Riv. Mat. Univ. Parma (4) 4 (1978), 73-78

OLUSOLA AKINYELE (*)

On a homomorphism between generalized semigroup algebras (**)

1. - Introduction.

Let G be a locally compact abelian group and A a complex commutative Banach algebra. Denote by $B^1(G, A)$ the Bochner integrable functions with respect to the Haar measure of G. In [2] the homomorphism T of $B^1(G, A)$ into $B^1(G, A')$ which are such that T keeps $L^1(G)$ « *pointwise* » invariant, have been characterized when: (i) G is an abelian group such that $\hat{G}$, the dual space of G is connected and $m(A)$, the space of regular maximal ideals of A, is totally disconnected; (ii) G is a compact abelian group.

Let S be a discrete abelian semigroup. In [3] Hewitt and Zuckerman defined $l_1(S)$ as an abelian convolution Banach algebra of complex-valued functions α on S, that vanish except on countable subsets of S and for which $\sum_{x\in S} |\alpha(x)| = \|\alpha\|$ is finite. Denote by $l_1(S, A)$ the set of all functions on S with values in A, that vanish except on a countable subset of S and for which $\|f\| = \sum_{s\in S} \|f(s)\|$ is finite. With convolution as multiplication $l_1(S, A)$ is a complex abelian Banach algebra (cfr. [1]$_1$). It is known [1]$_2$ that the space of maximal regular ideals of $l_1(S, A)$ is homeomorphic with $m(A)\times\hat{S}$, where $\hat{S}$ is the set of semicharacters of S. If φ_M is a complex homomorphism of A associated with $M\in m(A)$, then the author [1]$_1$ has shown that the Fourier transform of a function $f\in l_1(S, A)$ is represented by

$$\hat{f}(M, X) = \sum_{x\in S} \varphi_M(f(x))\,\chi(x)\,, \qquad (M, \chi)\in m(A)\times\hat{S}\,,$$

(*) Indirizzo: Dept. Math., Univ. Ibadan, Ibadan, Nigeria.
(**) Ricevuto: 20-VIII-1975.

where the series is absolutely convergent. Moreover if af denotes the function $af(x) = a\cdot f(x)$, $x \in S$, $a \in A$ and $f \in l_1(S)$, then $af \in l_1(S, A)$ and finite linear combinations of functions of this type are dense in $l_1(S, A)$ (cfr. $[\mathbf{1}]_1$).

The result given in this paper characterizes the continuous homomorphisms T from $l_1(S, A)$ into $l_1(S, A')$ such that $T(ef) = e'f$ for any $f \in l_1(S)$ and where e and e' are the identities of the complex abelian Banach algebras A and A' respectively. This characterization generalizes the result of [**2**] to discrete abelian semigroups.

2. - Preliminaries.

Let $M \in m(A)$. If we define a function $\varphi_M \colon l_1(S, A) \to l_1(S)$ by setting $\varphi_M f(x) = \varphi_M(f(x))$, $f \in l_1(S, A)$, $x \in S$, then φ_M defines a continuous homomorphism. We shall now establish a converse of this statement, which is a generalization of a lemma of [**2**] to discrete semigroups.

Proposition 1. Suppose S is a discrete abelian semigroup with the property $xy = x^2 = y^2$ implies $x = y$, $x, y \in S$. Let A be a complex abelian Banach algebra with identity e, and φ a continuous homomorphism of $l_1(S, A)$ into $l_1(S)$ such that $\varphi(ef) = f$ for all $f \in l_1(S)$. Then there exists an $M \in m(A)$ such that $(\varphi g)(x) = \varphi_M g(x)$ for any $g \in l_1(S, A)$.

Remark. The special case of Proposition 1 when S is a compact abelian group was already proved in [**2**]. The fact that $\chi^{-1} \in \hat{G}$ belongs to $L^1(G)$ was crucial to the proof of Proposition 1 for this case. Since this property is not available when G is locally compact or even for locally compact abelian, the proof in [**2**] cannot be carried over to locally compact groups. The compactness of G also ensures that the constant A-valued functions all belong to $B^1(G, A)$. For a semicharacter $\chi \in \hat{S}$, $0 \leqslant |\chi(x)| \leqslant 1$ for all $x \in S$, so that χ may not belong to $l_1(S)$ and in fact χ^{-1} may not even exist at some points of S, so that the method of [**2**] fails for Proposition 1. However, we shall use a simple method to prove Proposition 1 for discrete abelian semigroups. Our method will break down for locally compact abelian groups because the constant A-valued functions cannot be embedded in $B^1(G, A)$.

