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On pairwise s-regular spaces (**)

Introduction.

A subset A of a topological space X is *semi open* [4] if for some open set 0, $0 \subset A \subset \text{cl}\, 0$, where cl 0 denotes the closure of 0 in X. Every open set is semi open while the converse may be false [4]. A set N is said to be a *semi neighbourhood* [1] of a point $x \in X$ if there is a semi open set M such that $x \in M \subset N$. Complement of a semi open set is called *semi closed.* A point x of X is a *semi limit point* of $A \subset X$, if any semi open set containing x contains a point of A distinct from x. The union of the set A and the set of all the semi limit points of A is called the *semi closure* of A [2]. We denote it by scl A. It is the smallest semi closed set containing A. Infact [2], $A \subset B$ implies $\text{scl}\, A \subset \text{scl}\, B$; $\text{scl}\,(\text{scl}\, A) = \text{scl}\, A$ and A is semi closed iff $A = \text{scl}\, A$. In a bitopological space (X, P_1, P_2) by P_i-semi open set (resp. P_i-scl A) we mean a semi open set in X (respectively the semi closure of A) with respect to the topology P_i, $i = 1, 2$. A bitopological space (X, P_1, P_2) is *pairwise semi* T_2 $[5]_2$ if for any two distinct points x, y of X there exist disjoint P_i-semi open set U and P_j-semi open set V such that $x \in U$ and $y \in V$, $i \neq j$, $i, j = 1, 2$. A bitopological space (X, P_1, P_2) is *pairwise* T_0 [6], if for each pair of distinct points of X there is a set which is either P_1-open or P_2-open containing one of the points but not the other. The axioms of pairwise semi T_2 and pairwise T_0 are independent $[5]_2$. Also, a bitopological space (X, P_1, P_2) is said to be *pairwise regular* [3] if for every P_i-closed set F and a point $x \notin F$ there exist a P_j-open set V and a P_i-open set U such that $U \cap V = \emptyset$, $F \subset V$, $x \in U$, $i, j = 1, 2$, $i \neq j$.

In this paper we introduce and study pairwise s-regularity which is strictly weaker than pairwise regularity. Throughout the paper $X \sim B$ denotes the complement of B in X.

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1. - Pairwise *s*-regular spaces.

Definition. A bitopological space (X, P_1, P_2) is *pairwise s-regular* if for every P_i-closed set F and a point $x \notin F$, there exists a P_j-semi open set U and a P_i-semi open set V such that $U \cap V = \emptyset$, $F \subset U$, $x \in V$, $i \neq j$, $i, j = 1,2$.

It is evident that every pairwise regular space is pairwise s-regular. The converse need not be true.

Example 1. Let

$$X = \{a, b, c\}\,, \quad P_1 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}\,, \quad P_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}\,.$$

Then, (X, P_1, P_2) is pairwise s-regular but it is not pairwise regular.

Remark 1. Examples 2 and 3 below show that pairwise s-regular and pairwise T_0 are independent. In addition, example 2 shows that pairwise s-regular spaces may fail to be pairwise semi T_2.

Example 2.

$$X = \{a, b, c, d, e\}\,, \quad P_1 = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}\,, \quad P_2 = \{\emptyset, \{d, e\}, X\}\,.$$

Then, (X, P_1, P_2) is pairwise s-regular but it is not pairwise T_0. Note that it is not pairwise semi T_2 also.

Example 3. Let

$$X = \{a, b, c\}\,, \quad P_1 = \{\emptyset, \{a\}, \{b, c\}, X\}\,, \quad P_2 = \{\emptyset, \{b\}, \{b, c\}, X\}\,.$$

Then, the space (X, P_1, P_2) is pairwise T_0 but it is not pairwise s-regular. However we have the following theorem.

Theorem 1. *Every pairwise s-regular pairwise* T_0 *space* (X, P_1, P_2) *is pairwise semi* T_2.

Proof. Let $x, y \in X$ and $x \neq y$. X being pairwise T_0, let V be a P_i-open set, $i = 1$ or 2 which contains x but does not contain y. Then $X \sim V$ is a P_i-closed set containing y to which x does not belong. The conclusion now follows by pairwise s-regularity of (X, P_1, P_2).

Theorem 2. *In a bitopological space* (X, P_1, P_2), *the following conditions are equivalent (where in* $i \neq j$, $j, i = 1, 2$):

(a) (X, P_1, P_2) *is pairwise s-regular.*

(b) *For each* $x \in X$ *and each* P_i*-open set* U *containing* x *there is a* P_i*-semi open set* W *such that* $x \in W \subset P_j\text{-scl}\, W \subset U$.

(c) *Every* P_i*-closed set* A *is identical with the intersection of all the* P_i*-semi closed* P_j*-semi neighbourhoods of* A.

