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A class of non-hyponormal operators on Hilbert spaces (**)

1. – The purpose of this note is to introduce a new class of operators which extends the class of hyponormal operators and which is independent of the class of paranormal operators.

Let T be an operator on a Hilbert space H. Let $\sigma(T)$ denote the spectrum of T. T is called *hyponormal* if $T^*T \geqslant TT^*$; *paranormal* or an operator of class (N) if $\|T^2x\|\,\|x\| \geqslant \|Tx\|^2$ for all x in H; *k-paranormal* or an operator of class $(N; k)$ if $\|T^kx\|\,\|x\|^{k-1} \geqslant \|Tx\|^k$ for all x in H. If $(T - zI)^{-1}$ is normaloid for all $z \notin \sigma(T)$, then T is said to satisfy the growth condition (G_1).

We know that for a hyponormal operator T, $\|T^kx\|\,\|x\|^{k-1} \geqslant \|Tx\|^k$, for all x in H [**3**], where $k \geqslant 2$ is an integer and $\|Tx\| \geqslant \|T^*x\|$. It follows that $\|T^kx\| \cdot \|x\|^{k-1} \geqslant \|T^*x\|^k$ for all x in H. This motivates us to introduce the more general concept of a k-hyponormal operator or an operator class $(H; k)$ $(k \geqslant 2)$ defined as follows. An operator T is called *k-hyponormal* (or an operator of class $(H; k)$) if $\|T^kx\|\,\|x\|^{k-1} \geqslant \|T^*x\|^k$ for all x in H. Clearly $(H; 1)$ is the class of hypo-normal operators. In the present Note, we shall study the class $(H; 2)$ which turns out to be independent of the class (N).

2. – Firstly, we characterize the class $(H; 2)$ in

THEOREM 1. *An operator T is of class $(H; 2)$ if and only if* $T^{*2}T^2 - 2z(TT^*) + z^2 \geqslant 0$ $(z > 0)$.

PROOF. Since for positive real numbers b, c, $z^2 - 2bz + c \geqslant 0$ $(z > 0)$ if and only if $b^2 \leqslant c$, it follows that for all x in H, $\|T^2x\|\,\|x\| \geqslant \|T^*x\|^2$ if and only if $\|T^2x\|^2 - 2z\|T^*x\|^2 + z^2\|x\|^2 \geqslant 0$ $(z > 0)$. This proves the result.

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To determine the position of our class among the known classes, we prove Theorems 2 and 3.

Theorem 2. *Every operator of class $(H;2)$ is of class $(N;3)$.*

Proof. Let x be a vector in H. If $Tx=0$, then trivially $\|T^3x\|\,\|x\|^2 \geqslant \|Tx\|^3$. Therefore assume that $Tx\neq 0$. Then

$$\|T^3x\|\,\|x\|^2 = \|T^2(Tx)\|\,\|x\|^2 \geqslant \|T^*Tx\|^2\,\|x\|^2/\|Tx\|$$

$$= \langle (T^*T)^2x, x\rangle \|x\|^2/\|Tx\| \geqslant \langle (T^*T)x, x\rangle^2/\|Tx\| = \|Tx\|^3 .$$

This proves the desired assertion.

Remarks. (1) That every operator of class $(H;2)$ is k-paranormal for $k>3$ is not true. To see this, first observe that a weighted shift T with weights $\{\alpha_n\}$ is of class $(H;2)$ if and only if $|\alpha_{n-1}|^2 \leqslant |\alpha_n|\,|\alpha_{n+1}|$ for all n. If we let $\alpha_n=\frac{1}{2}$ $(n<0)$, $\alpha_0=1$, $\alpha_1=\frac{1}{2}$, $\alpha_2=2$, $\alpha_3=\frac{1}{2}$ and $\alpha_n=8$ $(n\geqslant 4)$, then $\alpha_{n-1}^2 \leqslant \alpha_n\alpha_{n+1}$; thus T with these weights is of class $(H;2)$. Since $\|T^4e_0\|<\|Te_0\|^4$, T fails to be 4-paranormal. (2) Using the technique of the above result, one can easily prove that every k-hyponormal operator is $k+1$-paranormal.

Theorem 3. *There is an operator of class $(H;2)$ which is not of class (N) and viceversa.*

Proof. Let T be a weighted shift operator of class $(H;2)$, as defined in Remark 1 following Theorem 2. Since T is not 4-paranormal, it follows that T is not of class (N), as every operator of class (N) is of class $(N;k)$.

Next we give an example of a paranormal operator which fails to be of class $(H;2)$. Let K be the direct sum of a denumerable copies of H. Let A and B be positive operators on H. Define an operator $T_{A,B,n}$ on K as follows

$$T_{A,B,n}\langle x_1, x_2, x_3, \ldots\rangle = \langle 0, Ax_1, Ax_2, \ldots, Ax_n, Bx_{n+1}, Bx_{n+2}, \ldots\rangle .$$

Then by Theorem 1, $T_{A,B,n}$ is of class $(H;2)$ if and only if

$$AB^2A-2zA^2+z^2\geqslant 0\,, \qquad B^4-2zA^2+z^2\geqslant 0 \qquad (z>0)\,.$$

If we take H to be a two-dimensional Hilbert space and $A=C^{\frac{1}{2}}$ and

$B = (C^{-\frac{1}{2}}DC^{-\frac{1}{2}})^{\frac{1}{2}}$, where

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix},$$

then as shown by Andô ([**1**], p. 172), $T_{A,B,n}$ is paranormal. Since for $z = 1$,

$$B^4 - 2zA^2 + z^2 = C^{-\frac{1}{2}}(DC^{-1}D - 2C^2 + C)C^{-\frac{1}{2}} \not\geqslant 0,$$

T fails to be of class $(H; 2)$. This proves the result.

