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Some results on Wiener-Hopf equations on finite intervals. (**)

1. - Introduction.

1.1. – We study integral equations of the I kind of the form

$$\int_E \varphi(y)\cdot K(x-y)\,\mathrm{d}y = f(x) \qquad x\in E\,, \tag{1.1}$$

where $K\in L^1_{\mathrm{loc}}(\boldsymbol{R})$, f is a tempered distribution and E is the half-line $\boldsymbol{R}^+$ or the interval $]0,1[$. We solve explicitly the equation (1.1) when $E=\boldsymbol{R}^+$, $K(x)=|x|^{-\alpha}$; $0<\alpha<1$, and f belongs to $H^r_s(\boldsymbol{R}^+)$ (a normed vector space that is isomorphic to the weighted Sobolev space $W^r_s(\boldsymbol{R}^+)$ when $r\in\overline{\boldsymbol{R}^+}$). When $E=\,]0,1[$ we shall prove that the kernel of the equation (1.1) is finite dimensional provided that K satisfies suitable hypotheses.

1.2. – There are some boundary value problems that can be reduced to an integral equation of the form (1.1). For example, consider the homogeneous equation

$$\operatorname{sgn}(x)\,|x|^p u_y - u_{xx} = 0 \qquad p>-1\,. \tag{1.2}$$

We are looking for a solution of (1.2) belonging to some weighted Sobolev space such that $u(x)=h(x)$ a.e. on $x\in\boldsymbol{R}^+$, where h is a given function. The

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problem was studied by Pagani in $[4]_1$, $[4]_2$ using an integral transformation technique, but it is possible to show that the problem can be reduced to solve the integral equation

$$\int_0^{+\infty} \frac{\varphi(y)}{|x-y|^\alpha}\,dy = f(x) \qquad x \in \boldsymbol{R}^+ ,$$

where $\alpha = 1 - 1/(p+2)$ and f is a known function depending on h. The unknown φ is the trace of $u_x(x, y)$ on $\{x = 0, y > 0\}$. Other boundary value problems bring to an equation of type (1.1) where E is a finite interval; for example we could get a solution $u \in H^m(\boldsymbol{R}^2_+)$ of the Helmholtz equation $u_{xx} + u_{yy} - u = 0$ when the trace of the solution u and of its derivative u_y are given respectively in $J_1 = \{x \mid 0 < x < 1\}$ and $J_2 = \{x \mid x > 0, x < 1\}$, solving an integral equation of type (1.1) where $E =]0, 1[$ and the convolution kernel $K(x)$ is the modified Bessel function of III kind [**8**]. One can find some results about equations on finite intervals in [**2**] and [**7**].

2. - The convolution equation when $E = \boldsymbol{R}^+$ and $K(x) = |x|^{-1+\alpha}$, $0 < \alpha < 1$.

2.1. - Consider the convolution equation of the I kind

$$\int_0^{+\infty} \frac{\varphi(y)}{|x-y|^{1-\alpha}}\,dy = f(x)\,, \qquad x \in \boldsymbol{R}^+, \qquad 0 < \alpha < 1\,. \tag{2.1}$$

Here (2.1) means exactly

$$\operatorname{supp}\varphi \subseteqq [0, +\infty[\,, \qquad \operatorname{supp}(f - |x|^{-1+\alpha} * \varphi) \subseteqq]-\infty, 0]\,. \tag{2.2}$$

We have the following

Theorem 2.1. *Let $f \in H^s_{s+q-\alpha}(\boldsymbol{R}^+)$: $s \in \boldsymbol{R}$, $0 < \alpha < 1$, $\alpha < q < 1$. Then there exists one and only one solution $\varphi \in H^{s-\alpha}_{s+q-\alpha}(\boldsymbol{R}^+)$ of the equation* (2.1). *φ satisfies the inequality $\|\varphi\|_{H^{s-\alpha}_{s+q-\alpha}} \leqq C\|f\|_{H^s_{s+q-\alpha}}$, where C depends only on α.*

Remarks. 1) Let

$$\mathscr{S}(\boldsymbol{R}) = \{u \in C^\infty(\boldsymbol{R}) \mid \forall h, k \in \boldsymbol{N}^0,\ p_{hk}(u) = \sup_{x \in \boldsymbol{R}} |x^h u^{(k)}(x)| < +\infty\}$$

topologized by the family of seminorms $\{p_{hk}\}$. For each $\xi \in \boldsymbol{R}$ let

$$\mathscr{S}_\xi(\boldsymbol{R}^+) = \{u \in C^\infty(]0, +\infty[) \mid \forall m, n \in \boldsymbol{N}^0,\ q^{(\xi)}_{mn}(u) = \sup_{x \in \boldsymbol{R}^+} |\log^m x \cdot x^{n-\xi+1} u^{(n)}(x)| < +\infty\}$$

