}^{-1} - m_{21}| < \varepsilon_1^1 \qquad \text{and} \qquad |\mu f|_{A_{22}}^{-1} - m_{22}| < \varepsilon_1^1 .$$

Next, we may decompose m_3 with respect to $\mu f|_{A_1 \cup A_{21}}^{-1}$ as $m_3 = m_{31} + m_{32}$ where m_{31} is topologically singular and m_{32} is absolutely continuous with respect to $\mu f|_{A_1 \cup A_{21}}^{-1}$ Then A_3 can be partitioned into $A_{31} \cup A_{32}$ such that

$$|\mu f|_{A_{31}}^{-1} - m_{31}| < \varepsilon_1^2 \qquad \text{and} \qquad |\mu f|_{A_{32}}^{-1} - m_{32}| < \varepsilon_1^2 .$$

Using this method we can decompose m_n with respect to $\mu f|_{A_1 \cup A_{21} \cup \ldots \cup A_{n-1,1}}^{-1}$ into the sum $m_n = m_{n1} + m_{n2}$, where m_{n1} is topologically singular and m_{n2} is absolutely continuous with respect to $\mu f|_{A_1 \cup A_{21} \cup \ldots \cup A_{n-1,1}}^{-1}$. Then A_n can be partitioned into $A_{n1} \cup A_{n2}$ such that

$$|\mu f|_{A_{n1}}^{-1} - m_{n1}| < \varepsilon_1^{n-1} \qquad \text{and} \qquad |\mu f|_{A_{n1}}^{-1} - m_{n2}| < \varepsilon_1^{n-1} .$$

Let $B_1^n = A_1 \cup A_{21} \cup \ldots \cup A_{n1}$ and $B_1^1 = A_1$, we have

$$m_{n2} \ll \mu f|_{B_1^{n-1}}^{-1} .$$

Now, we may decompose m_{32} with respect to m_{22} as $m_{32} = m_{321} + m_{322}$, where m_{321} is topologically singular and m_{322} is absolutely continuous with respect to m_{22}. Then Y can be partitioned into $C_0 \cup C_0'$ such that

$$m_{22}(C_0) < \varepsilon_2^1/2 , \qquad m_{322}(C_0) < \varepsilon_2^1/2 , \qquad m_{321}(C_0') < \varepsilon_2^1/2 .$$

Let $A_{321} = f|_{A_{32}}^{-1}(C_0)$ and $A_{322} = f|_{A_{32}}^{-1}(C_0')$. Then A_{321} and A_{322} form a partition of A_{32} and we have

$$|\mu f|_{A_{321}}^{-1} - m_{321}| < \varepsilon_1^2 + \varepsilon_2^1 < \varepsilon$$

and

$$|\mu f|_{A_{322}}^{-1} - m_{322}| < \varepsilon_1^2 + \varepsilon_2^1 < \varepsilon .$$

Letting $B_2^1 = A_{22}$ we have

$$|m_{22} - \mu f|_{B_2^1}^{-1}| < \varepsilon \qquad \text{and} \qquad m_{22} \ll \mu f|_{A_1}^{-1} = \mu f|_{B_1^1}^{-1} .$$

Letting $B_2^2 = A_{22} \cup A_{321}$, we have

$$|(m_{22} + m_{321}) - \mu f|_{B_2^2}^{-1}| < \varepsilon \qquad \text{and} \qquad (m_{22} + m_{321}) \ll \mu f|_{B_1^2}^{-1} .$$

If we decompose m_{42} with respect to $m_{22} + m_{321}$ we obtain two finitely additive measures m_{421} and m_{422} which are respectively singular and absolutely continuous with respect to $m_{22}+m_{321}$ and A_{42} can be partitioned into $A_{421}\cup A_{422}$ such that

$$|\mu f|^{-1}_{A_{421}} - m_{421}| < \varepsilon_1^3 + \varepsilon_2^3 < \varepsilon\,, \qquad |\mu f|^{-1}_{A_{422}} - m_{422}| < \varepsilon_1^3 + \varepsilon_2^2 < \varepsilon\,.$$

Let $B_2^3 = A_{22}\cup A_{321}\cup A_{421}$. We have

$$|(m_{22}+m_{321}+m_{421}) - \mu f|^{-1}_{B_2^3}| < \varepsilon, \qquad (m_{22}+m_{321}+m_{421}) \ll \mu f|^{-1}_{B_1^3}\,.$$

Similarly, if we decompose m_{n2} into m_{n21} and m_{n22} which are respectively singular and absolutely continuous with respect to $(m_{22}+m_{321}+\ldots+m_{n-1,21})$ we obtain a partition $A_{n2}=A_{n21}\cup A_{n22}$ such that

$$|\mu f|^{-1}_{A_{n2i}} - m_{n2i}| < \varepsilon_1^{n-1} + \varepsilon_2^{n-2} < \varepsilon\,, \qquad i = 1, 2\,.$$