Proof of Proposition 1. Let $x_0 \in S$ be fixed and denote by ξ_{x_0} the characteristic function of the point $x_0 \in S$. Then $\xi_{x_0} \in l_1(S)$ and for any $a \in A$, $a\,\xi_{x_0} \in l_1(S, A)$. Let $U = \{a\xi_{x_0} \colon a \in A\}$, then $U \subset l_1(S, A)$ and the mapping $a \to a\xi_{x_0}$ is an isometric isomorphism of A onto $U \subset l_1(S, A)$. Denote the functions in U by the elements of A. If $a = e \in A$, then by hypothesis $\varphi(e\xi_{x_0}) = \xi_{x_0}$ and so $\varphi \neq 0$. Moreover, for $a \in U$,

$$\varphi(a * e\xi_{x_0})(x) = \big(\varphi(a) * \varphi(e\xi_{x_0})\big)(x) = \big(\varphi(a) * \xi_{x_0}\big)(x)$$
$$= \sum_{u_0 v_0 = x} \varphi(a\xi_{x_0})(u_0)\, \xi_{x_0}(v_0) = \sum_{u_0 v_0 = x} \varphi\big(a\xi_{x_0}(u_0)\big)\, \xi_{x_0}(v_0) = \varphi(a)\,.$$

Hence φ maps each $a \in A$ into a constant function in $l_1(S)$. Finally,

$$\varphi(ab) = \varphi(ab * e\xi_{x_0})(x) = \varphi(a * be\xi_{x_0})(x) = \sum_{uv=x} \varphi(a\xi_{x_0})(u)\, \varphi(b\xi_{x_0})(v)$$
$$= \varphi(a)\, \varphi(b) \sum_{uv=x} \xi_{x_0}(u)\, \xi_{x_0}(v) = \varphi(a)\, \varphi(b)\,.$$

Hence φ is a non-zero continuous complex homomorphism of A, and by the Gelfand representation theorem $\exists$ a maximal regular ideal $M \in m(A)$, such that $\varphi(a) = \varphi_M(a)$ for any $a \in A$.

Let $f \in l_1(S)$ and $a \in A$. If $\chi \in \hat{S}$, then $\varphi(af) \in l_1(S)$ and

$$\varphi(\hat{af})(\chi) = \sum_{u \in S} \varphi(af)(u)\, \chi(u) = \sum_{u \in S} \varphi(aef)(u)\big(\chi(u)\big) = \sum_{u \in S} \varphi(a)\varphi\big(ef(u)\big)\, \chi(u)$$
$$= \sum_{u \in S} \varphi_M(a)\, f(u)\, \chi(u) = \varphi_M(a)\hat{f}(\chi)\,.$$

Since $l_1(S)$ is semisimple by hypothesis (cfr. [**3**]), then $\varphi(af) = \varphi_M(a)f$ any $f \in l_1(S)$. Suppose $\{a_k\}_{k=1}^n \subset A$, $\{f_k\}_{k=1}^n \subset l_1(S)$, then it is easy to see that

$$\varphi\Big(\sum_{k=1}^n a_k f_k\Big) = \varphi_M\Big(\sum_{k=1}^n a_k f_k\Big)\,.$$

For any $g \in l_1(S, A)$ there exists $\{g_n\}$ which is a finite linear combination of $a_k f_k$ such that $g_n \to g$ (cfr. [**1**]$_1$) and so $\varphi(g_n) = \varphi_M(g_n)$. Since $\varphi(g_n) \to \varphi(g)$ and $\varphi_M(g_n) \to \varphi_M(g)$ it follows that $\varphi(g) = \varphi_M(g)$ for any $g \in l_1(S, A)$ which concludes the proof.

3. - An homomorphism of $l_1(S, A)$.

In this section we shall state and prove the main theorem of this paper. We begin by stating the following theorem which was proved in [**2**].