topologized by the family of seminorms $\{q^{(\xi)}_{mn}\}$. Let $\mathscr{S}'(\boldsymbol{R})$ and $\mathscr{S}'_\xi(\boldsymbol{R}^+)$ the dual spaces of $\mathscr{S}(\boldsymbol{R})$ and $\mathscr{S}_\xi(\boldsymbol{R}^+)$ respectively. For each $g \in \mathscr{S}_{-\xi+1}$ we define $\tau_{-\xi+1} g \colon \boldsymbol{R} \ni t \mapsto \exp[\xi t] g(\exp[t])$. $\tau_{-\xi+1}$ is an algebraic and topological isomorphism of $\mathscr{S}_{-\xi+1}$ onto $\mathscr{S}$ (the space of quickly decreasing functions). Let

$$\forall g \in \mathscr{S}_{-\xi+1}, \qquad \tilde{g}_\xi(\eta) = (\mathscr{M}_\xi g)(\eta) = \int_0^{+\infty} x^{\xi-1-i\eta} g(x)\, dx = (\tau_{-\xi+1} g)^\wedge(\eta)$$

the Mellin transform of g. Obviously $\mathscr{M}_\xi$ is an algebraic and topological isomorphism of $\mathscr{S}_{-\xi+1}$ onto $\mathscr{S}$.

For each $S \in \mathscr{S}'_\xi$ we define $\sigma_\xi S \colon \mathscr{S} \ni f \mapsto S(\tau_\xi^{-1} f)$; σ_ξ is an algebraic and topological isomorphism of $\mathscr{S}'_\xi$ onto $\mathscr{S}'$ (the space of tempered distributions). $\mathscr{S}_{-\xi+1} \subseteq \mathscr{S}'_\xi$ with continuous injection. It is quite obvious that σ_ξ is the continuous extension of $\tau_{-\xi+1}$; we can also extend the Mellin transform to the whole space $\mathscr{S}'_\xi$ in this way: $\forall S \in \mathscr{S}'_\xi$: $\tilde{S}_\xi = (\mathscr{M}_\xi S) = (\sigma_\xi S)^\wedge$.

This extension is an algebraic and topological isomorphism of $\mathscr{S}'_\xi$ onto $\mathscr{S}'$.

Now let $H^s_r(\boldsymbol{R}^+) = \{u \in \mathscr{S}'_{s-r} \mid (1 + x^2)^{r/2} \tilde{u}_{s-r}(x) \in L^2(\boldsymbol{R})\}$ equipped by the norm

$$\|u\|_{H^s_r} = \Big(\int_{\boldsymbol{R}} (1 + x^2)^{r/2} |\tilde{u}_{s-r}(x)|^2\, dx \Big)^{\frac{1}{2}}.$$

It is easy to see that $\mathscr{S}'_{s-r}(\boldsymbol{R}^+) \supseteq H^s_r(\boldsymbol{R}^+) \supseteq \mathscr{S}_{(s-r)+1}(\boldsymbol{R}^+) \supseteq \mathscr{D}(\boldsymbol{R}^+)$ so that $\overline{\mathscr{S}_{-(s-r)+1}(\boldsymbol{R}^+)} = H^s_r(\boldsymbol{R}^+)$. Moreover $\forall \xi, t \in \boldsymbol{R}$, $\sigma_\xi \colon u \mapsto \sigma_\xi u$ is an isometry between $H^t_{\xi+t}$ and H^t. For other information about this argument see [**1**].

Let $0 < \alpha < 1$, $\alpha < q < 1$ and $s \in \boldsymbol{R}$. Let $\mathscr{S}_{-q+1}$ and $\mathscr{S}_{-(q-\alpha)+1}$ topologized respectively by the norm $\|\cdot\|_{H^{s-\alpha}_{s+q-\alpha}}$ and $\|\cdot\|_{H^s_{s+q-\alpha}}$. It is easy to see that the operator

$$T \colon \mathscr{S}_{-q+1}(\boldsymbol{R}^+) \ni \varphi \mapsto \int_0^{+\infty} \frac{\varphi(y)}{|x-y|^{1-\alpha}}\, dy \in \mathscr{S}_{-(q-\alpha)+1}\ (\boldsymbol{R}^+)$$

is continuous. Therefore the operator

$$(2.3) \qquad |x|^{-1+\alpha} * \colon \overline{\mathscr{S}_{-q+1}} = H^{s-\alpha}_{s+q-\alpha} \ni \varphi \mapsto |x|^{-1+\alpha} * \varphi \in H^s_{s+q-\alpha} = \overline{\mathscr{S}_{-(q-\alpha)+1}}$$

is the continuous extension of T to $\overline{\mathscr{S}_{-q+1}}$.

