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# A fixed point theorem in bi-metric spaces (**)

## 1. - Introduction.

Let $(X, d)$ be a metric space. A mapping $T: X \to X$ is called a contraction mapping with respect to $d$ if there exists an $\alpha$, $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leqslant \alpha d(x, y) \qquad \text{for all} \quad x, y \in X .$$

The well known Banach contraction principle states that a contraction mapping on a complete metric space has a unique fixed point.

We find that a contraction mapping is always continuous.

A triple $(X, d_1, d_2)$ where $d_1$ and $d_2$ are metrics on $X$ will be called a bi-metric space.

Maia [2] considered bi-metric space to find out a sufficient condition for the existence of a unique fixed point of a mapping $T$ which is a contraction with respect to one metric and continuous with respect to other. He proved the following

Theorem A. Let $(X, d_1, d_2)$ be a bi-metric space such that

(i) $d_1(x, y) \leqslant d_2(x, y)$ for all $x, y \in X$,

(ii) $T: X \to X$ is contraction with respect to $d_2$,

(iii) $T: X \to X$ is continuous with respect to $d_1$,

(iv) $X$ is complete with respect to $d_1$.

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Then there exists a unique fixed point of $T$ in $X$. It may be remarked that if the two metrics $d_1$ and $d_2$ are equal, the above theorem reduces to the classical Banach contraction principle.

**2.** – In this paper we generalize Theorem A by replacing conditions (ii), (iii) and (iv) by less restricted conditions. We prove the following

Theorem. *Let $(X, d_1, d_2)$ be a bi-metric space such that*

(i) $d_1(x, y) \leqslant d_2(x, y)$ *for all* $x, y \in X$,

(ii) $T$ *satisfies*: $d_2(T^{2p}x, T^{2p}y) \leqslant \alpha\ d_2(T^p x, T^{2p}x) + \beta d_2(T^p y, T^{2p}y)$ *for positive integer*, $p$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$,

(iii) $T^q$ *is a continuous for some positive integer* $q$ *at a point* $w$ *in* $(X, d_1)$,

(iv) *there exists a point* $x_0 \in X$ *such that the sequence of iterates* $\{T^n(x_0)\}$ *has a subsequence* $\{T^{n_i}(x_0)\}$ *converging to* $w$ *in* $(X, d_1)$.

*Then* $w$ *is a unique fixed point of* $T$.

For the proof of the Theorem, we need the following Lemmas.

Lemma 1. *Let $(X, d)$ be a metric space and $T$ a self mapping of $X$ satisfying*

$$d(T^{2p}x, T^{2p}y) \leqslant \alpha d(T^p x, T^{2p}x) + \beta d(T^p y, T^{2p}y)$$

*for all $x, y \in X$ and $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$, $p$ being a positive integer.*

*Then for any $x \in X$, the sequence of iterates $\{T^n(x)\}$ is a Cauchy-sequence.*

Proof. Let $x \in X$. Define $T^p(x) = x_0$ and $T^p(x_{n-1}) = x_n$. Put $K = \alpha/1 - \beta$. Then

$$d(x_1, x_2) = d(T^{2p}x, T^{2p}x_0) \leqslant \alpha d(T^p x, T^{2p}x) + \beta d(T^p x_0, T^{2p}x_0) \leqslant \alpha d(x_0, x_1) + \beta d(x_1, x_2),$$

hence $d(x_1, x_2) \leqslant K d(x_0, x_1)$. Again

$$d(x_2, x_3) = d(T^{2p}x_0, T^{2p}x_1) \leqslant \alpha d(T^p x_0, T^{2p}x_0) + \\ + \beta d(T^p x_1, T^{2p}x_1) \leqslant \alpha d(x_1, x_2) + \beta d(x_2, x_3).$$

Therefore $d(x_2, x_3) \leqslant K^2 d(x_0, x_1)$. In general, $d(x_n, x_{n+1}) \leqslant K^n\, d(x_0, x_1)$. Thus $\{x_n\}$ is a Cauchy-sequence.

Lemma 2. *If $T^n$ (n positive integer) has a unique fixed point $w$ in a metric space $X$, then $w$ is the unique fixed point of $T$ in $X$.*

Proof. Simple.

Remark. The converse of Lemma 2 is not necessarily true. For let $X = [0, 1]$, with the usual metric. Suppose $T: X \to X$ such that $T(x) = 1 - x$ for all $x \in X$. Then $T$ has a unique fixed point but $T^2$ has none.

Proof of the Theorem. Let $T^p(x) = x_0$, $T^p(x_{n-1}) = x_n$. Then $\{x_n\}$ is a Cauchy-sequence in $(X, d_2)$ and from (i) it is a Cauchy-sequence in $(X, d_1)$. By (iv) a subsequence of $\{x_n\}$ converges to $w$ in $(X, d_1)$. Now we have

$$\lim_{i\to\infty} x_{n_i} = w \qquad \text{in} \quad (X, d_1) .$$

Also $T^q$ is continuous at $w$ in $(X, d_1)$. Therefore

$$T^q(w) = T^q(\lim_{i\to\infty} x_{n_i}) = \lim_{i\to\infty} T^q(x_{n_i}) = \lim_{i\to\infty} (x_{n_i+q}) = w .$$

Now we shall show that $w$ is a unique fixed point of $T^q$. For if $x \neq y$, $T^q(x) = x$ and $T^q(y) = y$, we get

$$d_2(x, y) = d_2(T^{2pq}x, T^{2pq}y) \leqslant \alpha d_2(T^{pq}x, T^{2pq}x) + \beta d_2(T^{pq}y, T^{2pq}y)$$
$$\leqslant \alpha d_2(x, x) + \beta d_2(y, y) .$$

Thus $w$ is a unique fixed point of $T^q$. Therefore $w$ is a unique fixed point of $T$.

Remarks. (a) Condition (ii) can be replaced by any condition which gives a Cauchy sequence. (b) If $d_1 = d_2$ and $q = p$, our theorem reduces to a theorem of Gupta and Khan [**1**].

Example. Let $X = [0, 1]$ with usual metric space, and $T: X \to X$ be defined by $T(0) = T(1) = 0$, $T(x) = 1$ for all $x \in (0, 1)$. Then $T$ is not continuous so it does not satisfy the condition (ii) of Theorem A since a contraction mapping with respect to any metric has to be continuous.

But $T^2(x) = 0$ for all $x \in [0, 1]$. Hence $T^2$ is continuous and satisfies condition (iii) of our theorem.

## References

[1] V. K. Gupta and M. S. Khan, *Some fixed point theorems*, (to appear).

[2] M. G. Maia, *Un'osservazione sulle contrazioni metriche*, Rend. Sem. Mat. Univ. Padova **40** (1968), 139-143.

## Abstract

*A sufficient condition for the existence of a unique fixed point of a self mapping of a bi-metric space has been obtained.*

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