Theorem A. Suppose G is a compact abelian group with Haar measure normalized to 1 and A a complex commutative Banach algebra with identity e with no restrictions on $m(A)$. Suppose A' is a complex semisimple Banach algebra with identity e' and $T: B^1(G, A) \to B^1(G, A')$ is a continuous homomorphism such that $T(ef) = e'f$ for any $f \in L^1(G)$. Then there exists a continuous $\tau: A \to A'$ such that $Tg(x) = \tau(g(x))$ for any $g \in B^1(G, A)$.

Our main aim in this paper is to generalize Theorem A to discrete abelian semigroups. If in Theorem A, G is taken as a locally compact abelian group, then the theorem is false. If however, $\hat{G}$ is connected and $m(A)$ is totally disconnected then Theorem 1 of [**2**] is an analogue of Theorem A for locally compact groups. In [**2**] the proof of Theorem A, depends on the existence of an approximate identity in $L^1(G)$. The approximate identity was used to generate a convergent sequence of continuous functions, the limit of which induces the desired homomorphism. The proof also employed an analogue of Proposition 1 for compact abelian groups to show that the limit of the sequence is independent of $\chi \in \hat{G}$. For our semigroup, $l_1(S)$ has no approximate identity so that this method cannot be carried over. However, a series representation of the Fourier transform enables us to extend Theorem A to semigroups.

We now state and prove an analogoue of Theorem A for discrete abelian semigroups.

Theorem 1. *Let S and A be as in Proposition 1 and A' a complex semisimple Banach algebra with identity e'. Let $T: l_1(S, A) \to l_1(S, A')$ be a continuous homomorphism such that $T(ef) = e'f$ for any $f \in l_1(S)$. Then there exists a continuous homomorphism $\tau: A \to A'$ such that $Tg(x) = \tau g(x)$ for any $g \in l_1(S,A)$.*

Proof. Let $f, g \in l_1(S)$ and $a \in A$, then, $T(eg * af) = T(eg) * T(af) = e'g * * T(af)$. Also, $T(eg * af) = T(eag * ef) = T(ag) * T(ef) = T(ag) * e'f$. So for all $f, g \in l_1(S)$, $T(ag) * e'f = e'g * T(af)$. Now let $(M, \chi) \in m(A) \times \hat{S}$ be arbitrary Taking Fourier transforms [**1**]$_1$

$$\widehat{Tag}(\overline{M}, \chi) \sum_{x \in S} \varphi_{\overline{M}}(e'f(x))\, \chi(x) = \sum_{u \in S} \varphi_{\overline{M}}(e'g(u))\, \chi(u)\, \widehat{Taf}(M, \chi)\,.$$

So that $\varphi_M(\widehat{Tag}(\overline{M}, \chi)\, e'\hat{f}(\chi)) = \varphi_{\overline{M}}(\widehat{Taf}(\overline{M}, \chi)\, e'\hat{g}(\chi))$ and since A' is semisimple $\widehat{Tag}(\overline{M}, \chi)\, e'\hat{f}(\chi) = \widehat{Taf}(\overline{M}, \chi)\, e'\hat{g}(\chi)$ for any $(\overline{M}, \chi) \in m(A) \times \hat{S}$. Choose $f, g \in \in l_1(S) \ni \hat{g}(\chi) \neq 0$ and $\hat{f}(\chi) \neq 0$, then

$$\frac{\widehat{Taf}(\overline{M}, \chi)}{e'\hat{f}(\chi)} = \frac{\widehat{Tag}(M, \chi)}{e'\hat{g}(\chi)}\,.$$

Define a function $\tau_\chi^{\bar{M}}$ on A by setting

$$\tau_\chi^{\bar{M}}(a) = \frac{\widehat{Taf}\,(\bar{M}, \chi)}{e'\hat{f}(\chi)},$$

then clearly $\tau_\chi^{\bar{M}}$ is a mapping of A into A' which is independent of the choice of $f \in l_1(S)$. Let $\tilde{M} \in m(A')$ and define the composition mapping $\varphi_{\tilde{M}} \circ T$ from $l_1(S, A)$ into $l_1(S)$. Clearly $\varphi_{\tilde{M}} \circ T$ is a continuous homomorphism such that $(\varphi_{\tilde{M}} \circ T)(ef) = \varphi_{\tilde{M}}(Tef) = f$ for any $f \in l_1(S)$. By Proposition 1, there exists $M \in m(A)$ such that $\varphi_{\tilde{M}} \circ T(f) = \varphi_M(f)$ for any $f \in l_1(S, A)$ and moreover, for any $\chi \in \hat{S}$,

$$\widehat{\varphi_{\tilde{M}}(T(af))}(\chi) = [(\varphi_{\tilde{M}} \circ T)(af)]\hat{}\,(\chi) = \varphi_M(a)\hat{f}(\chi)\,.$$