2). Let $K(x) = \exp[x(q-\alpha)]/|\exp[x]-1|^{1-\alpha}$; then $K \in L^1(\boldsymbol{R})$ and

$$(2.4)\quad \hat{K}(\xi) = \int_{\boldsymbol{R}} \exp[-i\xi x]K(x)\,\mathrm{d}x = \Gamma(\alpha)\frac{\Gamma(q-\alpha-i\xi)}{\Gamma(q-i\xi)}\left\{1+\frac{\sin\pi(q-\alpha-i\xi)}{\sin\pi(q-i\xi)}\right\}.$$

By routine computations we have

$$(2.5)\qquad \forall\,\xi\in\boldsymbol{R},\ \hat{K}(\xi)\neq 0;\ C_1 \leqq |\hat{K}(\xi)|(1+\xi^2)^{\alpha/2} \leqq C_2\,, \qquad C_1,\ C_2\in\boldsymbol{R}^+.$$

The following lemmas will be used in the proof of Theorem 2.1.

Lemma 2.1. *Let* $s\in\boldsymbol{R}$, K *the same as in* (2.4) *and* $\psi\in H^s(\boldsymbol{R})$. *Then the convolution equation* $\psi * K = 0$ *has only the trivial solution* $\psi = 0$.

Lemma 2.2. *Let* $s\in\boldsymbol{R}$, K *the same as in* (2.4) *and* $g\in H^s(\boldsymbol{R})$ *given. Then the convolution equation*

$$(2.6)\qquad \psi * K = g$$

has one and only one solution $\psi\in H^{s-\alpha}(\boldsymbol{R})$, *given by* $\hat{\psi} = \hat{g}/\hat{K}$.

Proof. By Fourier transform on (2.6) and from (2.5) we have

$$(2.7)\qquad \|\psi\|^2_{H^{s-\alpha}} = \int_{\boldsymbol{R}} (1+\xi^2)^{(s-\alpha)/2}|\hat{\psi}(\xi)|^2\,\mathrm{d}\xi = \int_{\boldsymbol{R}} (1+\xi^2)^{(s-\alpha)/2}\left|\frac{\hat{g}(\xi)}{\hat{K}(\xi)}\right|^2\mathrm{d}\xi$$

$$\leqq \frac{1}{C_1}\int_{\boldsymbol{R}} (1+\xi^2)^{s/2}|\hat{g}(\xi)|^2\,\mathrm{d}\xi = \frac{1}{C_1}\|g\|_{H^s} < +\infty\,.$$

Proof of Theorem 2.1. Let us consider equation (2.6); by Lemma 2.2 the unique solution ψ satisfies the inequality

$$(2.8)\qquad \|\psi\|_{H^{s-\alpha}} \leqq \frac{1}{C_1}\|g\|_{H^s}\,.$$

Let $\varphi = \sigma_q^{-1}\psi$ and $f = \sigma_{q-\alpha}^{-1}g$. As $\hat{\psi} = (\sigma_q\varphi)\hat{} = \tilde{\varphi}_q$ and $\hat{g} = (\sigma_{q-\alpha}f)\hat{} = \tilde{f}_{q-\alpha}$, from (2.7) we have $\varphi\in H^{s-\alpha}_{s+q-\alpha}$ and $f\in H^s_{s+q-\alpha}$.

The equation (2.6) becomes

$$(2.9)\qquad (\sigma_q\varphi)*K = \sigma_{q-\alpha}f\,,\qquad \sigma_{q-\alpha}^{-1}[(\sigma_q\varphi)*K] = f\,,\qquad A\varphi = f\,,$$

where A is a continuous, with a continuous inverse and one-to-one operator from $H^{s-\alpha}_{s+q-\alpha}$ onto $H^{s}_{s+q-\alpha}$ (in fact $(K*)$ is a continuous, with a continuous inverse and one-to-one operator from $H^{s-\alpha}$ onto H^{s}). We can conclude that the functional equation (2.9) where f is given in $H^{s}_{s+q-\alpha}(\boldsymbol{R}^+)$ has one and only one solution $\varphi \in H^{s-\alpha}_{s+q-\alpha}(\boldsymbol{R}^+)$ given by $\tilde{\varphi}_q = \tilde{f}_{q-\alpha}/\hat{K}$ and which satisfies the inequality $\|\varphi\|_{H^{s-\alpha}_{s+q-\alpha}} \leqq (1/C_1)\, \|f\|_{H^{s}_{s+q-\alpha}}$.

We now show that A is the convolution operator (2.3).

We know that there exists a sequence $\{g_m\}_{m\in \boldsymbol{N}^0}$ in $\mathscr{S}(\boldsymbol{R})$ converging to g in the norm of $H^s(\boldsymbol{R})$.