Remark. It follows from the above theorem that the following inclusions are proper:

(i) Hyponormal $\subseteq$ class $(H; 2)$ $\subseteq$ class $(N; 3)$.

(ii) Hyponormal $\subseteq$ class (N) $\subseteq$ class $(N; 3)$.

We know that every power of a paranormal operator is again paranormal ([**2**], Theorem 1). However, the corresponding assertion does not hold for operators of class $(H; 2)$.

Theorem 4. *There exists a hyponormal operator whose square is not an operator of class* $(H; 2)$.

Proof. Let $T = T_{A,B,n}$ where A and B are the positive square roots of positive operators

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

defined on a two-dimensional Hilbert space H. Since $B^2 \geqslant A^2$, it follows that T is hyponormal. If T^2 is of class $(H; 2)$, then $B^8 - 2zA^4 + z^2 \geqslant 0$ $(z > 0)$. But for $z = 1$, we have

$$B^8 - 2zA^4 + z^2 = \begin{bmatrix} -8 & -6 \\ -6 & -3 \end{bmatrix} \not\geqslant 0.$$

This shows that T^2 is not of class $(H; 2)$.

The preceding theorem shows that the product of two commuting operators of class $(H; 2)$ is not necessarily of class $(H; 2)$. This suggests the following problem. Does the product of two doubly commuting operators of class $(H;2)$ turn out to be an operator of class $(H;2)$. The following result provides the answer in negative.

Theorem 5. *There is a pair of doubly commuting operators of class* $(H;2)$ *such that their product is not of class* $(H;2)$.

Proof. Define $T=T_{A,B,2}$ on K as follows

$$T\langle x_1, x_2, x_3, \ldots\rangle = \langle 0, Ax_1, Bx_2, Bx_3, \ldots\rangle .$$

Then T is of class $(H;2)$ if and only if: $B^4-2zA^2+z^2\geqslant 0$ $(z>0)$.

If $\dim(H)=2$ and

$$A^2=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix} \qquad \text{and} \qquad B^4=\begin{bmatrix}1 & 2\\ 2 & 8\end{bmatrix},$$

then, clearly $B^4-2zA^2+z^2\geqslant 0$ $(z>0)$; thus T is of class $(H;2)$. Let $T_1= 1\otimes T$ and $T_2=T\otimes 1$. Then T_1 and T_2 will be two doubly commuting operators of class $(H;2)$. If T_1T_2 is of class $(H;2)$, then: $B^4\otimes B^4-2z\cdot(A^2\otimes A^2)+z^2\geqslant 0$ $(z>0)$, which is not true for $z=1$.

Remark. As argued in ([**1**], theorem 4), one can easily show that if an operator of class $(H;2)$ doubly commutes with a hyponormal operator, then their product is an operator of class $(H;2)$.

It is well-known that for any complex number z, $T+zI$ is hyponormal whenever T is hyponormal. However, the analogous assertion does not hold either for paranormal operators ([**1**], p. 174), or, as we shall show in the following result, for operators of class $(H;2)$.

Theorem 6. *The sum of an operator of class* $(H;2)$ *and a scalar is not necessarily of class* $(H;2)$.

Proof. Let T be a unilateral weighted shift with weights $\{\alpha_n\}$, where $\alpha_0=1/\sqrt{2}$, $\alpha_1=1/3$ and $\alpha_n=2-1/n$ $(n\geqslant 2)$. Since $\alpha_{n-1}^2\leqslant\alpha_n\alpha_{n+1}$ for all $n\geqslant 1$, it follows that T is an operator of class $(H;2)$. If $T+zI$ is also of class $(H;2)$, then a computation shows that: $\alpha_{n+1}^2\alpha_n^2+4|z|^2\alpha_n^2\geqslant\alpha_{n-1}^4+2|z|^2\alpha_{n-1}^2$ for all $n\geqslant 1$. But for $n=1$, the inequality fails if z is non-zero. This proves the desired assertion.

It has been shown in Theorem 1 that the inverse of a non-singular paranormal operator is again paranormal. However, for operators of class $(H;2)$, we have

Theorem 7. *There is a non-singular operator of class* $(H; 2)$ *whose inverse is not normaloid.*

Proof. Let T be a bilateral weighted shift of class $(H; 2)$ defined in Remark 1 of Theorem 2. Then since $\|T^{-1}\| = 4$ and $\|T^{-2}\| = 4$, it follows that $\|T^{-2}\| < \|T^{-1}\|^2$ thus T^{-1} fails to be normaloid.

Remark. This results shows that, unlike hyponormal operators, operators of class $(H; 2)$ and hence of class $(N; 3)$ need not satisfy the growth condition (G_1). However whether these operators are convexoid remains as an open problem.

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