For an arbitrary $\tilde{M} \in m(A')$, $\hat{e}'(\tilde{M}_0) = \hat{e}'(\tilde{M})$ for any $\tilde{M}_0$, and so,

$$\begin{aligned}\varphi_M(a)\hat{f}(\chi) &= \sum_{x \in S} \varphi_{\tilde{M}}(e'(\widehat{Taf})(x))\,\chi(x) = \sum_{x \in S} \varphi_{\tilde{M}}(e')\varphi_{\tilde{M}}(\widehat{Taf}\,(x))\,\chi(x) \\ &= \varphi_{\tilde{M}_0}(e'\,\widehat{Taf}\,(\tilde{M}, \chi)) = \varphi_{\tilde{M}_0}(\tau_\chi^{\tilde{M}}(a))\hat{f}(\chi)\,.\end{aligned}$$

Since $l_1(S)$ is semisimple, it follows that $\varphi_{\tilde{M}_0}(\tau_\chi^{\tilde{M}}(a)) = \varphi_M(a)$ for $a \in A$. Let $\chi_1 \neq \chi_2$ and $M_1 \neq M_2$, then $\varphi_{\tilde{M}_0}(\tau_\chi^{\tilde{M}}(a)) = \varphi_M(a) = \varphi_{\tilde{M}_0}(\tau_\chi^{\tilde{M}}(a))$ and $\varphi_{\tilde{M}_0}(\tau_\chi^{M_1}(a)) = \varphi_M(a) = \varphi_{\tilde{M}_0}(\tau_\chi^{M_2}(a))$ and since A' is semisimple, $\tau_\chi^{\tilde{M}}(a) = \tau(a)$ for any $a \in A$ is independent of $(\tilde{M}, \chi) \in m(A') \times \hat{S}$. It is easy to show that τ is a continuous homomorphism and by definition

$$\tau(a) = \frac{\widehat{Taf}\,(\tilde{M}, \chi)}{e'\hat{f}(\chi)} \quad \text{for all } (M, \chi) \in m(A') \times \hat{S},$$

$l_1(S, A')$ is semisimple [**1**]$_1$, hence $Taf = \tau(a)f$ for any $f \in l_1(S)$, $a \in A$. Since finite linear combinations of af are dense in $l_1(S, A)$, and T is continuous, it follows that $Tf(x) = \tau(f(x))$ for all $f \in l_1(S, A)$.

Corollary 1. *If T is an isomorphism in Theorem* 1, *from $l_1(S, A)$ onto $l_1(S, A')$, then τ is an isomorphism from A onto A'.*

Proof. The same technique of theorem 3 of [**2**] shows that τ is $1-1$ and onto.

The author wishes to thank the referee for suggestions which have been incorporated in the revision made to this paper.

References

[1] O. Akinyele: [•]$_1$ *A generalisation of the l_1-algebra of a commutative semigroup*, Atti Accad. Naz. Lincei, Rend. **49** (1970), 17-22; [•]$_2$ *Banach algebras of the type $l_1(S, A)$*, Atti Accad. Naz. Lincei, Rend. **52** (1972), 638-643.

[2] A. Hausner, *On a homomorphism between generalized group algebras*, Bull. Amer. Math. Soc. **67** (1961), 138-141.

[3] E. Hewitt and H. S. Zukerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. **83** (1956), 70-97.

Riassunto

Sia G un gruppo commutativo compatto e A, A' algebre di Banach commutative con identità e, e'. Hausner [**2**] *ha discusso gli omomorfismi T di $B^1(G, A)$ e $B^1(G, A')$ tale che $T(ef) = e'f$ per $f \in L^1(G)$, dove $B^1(G, A)$ è costituita da tutte le funzioni di Bochner integrabili definite in G avente valori in A. Lo scopo del presente lavoro è di generalizzare i risultati in* [**2**] *all'algebra $l_1(S, A)$ discussa in* [**1**]$_1$, *dove S è un semigruppo commutativo discreto.*

* * *