For each m consider the equation $(q \in \boldsymbol{R})$

$$(2.10) \qquad (K * \psi_m)(a) = g_m(a)\,, \qquad a \in \boldsymbol{R}\,.$$

The solution ψ_m belongs to $\mathscr{S}(\boldsymbol{R})$ and $\{\psi_m\}_{m\in \boldsymbol{N}^0}$ converges to ψ in the norm of $H^{s-\alpha}(\boldsymbol{R})$. For each m there exist two functions: $\varphi_m \in \mathscr{S}_{-q+1}(\boldsymbol{R}^+)$ and $f_m \in \mathscr{S}_{-(q-\alpha)+1}(\boldsymbol{R}^+)$ such that $\sigma_q \varphi_m = \psi_m$ and $\sigma_{q-\alpha} f_m = g_m$. Moreover $\{\varphi_m\}_{m\in \boldsymbol{N}^0}$ converges to $\varphi = \sigma_q^{-1}\psi$ in the norm of $H^{s-\alpha}_{s+q-\alpha}$ and $\{f_m\}_{m\in \boldsymbol{N}^0}$ converges to $f = \sigma_{q-\alpha}^{-1} g$ in the norm of $H^{s}_{s+q-\alpha}$.

Now we have $(\forall m \in \boldsymbol{N}^0,\ a \in \boldsymbol{R},\ x \in \boldsymbol{R}^+)$

$$(K * \psi_m)(a) = g_m(a) \Leftrightarrow (K * \sigma_q \varphi_m)(a) = (\sigma_{q-\alpha} f_m)(a) \Leftrightarrow$$

$$\int_{\boldsymbol{R}} \frac{\exp[(a-b)(q-\alpha)]}{|\exp[a-b]-1|^{1-\alpha}} \exp[qb]\varphi_m(\exp[b])\, db = \exp[(q-\alpha)a] f_m(\exp[a]) \Leftrightarrow$$

$$\int_0^{+\infty} \frac{1}{|x-y|^{1-\alpha}}\, \varphi_m(y)\, dy = f_m(x)\,.$$

So the operator $(1/|x|^{1-\alpha}*)$ is the restriction to $\mathscr{S}_{q-1}(\boldsymbol{R}^+)$ of the continuous operator A. The proof is now complete.

2.2. – The equation (2.1) can be solved in the Sobolev space $H^s(\boldsymbol{R})$. We have the following

Theorem 2.2. *Let $f \in H^s(\boldsymbol{R})$ and $s - \alpha/2 > -\frac{1}{2}$. Then there esists a solution φ of* (2.1) *such that*

$$(2.11) \qquad \varphi \in H^{s-\alpha}(\boldsymbol{R})\,, \qquad \operatorname{supp} \varphi \subseteqq [0, +\infty[\,,$$

if the following conditions are satisfied

$$(2.12) \qquad -\tfrac{1}{2} < s - \alpha/2 < \tfrac{1}{2}\,;$$

there exists a positive integer n such that

$$(2.13)\qquad n-\tfrac{1}{2}<s-\alpha/2<n+\tfrac{1}{2}\quad \textit{and}\quad g^{(k)}(0)=0\,,\quad (k=0,\dots,n-1)\,;$$

$$(2.14)\qquad s-\alpha/2=\tfrac{1}{2}\ \textit{and for some}\ \varepsilon>0\int\limits_0^\varepsilon |g(x)|^2\frac{dx}{x}<+\infty\,;$$

$$(2.15)\qquad s-\alpha/2=n+\tfrac{1}{2}\ \textit{and}\ g^{(k)}(0)=0\ \ (k=0,\dots,n-1)\ \textit{and for some}\ \varepsilon>0$$

$$\int\limits_0^\varepsilon |g^{(n)}(x)|^2\frac{dx}{x}<+\infty\,.$$

Here g is the distribution defined by

$$(2.16)\qquad \hat{g}=-\frac{\hat{f}}{K_+}$$

and K_+ is defined in this way

$$K(x)=|x|^{-1+\alpha}\,;\qquad \hat{K}(\xi)=c_\alpha|\xi|^{-\alpha}\,,\qquad c_\alpha=(2\pi)^{\frac{1}{2}}\,2^{\alpha-\frac{1}{2}}\,\frac{\Gamma(\alpha/2)}{\Gamma((1-\alpha)/2)}$$

$$\hat{K}(\zeta)=K_+(\zeta)K_-(\zeta)\,,\qquad K_+(\zeta)=\sqrt{c_\alpha}\,\zeta^{-\alpha/2}\,;\qquad K_-(\zeta)=\sqrt{c_\alpha}\,\zeta^{-\alpha/2}\,;\qquad \zeta=\xi+i\eta\,.$$

K_+ and K_- are holomorphic respectively in $\eta>0$ and $\eta<0$, and

$$K_+(\xi)=\sqrt{c_\alpha}\begin{cases}\exp[-i\pi\alpha/2]\,|\xi|^{-\alpha/2}\,, & \xi<0\,;\\ \xi^{-\alpha/2}\,, & \xi>0\,;\end{cases}$$

$$K_-(\xi)=\sqrt{c_\alpha}\begin{cases}\exp[i\pi\alpha/2]\,|\xi|^{-\alpha/2}\,, & \xi<0\,;\\ \xi^{-\alpha/2}\,, & \xi>0\,.\end{cases}$$

