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Construction of fixed points for densifying mappings (**)

Introduction.

The notion of measure of noncompactness was introduced by C. Kuratowskii [6]. It was Darbo [2] who defined the concept of k-set contraction using the notion of measure of noncompactness and proved the following fixed point Theorem (see Theorem 1.1). Darbo's theorem was extended by Furi and Vignoli [3] for densifying mappings. Without being aware of Furi and Vignoli's result Sadovskii [10] also extended the theorem of Darbo [2], but using different kind of measure of noncompactness (usually called ball measure of noncompactness, see Definition 1.3). Although these two measure of noncompactness share few properties in common (a counter example for the case where they differ may be found in Nussbaum [7], p. 127). Nussbaum [7] using the measure of noncompactness that of C. Kuratowskii [6] developed the degree theory for k-set contraction with $k < 1$, and later extended for densifying mappings. Since the densifying mappings are so general that the generalization of classical fixed point theorems for such kind of mappings is of continuing interest.

In the present paper we prove two fixed point theorems for densifying mappings using the degree theory developed by Nussbaum [7]. Theorem 2.2 generalizes almost all theorems, available in literature about fixed points for the sum of two mappings. Finally we have included an application of Theorem 2.1 for densifying verctorfields (see Definition 1.6). The elegant work of Petryshyn $[8]_1$, $[8]_2$ contains the applications of densifying mappings. As a corollary of Corollary 2.1 has been obtained a theorem due to Biepecki and Tadcusi [1].

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Preliminary definitions and results.

Let X be Banach space. Let D be an open bounded subset of X. Let $\bar{D}$ and ∂D be respectively the closure and boundary of D.

Definition 1.1. (*Kuratowskii*). Let X be a real Banach space and D be a bounded subset of X. *The measure of noncompactness* of D, denoted by $\gamma(D)$, is defined as follows:

$$\gamma(D) = \inf\{t > 0, \text{ such that } D \text{ can be covered by a finite number of subsets of diameter } < t\}.$$

$\gamma(D)$ has the following properties

(*a*) $0 \leqslant \gamma(D) \leqslant d(D)$, where $d(D)$ is the diameter of D,

(*b*) $\gamma(D) = 0$ if and only if D is precompact (i.e. $\bar{D}$ is compact),

(*c*) $C \subset D \Rightarrow \gamma(C) \leqslant \gamma(D)$,

(*d*) $\gamma(C \cup D) = \max\{\gamma(C), \gamma(D)\}$,

(*e*) $\gamma\big(C(D, r)\big) \leqslant \gamma(D) + 2r$, where $C(D, r) = \{x \text{ in } X/d(x, D) < r\}$,

(*f*) $\gamma(C + D) \leqslant \gamma(C) + \gamma(D)$, where $C + D = \{c + d/c \text{ in } C \text{ and } d \text{ in } D\}$.

Definition 1.2. (*Darbo*). Let X be a Banach space. Let $T: X \to X$ be a continuous mapping. T is said to be a *k-set contraction* if given any bounded subset D of X we have: $\gamma\big(T(D)\big) \leqslant k\gamma(D)$ for some $\gamma \geqslant 0$.

In case $\gamma\big(T(D)\big) < \gamma(D)$, for any bounded subset D of X such that $\gamma(D) > 0$, then T is called a *densifying mapping* [**3**].

Definition 1.3. (*Sadovskii*). Let X be a real Banach space. Let T: $X \to X$ be a continuous mapping. T is said to be *densifying* (*condensing*) if for any bounded subset D of X with $\chi(D) > 0$ we have $\chi\big(T(D)\big) < \chi(D)$, where $\chi(D)$ denotes the infimum of all real numbers $\varepsilon > 0$ such that D admits a finite ε-net.

Definition 1.4. Let X and Y be two Banach spaces. Let D be a closed and convex subset of X. A mapping $T: D \to Y$ is said to be *compact* if it is continuous and maps bounded sets into relatively compact sets.

A mapping $T: D \to Y$ is said to be *completely continuous* if it takes each weakly convergent sequence into strongly convergent sequence.