Let $B_2^n = B_2^{n-1}\cup A_{n+1,21}$. We have

$$|(m_{22}+m_{321}+\ldots+m_{n+1,21}) - \mu f|^{-1}_{B_2^n}| < \varepsilon$$

and

$$(m_{22}+m_{321}+\ldots+m_{n+1,21}) \ll \mu f|^{-1}_{B_1^n}\,.$$

So far we obtain two sequences, (B_1^n) and (B_2^n) of Borel sets in $\mathscr{B}(X)$ such that $\mu f|^{-1}_{B_2^n}$ is ε-approximately absolutely continuous with respect to $\mu f|^{-1}_{B_1^n}$. Furthermore B_1^n is a finite disjoint union of Borel sets: $B_1^n = A_1\cup A_{21}\cup\ldots\cup A_{n1}$, where f is one-one on each set A_{n1}. Also B_2^n is a finite disjoint union of Borel sets: $B_2^n = A_{22}\cup A_{321}\cup\ldots\cup A_{n+1,21}$, where f is one-one on each $A_{n,21}$. Moreover $\mu f|^{-1}_{A_{ni}}$ $(i = 1, 2)$ is ε-approximately equal to m_{ni}, where m_{n1} and m_{n2} are respectively the singular part and the absolutely continuous part in the Lebesgue decomposition of $m_n = \mu f|^{-1}_{A_n}$ with respect to $\mu f|^{-1}_{A_1\cup A_{21}\cup\ldots\cup A_{n-1,1}}$. In addition $\mu f|^{-1}_{A_{n,2i}}$ $(i = 1, 2)$ is ε-approximately equal to $m_{n,2i}$ where $m_{n,21}$ and $m_{n,22}$ are respectively the singular part and the absolutely continuous part in the Lebesgue decomposition of m_{n2} with respect to $(m_{22}+m_{321}+\ldots+m_{n-1,21})$.

In the next step we decompose m_{422} with respect to m_{322} as $m_{422} = m_{4221} + m_{4222}$, where m_{4221} and m_{4222} are topologically singular and absolutely continuous with respect to m_{322}, respectively. We obtain the partition $A_{4221}\cup A_{4222}$ of A_{422} such that $\mu f|^{-1}_{A_{422i}}$ $(i = 1, 2)$ is ε-approximately equal to m_{422i}. With $B_3^1 = A_{322}$ and $B_3^2 = A_{332}\cup A_{4221}$ we see that $\mu f|^{-1}_{B_3^1}$ is ε-approximately absolutely continuous with respect to $\mu f|^{-1}_{B_2^1}$ and $\mu f|^{-1}_{B_3^2}$ is ε-approximately absolutely continuous with respect to $\mu f|^{-1}_{B_2^2}$. Now decompose $m_{522} = m_{5221} + m_{5222}$ with respect to $m_{322}+m_{4221}$, $m_{622} = m_{6221} + m_{6222}$ with respect to $m_{322}+m_{4221}+$

$+ m_{5221}$ and so forth, we obtain partitions $A_{522} = A_{5221} \cup A_{5222}$, $A_{622} = A_{6221} \cup \cup A_{6222}$, etc., such that $\mu f|^{-1}_{A_{n22i}}$ $(i = 1, 2)$ is ε-approximately equal to m_{n22i}. Let $B_3^n = A_{322} \cup A_{4221} \cup \ldots \cup A_{n+2,221}$. We see that $\mu f|^{-1}_{B_3^n}$ is ε-approximately absolutely continuous with respect to $\mu f|^{-1}_{B_2^n}$.

Continuing this procedure we can construct a double sequence

$$B_K^n = A_{K22\ldots 2} \cup A_{K+1,22\ldots 2:1} \cup \ldots \cup A_{n+K-1,22\ldots 2:1},$$

where the A's are disjoint Borel sets of $\mathscr{B}(X)$ satisfying the following properties:

(1) On each A, the function f is one-one.

(2) $A_{n+K,22\ldots 2} = A_{n+K,22\ldots 2:1} \cup A_{n+K,22\ldots 2:2}$ $(n > K)$.

(3) $\mu f|^{-1}_{A_{n+K,22\ldots 2:i}}$ $(i=1, 2)$ is ε-approximately equal to $m_{n+K,22\ldots 2:i}$ and $m_{n+K,22\ldots 2} = m_{n+K,22\ldots 21} + m_{n+K,22\ldots 22}$, where $m_{n+K,22\ldots 21}$ is topologically singular and $m_{n+K,22\ldots 22}$ is absolutely continuous with respect to $(m_{K,22\ldots 2} + m_{K+1,22\ldots 21} + + \ldots + m_{n+K-1,22\ldots 21})$.

(4) $\mu f|^{-1}_{B_K^n}$ is ε-approximately equal to $m_{K,22\ldots 2} + m_{K+1,22\ldots 21} + \ldots + + m_{n+K-1,22\ldots 21}$.

(5) $(m_{K+1,22\ldots 2:2} + m_{K+2,22\ldots 2:21} + \ldots + m_{n+K,22\ldots 2:21})$ is absolutely continuous with respect to $m_{K,22\ldots 2} + m_{K+1,22\ldots 2:1} + \ldots + m_{n+K-1,22\ldots 2:1})$.

(4) and (5) show that $\mu f|^{-1}_{B_{K+1}^n}$ is ε-approximately absolutely continuous with respect to $\mu f|^{-1}_{B_K^n}$.

We end the proof by putting $D_i = \bigcup_{n=1}^{\infty} B_i^n$.

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Abstract.

Many problems related to countably additive measures have been extended to finitely additive set functions. For example, the Hewitt-Yosida decomposition was generalized to s-bounded finitely additive set functions by using a variation of the Caratheodory process. Previously the author has obtained generalizations of the Hewitt-Yosida decomposition and the Lebesgue decomposition to finitely additive vector measures satisfying some continuity condition. The present work generalizes a Hellinger-Hahn decomposition of the domain X of a Borel function (see M. G. *Nadkarni, Hellinger-Hahn type decomposition of the domain of a Borel function,* Studia Math. **47** (1973), 51-62) *in the case where the measure μ on the Borel σ-algebra of X is only finitely additive.*

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