Moreover if $s-\alpha/2-\frac{1}{2}$ is not an integer and (2.12) or (2.13) holds, we get

$$(2.17)\qquad \|\varphi\|_{H^{s-\alpha}(R)}\leq\frac{C_1}{c_\alpha}\,\|f\|_{H^s(R^+)}\,.$$

If $s - \alpha/2 - \frac{1}{2}$ is an integer and (2.14) or (2.15) holds, we get

$$\|\varphi\|_{H^{s-\alpha}(R)} \leqq C_2 \left\{ \frac{1}{c_\alpha} \|f\|_{H^s(R^+)} + \left[\int_0^{+\infty} |g^{(n)}(x)|^2 \frac{dx}{x} \right]^{\frac{1}{2}} \right\}, \tag{2.18}$$

where C_1 and C_2 depend only on s.

Remark. The theorem is a slight modification of the Theorem 3.1 in [**5**]. Here the kernel $K(x) = 1/|x|^{1-\alpha}$ is not in $L^1(\boldsymbol{R})$ and its Fourier transform $\hat{K}(\xi)$ can be factorized in the product of two functions $K_+(\zeta)$ and $K_-(\zeta)$ such that $K_+(\zeta)$ and $K_-(\zeta)$, ($\zeta = \xi + i\eta$) are holomorphic respectively in the upper half-plane $\eta > 0$ and in lower half-plane $\eta < 0$, but are not continuous in the closure of these half-planes. Anyway the line of the proof works again with some observations based on very simple inequalities.

3. - The convolution equation when $E =]0,1[$.

Consider equation

$$\int_0^1 \varphi(x) K(x-y)\, dy = f(x) \qquad (0 < x < 1)\,, \tag{3.1}$$

where $K \in L^1(\boldsymbol{R})$, $\operatorname{supp} K \subseteqq [-1, 1]$ and φ and f are tempered distributions. The (3.1) means exactly

$$\{\operatorname{supp} \varphi \subseteqq [0, 1];\ \operatorname{supp}(K * \varphi - f) \subseteqq]-\infty, 0] \cup [1, +\infty[\}\,. \tag{3.2}$$

We'll show under suitable hypotheses on K that the operator

$$(K *): H^{s-r}([0, 1]) \ni \varphi \mapsto K * \varphi \in H^s(]0, 1[) \tag{3.3}$$

has closed range and its kernel is finite dimensional. Here r, s are convenient numbers that will be precised below.

For $H^z([0, 1])$, $z \in \boldsymbol{R}$ we mean the closed linear subspace of distributions $\gamma \in H^z(\boldsymbol{R})$ such that $\operatorname{supp} \gamma \subseteqq [0, 1]$, equipped with the norm $\|\cdot\|_{H^z(R)}$. $H^z([0, 1])$ is an Hilbert space [**9**].

Precisely we shall prove the following

Theorem 3.1. *Let $\hat{K}(\xi)$ be the Fourier transform of K. Suppose there exist two functions $K_+(\xi)$, $K_-(\xi)$ such that:*

(i) $\hat{K}(\xi) = K_+(\xi)$, $K_-(\xi)$ *and* $\zeta = \xi + i\eta \mapsto K_+(\zeta)$ *is holomorphic in* $\eta > 0$ *and continuous in* $\eta \geqq 0$; $\zeta = \xi + i\eta \mapsto K_-(\zeta)$ *is holomorphic in* $\eta < 0$ *and continuous in* $\eta \leqq 0$.

(ii) *Suppose there exists* p, q, C_1, C_2, C_3, C_4, *real constants, such that*

$$(3.4)\quad \begin{cases} 0 < C_1 \leqq (1 + |\zeta|^2)^{p/2} |K_+(\zeta)| \leqq C_2\,, & \forall \zeta \text{ in } \eta \geqq 0\,; \\ 0 < C_3 \leqq (1 + |\zeta|^2)^{q/2} |K_-(\zeta)| \leqq C_4\,, & \forall \zeta \text{ in } \eta \leqq 0\,;\ p + q > 0\,. \end{cases}$$

(iii) *Let* $\lambda > 0$, *arbitrarily small, be fixed*; *let* $0 < \beta$, *arbitrarily large, be fixed*

$$J_1 = \{(\xi, \eta) \,|\, \eta = \omega\xi,\ 1 - \lambda < \omega < 1 + \lambda;\ \xi^2 + \eta^2 > \beta^2\}\,.$$

Suppose there exists $C_5 > 0$, $\delta > 0$, $\varepsilon > 0$ *such that*

$$(3.5)\qquad \forall (\xi, \eta) \in J_1 \,|\hat{K}(\xi) - \hat{K}(\eta)| \leqq C_5 \frac{|\xi - \eta|^{\delta} (1 + \xi^2 + \eta^2)^{(r-\varepsilon)/2}}{(1 + \xi^2)^{r/2} (1 + \eta^2)^{r/2}}\,,$$

where $r = p + q$.