Remark 1.1. These two classes of mappings are not comparable. That is neither one is contained in the other. The counter examples demonstrating the difference may be found in Vainberg ([**12**], pp. 14-16). Infact Vainberg [**12**] these two mappings heve been refered as completely continuous and strongly continuous respectively.

Definition 1.5. (*Granas*, p. 30). Let X and Y be two Banach spaces. A mapping $T: X \to Y$ is said to be *completely continuous vectorfield* on X, if it can be represented as

$$T(x) = x - F(x) , \tag{1}$$

where $F: X \to Z$, where Z is an arbitrary but fixed Banach space and F is completely continuous mapping.

Remark 2.1. The sum of two t-set contractions is again a t-set contraction. Completely continuous mapping is zero-set contraction. Contraction mapping (a mapping of a closed, bounded and convex subset D of a Banach into itself satisfying the condition $\|T(x) - T(y)\| \leqslant k\|x - y\|$, where $k < 1$ and contractive mappings i.e. $\|T(x) - T(y)\| < \|x - y\|$) are respectively examples of t-set contraction with $t < 1$ and densifying mappings.

Theorem 1.1. (*Darbo*). *Let C be a closed, bounded and convex subset of a Banach space X. Let $T: C \to C$ be a k-set contraction with $k < 1$. Then T has a fixed point.*

Theorem 1.2. (*Sadovskii* [**10**], *Furi and Vignoli* [**3**]). *Let C be nonempty closed, bounded and convex subset of a Banach space X. Let $T: C \to C$ be densifying (condensing). Then T has a fixed point.*

Definition 1.6. Let X and Y be two Banach spaces. Let $T: X \to Y$ be densifying. T is said to *densifying vectorfield* on X, provided if T can be expressed in the form $T(x) = x - F(x)$, where $F: X \to Y$ is densifying.

The set of all densifying vectorfields will be denoted by $L(Y^X)$.

Definition 1.7. Two densifying vectorfields A and B in $L(Y^X)$ are said to be homotopic, provided there exists a homotopy $h(x, t): X \times I \to Y$ between A and B which can be represented by $h(x, t) = x - H(x, t)$, where the mapping $H(x, t): X \times I \to Y$ is densifying.

Theorem 2.1. *Let X be a Banach space. Let D be an open bounded subset of X. Let A and $B: \bar{D} \to X$ be two densifying mappings such that $\|A(x) - B(x)\| \leq \|A(x) - x\|$ for all x in ∂D. If* $\operatorname{Deg}(I - B, D. O)$ *is defined so is* $\operatorname{Deg}(I - A, D, O)$. *Moreover* $\operatorname{Deg}(I - A, D, O) = \operatorname{Deg}(I - B, D, O)$. *Furthermore, if* $\operatorname{Deg}(I - B, D, O) \neq 0$, *then A has a fixed point.*

Proof. It is enough to show that $\operatorname{Deg}(I - A, D, O) \neq 0$. To do so we define the homotopy $H(x, t): \bar{D} \times I \to X$ as follows $H(x, t) = tB(x) + (1 - t)A(x)$, x in $\bar{D}$, t in I, where $I = [0, 1]$. Then $H(x, t)$ is densifying. Clearly $H(x, t)$ being convex combination of two continuous mappings is continuous. Let C be any bounded but not precompact subset of D, then by definition $H(x, t)$ we have $H(C, t) = tB(C) + (1 - t)A(C)$. Hence

$$\gamma(H(C, t)) = \gamma(tB(C) + (1 - t)A(C)) < t\gamma(C) + (1 - t)\gamma(C) = \gamma(C) .$$

Moreover, $H(x, t)$ is uniformly continuous in t for t in I. Indeed, let $|t - s| < \delta/(R + S)$, we need to show that $\|H(x, t) - H(x, s)\| < \varepsilon$ for all x in D. Now by definition of $H(x, t)$ we have

$$\|H(x, t) - H(x, s)\| = \|(t - s)B(x) + (1 - t)A(x) - (1 - s)A(x)\| =$$

$$= \|(t - s)B(x) - (t - s)A(x)\| \leq |(t - s)|(\|B(x)\| + A(x)\|) < |t - s|(R + S) = \delta ,$$

$R \geq \|B(x)\|$ and $S \geq \|A(x)\|$. Letting $\varepsilon = \delta$ we have $\|H(x, t) - H(x, s)\| < \varepsilon$, as was claimed.