(d) *For every set* A *and every* P_i*-open set* B *such that* $A \cap B \neq \emptyset$, *there exists a* P_i*-semi open set* 0 *for which* $A \cap 0 \neq \emptyset$ *and* $P_j\text{-scl}\, 0 \subset B$.

(e) *For every nonempty set* A *and* P_i*-closed set* B *such that* $A \cap B = \emptyset$, *there exists disjoint sets* G *and* H *such that* G *is* P_i*-semi open,* H *is* P_j*-semi open,* $A \cap G \neq \emptyset$, *and* $B \subset H$.

Proof. (a) $\Rightarrow$ (b). Let U be a P_i-open set such that $x \in U$. Then $X \sim U$ is P_i-closed and $x \notin (X \sim U)$. Now by (a) there exists a P_j-semi open set V and P_i-semi open set W such that $x \in W$, $X \sim U \subset V$ and $W \cap V = \emptyset$. And so, $x \in W \subset P_j\text{-scl}\, W \subset U$.

(b) $\Rightarrow$ (c). Let A be P_i-closed and $x \notin A$. Then $X \sim A$ is P_i-open and cintains x. By the hypothesis there is a P_i-semi open set W such that $x \in W \subset$ $\subset P_j\text{-scl}\, W \subset X \sim A$. This gives that $X \sim W \supset X \sim P_j\text{-scl}\, W \supset A$. Therefore, $X \sim W$ is P_i-semi closed P_j-semi-neighbourhood of A to which x does not belong. Thus (c) holds.

(c) $\Rightarrow$ (d). Let $A \cap B \neq \emptyset$ and B is P_i-open. Let $x \in A \cap B$. Since $x \notin (X \sim B)$ which is P_i-closed, in view of (c) let V be a P_i-semi closed P_j-semi neighbourhood of $X \sim B$, such that $x \notin V$. Now let U be P_j-semi open set such that $X \sim B \subset U \subset V$. Then, $0 = X \sim V$ is P_i-semi open and if fulfills the requirements of (d).

(d) $\Rightarrow$ (e). Let $A \cap B = \emptyset$, A is nonempty and B is P_i-closed. Then $A \cap (X \sim B) \neq \emptyset$ and $X \sim B$ is P_i-open. By (d), let G be a P_i-semi open such that $A \cap G \neq \emptyset$, $G \subset P_j\text{-scl}\, G \subset X \sim B$. Put $H = X \sim P_j\text{-scl}\, G$. Then, H is P_j-semi open, $B \subset H$ and $G \cap H = \emptyset$.

(e) $\Rightarrow$ (a). Obvious.

Remark 2. Pairwise s-regularity is not hereditary. For, $\{b, c\}$ as a subspace of the pairwise s-regular space (X, P_1, P_2) of example 1, is not pairwise s-regular. However:

Theorem 3. *Every biopen subspace of a pairwise s-regular space* (X, P_1, P_2) *is pairwise s-regular.*

To prove the theorem we require the following

Lemma. *If* (Y, T_1, T_2) *is a* P_j*-open subspace of a pairwise space* (X, P_1, P_2), *then, for any subset* B *of* Y, $T_j\text{-scl}\, B = (P_j\text{-scl}\, B) \cap Y$, $j = 1, 2$.

Proof of the lemma. Let $x \in T_j\text{-scl}\, B$, $j = 1$ or 2. Then $x \in Y$. Let V be any P_j-semi open set containing x. Now Y being P_j-open in X, $V \cap Y$ is T_j-semi open in Y (cfr. [**5**]$_1$, theorem 2.3: if Y is open in a topological space X and U is semi open in X, then $Y \cap U$ is semi open in Y), and contains x. And so $V \cap Y$ meets B. Consequently V meets B. Thus, $x \in P_j$-$\text{scl}\, B$. Hence $x \in (P_j\text{-scl}\, B) \cap Y$. Now let $y \in (P_j\text{-scl}\, B) \cap Y$ and let 0 be a T_j-semi open set containing y. Then 0 is a P_j-semi open set for Y is P_j-semi open (cfr. [**5**]$_1$, theorem 2.4: if Y is a subspace of a topological space X, then A is semi open in Y is semi open in X iff Y is semi open in X). Consequently 0 meets B for $y \in P_j\text{-scl}\, B$. Hence $y \in T_j\text{-scl}\, B$.

Proof of the theorem. Let (Y, T_1, T_2) be a biopen subspace of (X, P_1, P_2). Let A be T_i-open in Y, $i = 1$ or 2 and $x \in A$. Y being P_i-open, A is P_i-open. Since X is pairwise s-regular there is a P_i-semi open set U in X such that $x \in U \subset P_j\text{-scl}\, U \subset A$. Therefore, $x \in U \subset (P_j\text{-scl}\, U) \cap Y \subset A$. And so, $x \in U \subset T_j\text{-scl}\, U \subset A$, by the lemma, for Y is P_j-open. The theorem now follows by Theorem 2 (b), since U is T_i-semi open ([4], theorem 6).

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