(iv) *Let* $f \in H^s(\boldsymbol{R})$, $-\frac{1}{2} < s - p < \frac{1}{2}$. *Then every solution* φ *of* (3.1) *satisfies the following inequality*

$$(3.6)\qquad \|\varphi\|_{H^{s-r}([0,1])} \leqq C\{\|f\|_{H^s(]0,1[)} + \|\varphi\|_{H^{s-r-\alpha}([0,1])}\}\,,$$

where $\alpha = \min(r, \varepsilon)$ *and* C *is a positive constant not depending on* φ.

Remarks.

1). By (3.4), $|\hat{K}(\xi)| \leqq (C_2 C_4)/(1 + \xi^2)^{r/2}$ and it is easy to see that

$$\forall \xi, \eta \quad |\hat{K}(\eta) - \hat{K}(\xi)| \leqq C_6 \frac{(1 + \xi^2 + \eta^2)^{r/2}}{(1 + \xi^2)^{r/2} (1 + \eta^2)^{r/2}}\,.$$

2). $\forall n \in \boldsymbol{N}_0$: $x^n K(x) \in L^1(\boldsymbol{R})$. Then $\hat{K}(\xi) \in C^\infty(\boldsymbol{R}) \wedge \forall n$: $\hat{K}^{(n)}(\xi) \to 0$, when $\xi \to \infty$.

3). If there exists $\varepsilon > 0$ and $C_7 > 0$ such that

$$(3.7)\qquad |\hat{K}'(\xi)| \leqq \frac{C_7}{(1 + \xi^2)^{(r+\varepsilon)/2}}\,,$$

then (3.5) holds. In fact let J_1 the set in (iii) of Theorem 3.1; we can also write

$$J_1 = \left\{\xi, \omega\xi \,|\, 1-\lambda \leqq \omega \leqq 1+\lambda;\ |\xi| > \left(\frac{\beta^2}{1+\omega^2}\right)^{\frac{1}{2}}\right\}.$$

We have equivalently

$$(3.8)\qquad \forall \xi, \eta \in J_1\colon |\hat{K}(\eta) - \hat{K}(\xi)| = |\eta - \xi| \cdot |\hat{K}'(\bar{\xi})|, \qquad \text{where } \bar{\xi} = \bar{\xi}(\xi, \omega),$$

$$(3.8)\qquad \forall \xi, \omega\xi \in J_1\colon |\hat{K}(\omega\xi) - \hat{K}(\zeta)| = |\omega\xi - \xi| \cdot |\hat{K}'(\tilde{\omega}\xi)|, \quad \text{where } \tilde{\omega} = \tilde{\omega}(\xi, \omega).$$

Obviously: $1-\lambda < \tilde{\omega} < 1+\lambda$. By (3.7) we have

$$\forall \xi, \omega\xi \in J_1\colon\ |\hat{K}'(\tilde{\omega}\xi)| \leqq \frac{C_7}{(1+\tilde{\omega}^2\xi^2)^{(r+\varepsilon)/2}} \leqq \frac{1}{(1-\lambda)^2}\ \frac{C_7}{(1+\xi^2)^{(r+\varepsilon)/2}}.$$

Moreover

$$\forall \omega\colon \frac{1}{(1+\xi^2)^{\varepsilon/2}} \leqq (1+\omega^2)^{\varepsilon/2}\, \frac{1}{(1+\xi^2+\omega^2\xi^2)^{\varepsilon/2}}$$

and it is trivial that

$$\forall \omega\colon 1 \leqq \left\{\frac{1+\xi^2+\omega^2\xi^2}{1+\omega^2\xi^2}\right\}^{r/2}.$$

Thus

$$\forall \xi, \omega\xi \in J_1\colon |\hat{K}'(\tilde{\omega}\xi)| \leqq C_8\, \frac{(1+\omega^2\xi^2+\xi^2)^{(r-\varepsilon)/2}}{(1+\xi^2)^{r/2}(1+\omega^2\xi^2)^{r/2}}$$

and by (3.8) or (3.8)′

$$\forall \xi, \eta \in J_1\colon |\hat{K}(\eta) - \hat{K}(\xi)| \leqq C_9\, \frac{|\xi-\eta|\,(1+\xi^2+\eta^2)^{(r-\varepsilon)/2}}{(1+\xi^2)^{r/2}(1+\eta^2)^{r/2}}.$$

4). In $\boldsymbol{R}^2 \setminus J_1$ we have

$$(3.9)\qquad \forall \sigma, 0<\sigma\leqq 1,\ \forall \varepsilon > 0\colon |\hat{K}(\eta) - \hat{K}(\xi)| \leqq C_{10}\, \frac{|\eta-\xi|^{\sigma}(1+\xi^2+\eta^2)^{(r-\varepsilon)/2}}{(1+\eta^2)^{r/2}(1+\xi^2)^{r/2}}.$$

This follows easily from the arguments above.