To complete the proof of Theorem 2.1 it suffices to show that $x - H(x, t) \neq 0$. Without loss of generality we may assume that $x - H(x, 1) \neq 0$, otherwise we get a contradiction to the hypothesis that $\operatorname{Deg}(I - B, D, O) \neq 0$ (i.e. B has a fixed point). Furthermore, let us assume that $A(x) \neq x$, otherwise we are done.

Now let us suppose that $x - H(x, t) = 0$ for some x in ∂D and t in I, then we have the following three cases.

Case 1. If $t = 0$, then $x - H(x, 0) = 0$ implies that $x - A(x) = 0$, which in turns implies $A(x) = x$, a contradiction to our assumption.

Case 2. If $t = 1$, then $x - H(x, 1) = 0$ implies that $x - B(x) = 0$, which in turns implies that $B(x) = x$, a contradiction to our assumption.

Case 3. If $0 < t < 1$, then

$$x - H(x, t) = 0 \Rightarrow x - tB(x) - (1 - t)A(x) = 0 ,$$

$$x - tB(x) - A(x) + tA(x) = 0 \quad \Rightarrow \quad x - A(x) = t\big(B(x) - A(x)\big) .$$

Hence

$$\|A(x) - x\| = t\|B(x) - A(x)\| \quad \text{or} \quad \|B(x) - A(x)\| \geqslant \|A(x) - x\| \quad (0 < t < 1) ,$$

a contradiction to the hypothesis. Thus $H(x, t)$ is well defined homotopy. Therefore by homotopy theorem [7] $\operatorname{Deg}\big(I - H(\cdot, t)\big)$ is constant in t for t in I. Hence

$$\operatorname{Deg}\big(I - H(\cdot, 0), D, O\big) = \operatorname{Deg}\big(I - H(\cdot, 1), D, O\big)$$

or

$$\operatorname{Deg}(I - A, D, O) = \operatorname{Deg}(I - B, D, O) \neq 0 .$$

Hence there exists a x in D such that $(I - A)(x) = 0$ or equivalently $A(x) = x$. Thus the Theorem.

Corollary 2.1. *Let X be a Banach space. Let f and g be two densifying vectorfields on X. Let us assume that the following inequality*

$$\|f(x) - g(x)\| \leqslant \|f(x)\| \tag{2}$$

holds for all x in X. Then the densifying vectorfields f and g are homotopic.

Proof. Since f and g are densifying vectorfields on X, therefore by Definition 1.5 we can write $f(x)$ and $g(x)$ as follows

$$f(x) = x - A(x) , \qquad g(x) = x - B(x) ,$$

where $A, B\colon X \to X$ are densifying. Now $A(x) - B(x) = g(x) - f(x)$. Hence (by [2])

$$\|A(x) - B(x)\| = \|g(x) - f(x)\| = \|f(x) - g(x)\| \leqslant \|f(x) = \|A(x) - x\| .$$

Thus the conditions of Theorem 2.1 are satisfied and the homotopy of Theorem 2.1 serves the desired purpose.

As a corollary of Corollary 2.1 we have the following theorem due to Biepecki and Tadcusi [1].

Corollary 2.2. *Let X be a Banach space. Let $S = \{x \text{ in } X / \|x\| = R\}$ be a sphere in X. Let $h(x): S \to X$ be a completely continuous mapping. Let $g(x) = x - h(x)$ be a completely continuous vectorfield. Let $f(x) = x - r(x)$ be another completely continuous vectorfield on S. Furthermore suppose*

$$\|r(x) - h(x)\| \leqslant \|x - h(x)\| .$$

Then $\mathrm{Deg}(g, S, O) = \mathrm{Deg}(f, S, O)$. *Moreover if* $\mathrm{Deg}(g, S, O) \neq 0$ *for* $\|x\| = R$, *then the equation $x = r(x)$ has at least one solution y in S.*

Theorem 2.2. *Let X be a Banach space. Let D be an open, bounded subset of X. Let $A, B: D \to X$ be respectively p-set contraction and q-set contraction such that $p + q \leqslant 1$. Let A, B satisfy the following inequality*

$$\|A(x) - B(x) - x\| \geqslant \|A(x)\| .$$

Furthermore suppose that $\mathrm{Deg}(I - B, D, O) \neq 0$. *Then there exists a x in $\bar{D}$ such that $A(x) + B(x) = x$.*

Proof. It is enough to show that $\mathrm{Deg}(I - (A + B), D, O) \neq 0$. Let us define the homotopy $H(x, t): \bar{D} \times I \to X$ as follows $H(x, t) = (1 - t) A(x) + B(x)$, for all x in $\bar{D}$ and t in I. Then $H(x, t)$ is densifying. Clearly $H(x, t)$ being convex combination of continuous mappings is continuous. Let C be any bounded but not precompact subset of D, then by definition of $H(x, t)$ we have

$$H(C, t) = (1 - t) A(C) + B(C) ,$$

$$\gamma H(C, t) = [(1 - t) A(C) + B(C)] \leqslant (1 - t)\gamma A(C) + \gamma B(C) \leqslant (1 - t) p\gamma(C) + q\gamma(C)$$
$$\leqslant (p + q - pt)\gamma(C) = r\gamma(C) < \gamma(C), \text{ when } [r = p + v - bt < 1] .$$

Finally to show that $H(x, t)$ is well defined homotopy it remains to show that $H(x, t)$ is uniformly continuous in t for t in I. Indeed, let $|t - s| < m/M$, then by definition of $H(x, t)$ we have $(M \geqslant \|A(x)\|)$

$$\|H(x, t) - H(x, s)\| = \|(s - t) A(x)\| = |(s - t)| \|A(x)\|$$
$$\leqslant |(t - s)| M < (m/M)(M) = m .$$

Taking $\varepsilon = m$, we have $\|H(X, t) - H(x, s)\| < \varepsilon$. Thus $H(x, t)$ is well defined homotopy.

Now we claim that $x - H(x, t) \neq 0$, for all x in ∂D and t in I. Without loss of generality we may assume that $x \neq A(x) + B(x)$, otherwise we are done. Furthermore let us assume that $x - H(x, 1) \neq 0$, otherwise we have a contradiction to the assumption that $\operatorname{Deg}(I - B, D, O) \neq 0$. Let us suppose that $x - H(x, t) = 0$ for some x in ∂D and t in I. Now we have the following three cases.

Case 1. If $t = 0$, then $x - H(x, 0) = 0$ implies that $x - (A(x) + B(x)) = 0$ which in turns implies that $x = A(x) + B(x)$, a contradiction to our assumption.

Case 2. If $t = 1$, then $x - H(x, 1) = 0$ implies that $x - B(x) = 0$ which in turns implies that $x = B(x)$, a contradiction to our assumption.

Case 3. If $0 < t < 1$, then $x - H(x, t) = 0$ implies that $x - (1 - t)A(x) - B(x) = 0$. Thus we have

$$x - A(x) + tA(x) - B(x) = 0 \qquad \text{or} \qquad tA(x) = A(x) + B(x) - x .$$

Hence

$$t\|A(x)\| = \|A(x) + B(x) - x\| \quad \text{or} \quad \|A(x)\| \geqslant \|A(x) + B(x) - x\| \qquad (0 < t < 1),$$

a contradiction to the hypothesis. Thus by homotopy Theorem $\operatorname{Deg}(I - H(\cdot, t), D, O)$ is constant in t for t in I. Hence

$$\operatorname{Deg}(I - H(\cdot, 0), D, O) = \operatorname{Deg}(I - H(\cdot, 1), D, O)$$

or

$$\operatorname{Deg}(I - (A + B), D, O) = \operatorname{Deg}(I - B, D, O) \neq 0 .$$

Therefore there exists a x in D such that $(I - (A + B))(x) = 0$, i.e. $A(x) + B(x) = x$. Thus Theorem 2.2.

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Summary

See Introduction.

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