5). For a result in [**6**] the (3.6) assures that the operator (3.3) has closed range and its kernel is finite dimensional.

Proof of Theorem 3.1. Suppose φ is a solution of (3.1). Let $\frac{1}{2} > h > 0$ and $\psi_1, \psi_2 \in C_0^\infty$ such that

$$\text{(3.10)} \qquad \operatorname{supp} \psi_1 \subseteqq [-h, 2h], \qquad \operatorname{supp} \psi_2 \subseteqq [h, 1+h],$$

$$\forall x \in [0, 1]: \psi_1(x) + \psi_2(x) = 1 .$$

We write (3.1) in this way

$$\int_0^1 \varphi(y) K(x-y)\, dy - f(x) = \theta(x) .$$

By (3.2) $\operatorname{supp} \theta \subseteqq]-\infty, 0] \cup [1, +\infty[$.

We have

$$\int_0^1 \varphi(y) K(x-y)\, dy - f(x) = \theta(x) ,$$

$$\text{(3.11)} \qquad \psi_j(x) \int_0^1 \varphi(y) K(x-y)\, dy - \int_0^1 \psi_j(y)\varphi(y) K(x-y)\, dy +$$

$$+ \int_0^1 \psi_j(y)\varphi(y) K(x-y)\, dy - \psi_j(x) f(x) = \psi_j(x)\theta(x) ,$$

where $j = 1, 2$. For $j = 1$ we write the (3.11) in this way

$$\text{(3.12)} \qquad \int_0^1 \varphi_1(y) K(x-y)\, dy - F_1(x) = \theta_1(x) ,$$

where

$$\varphi_1 = \psi_1 \varphi \quad \operatorname{supp} \varphi_1 \subseteqq [0, 2h] , \quad F_1 = \psi_1 f + \mathrm{Kom} , \quad \theta_1 = \psi_1 \theta , \quad \operatorname{supp} \theta_1 \subseteqq]-\infty, 0] ,$$

$$\mathrm{Kom}(x) = \int_0^1 [\psi_1(y) - \psi_1(x)]\varphi(y) K(x-y)\, dy = \{K * (\psi_1\varphi) - \psi_1(K * \varphi)\}(x) .$$

For $j = 2$ we have the analogous formula of (3.12).

For a result in [5], if $-\frac{1}{2} < s - p < \frac{1}{2}$, the following inequality holds $\|\varphi_1\|_{H^{s-r}([0,1])} \leqq C_{11} \|F_1\|_{H^s(R^+)}$, where $r = p + q$. Now

$$\|F_1\|_{H^s(R^+)} \leqq \Big\| \int_0^1 [\psi_1(y) - \psi_1(x)]\varphi(y) K(x-y)\, \mathrm{d}y \Big\|_{H^s(R^+)} + \|\psi_1 f\|_{H^s(R^+)} .$$

By hypotheses supp $\psi_1 \subseteqq [-h, 2h]$, $\psi_1 \in C^\infty([0, 2h])$; then the moltiplication for ψ_1 is a continuous operator in $H^s(]0, 2h[)$ [3]. We get from this and other easy considerations $\|\psi_1 f\|_{H^s(R^+)} \leqq C_{12} \|f\|_{H^s(]0,1[)}$. Consider now

$$\begin{aligned}
\|\mathrm{Kom}\,(x)\|^2_{H^s(R^+)} &\leqq \int_R (1+\xi^2)^s |\widehat{\mathrm{Kom}}\,(\xi)|^2\, \mathrm{d}\xi \\
&\leqq \int_R (1+\xi^2)^s \Big| \int_R \hat{\psi}_1(\xi-\eta)\hat{\varphi}(\eta)[\hat{K}(\xi) - \hat{K}(\eta)]\, \mathrm{d}\eta \Big|^2 \mathrm{d}\xi \\
&\leqq \int_R \mathrm{d}\xi (1+\xi^2)^s \Big\{ \int_R |\hat{\varphi}(\eta)|^2 |\hat{K}(\xi) - \hat{K}(\eta)|^2 |\hat{\psi}_1(\xi-\eta)\, \mathrm{d}\eta \times \\
&\qquad\qquad \times \int_R |\hat{\psi}_1(\xi-\mu)|\, \mathrm{d}\mu \Big\} = B_1 .
\end{aligned}$$

As

$$\psi_1 \in C_0^\infty(\boldsymbol{R}) \Rightarrow \hat{\psi}_1 \in \mathscr{S}(\boldsymbol{R}) \Rightarrow \forall \xi \in \boldsymbol{R} \int_R |\hat{\psi}_1(\xi-\mu)|\, \mathrm{d}\mu \leqq C_{13} ,$$

then from (3.5) and (3.9) $\forall \sigma$ $(0 < \sigma \leqq 1)$, $\forall m \in \boldsymbol{N}^0$:

$$\begin{aligned}
B_1 &\leqq C_{13} \int_{R^2} \mathrm{d}\xi\, \mathrm{d}\eta\, |\hat{\varphi}(\eta)|^2 (1+\xi^2)^s |\hat{\psi}_1(\xi-\eta)| \cdot |\hat{K}(\xi) - \hat{K}(\eta)|^2 \\
&\leqq C_{13} \Big\{ \int_{J_1} + \int_{R^2 \setminus J_1} \Big\} \\
&\leqq C_{14} \Bigg\{ \int_{J_1} \mathrm{d}\xi\, \mathrm{d}\eta\, \frac{|\hat{\varphi}(\eta)|^2 (1+\xi^2)^s |\xi-\eta|^{2\delta} (1+\xi^2+\eta^2)^{r-\varepsilon}}{(1+\xi^2)^r (1+\eta^2)^r [1+(\xi-\eta)^2]^m} + \\
&\quad + \int_{R^2 \setminus J_1} \mathrm{d}\xi\, \mathrm{d}\eta\, \frac{|\hat{\varphi}(\eta)|^2 (1+\xi^2)^s |\xi-\eta|^{2\delta} (1+\xi^2+\eta^2)^{r-\varepsilon}}{(1+\xi^2)^r (1+\eta^2)^r [1+(\xi-\eta)^2]^m} \Bigg\} = B_2 .
\end{aligned}$$

Here obviously C_{14} depends only on s, r

$$\begin{aligned}
B_2 \leqq C_{15} \int_{R^2} &\mathrm{d}\xi\, \mathrm{d}\eta\, |\hat{\varphi}(\eta)|^2 (1+\xi^2)^{s-r} (1+\eta^2)^{-r} \times \\
&\times (1+\xi^2+\eta^2)^{r-\varepsilon} \frac{[|\xi-\eta|^{2\sigma} + |\xi-\eta|^{2\delta}]}{[1+(\xi-\eta)^2]^m} = B_3 .
\end{aligned}$$

Now [**5**]

$$\forall t, \xi, \eta \in \boldsymbol{R}: (1+\xi^2)^t \leqq (1+\eta^2)^t \{\tfrac{1}{2}|\xi-\eta| + [1+\tfrac{1}{4}(\xi-\eta)^2]^{\frac{1}{2}}\}^{2|t|};$$

hence, putting $\xi-\eta=\tau$, we have

$$B_3 \leqq C_{15} \int_{\boldsymbol{R}} d\eta\, |\hat{\varphi}(\eta)|^2 (1+\eta^2)^{s-2r} \int_{\boldsymbol{R}} d\tau [\,|\tau|^{2\sigma} + |\tau|^{2\delta}] \times$$

$$\times [1+(\tau+\eta)^2+\eta^2]^{r-\varepsilon} \frac{\{\frac{1}{2}|\tau| + (1+\tau^2/4)^{\frac{1}{2}}\}^{2|s-r|}}{[1+\tau^2]^m} = B_4 .$$

If $r-\varepsilon \leqq 0$ we get at once $B_4 \leqq C_{16} \|\varphi\|^2_{H^{s-2r}([0,1])}$.

If $r-\varepsilon > 0$, by the trivial inequality $1+(\tau+\eta)^2+\eta^2 \leqq 3(1+\tau^2)(1+\eta^2)$, we get $B_4 \leqq C_{17} \|\varphi\|^2_{H^{s-r-\varepsilon}([0,1])}$. Finally we have

$$\|\varphi_1\|_{H^{s-r}([0,1])} \leqq C_{18} \{\|f\|_{H^s(]0,1[)} + \|\varphi\|_{H^{s-r-\alpha}([0,1])}\} ,$$

where $\alpha = \min(r, \varepsilon)$. In an analogous way we can get

$$\|\varphi_2\|_{H^{s-r}([0,1])} \leqq C_{19} \{\|f\|_{H^s(]0,1[)} + \|\varphi\|_{H^{s-r-\alpha}([0,1])}\} .$$

But $\varphi = \varphi_1 + \varphi_2$; then the (3.6) follows.

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Summary.

In this work we consider integral equations of Wiener-Hopf type on finite intervals. We prove Peetre inequality for solutions in Sobolev spaces H^s. We also solve a Wiener-Hopf equation on the half-line in the weighted spaces $H^s_r(\mathbf{R}^+)$ when the convolution kernel is $|x|^{-\alpha}$, $0<\alpha<1$.

* * *