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On a class of functions with positive real part (**)

1. - Introduction.

Let P denote the class of functions $p(z) = 1 + b_1 z + \ldots$ which are analytic and satisfy $\text{Re}\,(p(z)) > 0$ for all z in $E \equiv \{z: |z| < 1\}$. Considerable work has been done to study the various aspects of the above mentioned class (see [7], $[8]_1$, $[8]_2$ etc.). The study has been extended to the class $P(\alpha)$ of functions $p(z) = 1 + b_1 z + \ldots$ analytic in E and whose real part is not less than $\alpha (0 \leqslant \alpha < 1)$ in E by Padmanabhan [6], Tonti and Trahan [11], McCarty [4] etc. and used to obtain various important results. Recently a subclass of $P(\alpha)$ was investigated by Shaffer [9]. In this paper we consider the class $P(\alpha, \beta)$ of functions $p(z) = 1 + b_1 z + \ldots$ analytic in E, satisfying for all z in E the condition $|(p(z) - 1)/(p(z) + (1 - 2\alpha))| < \beta$ for some $\alpha, \beta (0 \leqslant \alpha < 1, 0 < \beta \leqslant 1)$. It is easily seen that, for $p \in P(\alpha, \beta)$ the values $p(z)$ lie inside the circle in the right half-plane with centre $(1 + (1 - 2\alpha)\beta^2)/(1 - \beta^2)$ and radius $2\beta(1 - \alpha)/(1 - \beta^2)$. Further it follows from Schwarz's lemma that if $p \in P(\alpha, \beta)$ then

$$p(z) = (1 + (2\alpha - 1)\beta z\varphi(z))/(1 + \beta z\varphi(z)) ,$$

where $\varphi(z)$ is analytic and $|\varphi(z)| \leqslant 1$ in E. Sharp coefficient estimates and a sufficient condition for a function $p(z)$ to belong to $P(\alpha, \beta)$ is obtained. Moreover we define

$$P_{2a}(\alpha, \beta) = \{p(z) = 1 + 2az + a_2 z^2 + \ldots : p \in P(\alpha, \beta)\} \qquad |a| < (1 - \alpha)\beta ,$$

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and investigate the effect of a on the radius of starlikeness of $f(z) = p(z) - 1$ where $p \in p_{2a}(\alpha, \beta)$, which generalizes the corresponding results obtained by Singh and Goel [**10**], Gupta [**3**] and Bajpai [**1**].

2. - A coefficient formula.

Theorem 1. *If* $p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ *is in* $P(\alpha, \beta)$ *for some* α, β $(0 \leqslant \alpha < 1,$ $0 < \beta \leqslant 1)$, *then* $|b_n| \leqslant 2(1-\alpha)\beta$, $n \geqslant 1$. *The estimates are sharp.*

Proof. Since $p(z) \in P(\alpha, \beta)$, we have

$$p(z) = \frac{1 + (2\alpha - 1)\omega(z)}{1 + \omega(z)}, \tag{2.1}$$

where $\omega(z) = \sum_{k=1}^{\infty} s_k z^k = z\varphi(z)$ is analytic and satisfies the condition $|\omega(z)| < \beta$ for $z \in E$. Then (2.1) gives $\big(p(z) + (1 - 2\alpha)\big)\,\omega(z) = 1 - p(z)$, or

$$\Big[2(1-\alpha) + \sum_{k=1}^{\infty} b_k z^k\Big]\Big[\sum_{k=1}^{\infty} s_k z^k\Big] = -\sum_{k=1}^{\infty} b_k z^k . \tag{2.2}$$

Equating corresponding coefficients on both sides of (2.2) we see that the coefficient b_n on the right of (2.2) depends only on $b_1, b_2, \ldots, b_{n-1}$ on the left of (2.2). Hence for $n \geqslant 1$, it follows from (2.2) that

$$\Big[2(1-\alpha) + \sum_{k=1}^{n-1} b_k z^k\Big]\omega(z) = -\Big[\sum_{k=1}^{n} b_k z^k + \sum_{k=n+1}^{\infty} c_k z^k\Big].$$

Since $|\omega(z)| < \beta$, we get

$$\beta\,\Big|2(1-\alpha) + \sum_{k=1}^{n-1} b_k z^k\Big| \geqslant \Big|\sum_{k=1}^{n} b_k z^k + \sum_{k=n+1}^{\infty} c_k z^k\Big| . \tag{2.3}$$

Squaring both sides of (2.3) and integrating about $|z| = r$, $0 < r < 1$, we obtain

$$\beta^2\Big\{4(1-\alpha)^2 + \sum_{k=1}^{n-1} |b_k|^2 r^{2k}\Big\} \geqslant \sum_{k=1}^{n} |b_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |c_k|^2 r^{2k} .$$

If we take the limit as r approaches 1, then

$$\beta^2\{4(1-\alpha)^2+\sum_{k=1}^{n-1}|b_k|^2\}\geqslant\sum_{k=1}^{n}|b_k|^2 \qquad \text{or} \qquad (1-\beta^2)\sum_{k=1}^{n-1}|b_k|^2+|b_n|^2\leqslant 4(1-\alpha)^2\beta^2 .$$

Since $0<\beta\leqslant 1$, $|b_n|^2\leqslant 4(1-\alpha)^2\beta^2$, whence follows that $|b_n|\leqslant 2(1-\alpha)\beta$, $n\geqslant 1$. The bounds are sharp for the functions

$$p(z)=\frac{1+(1-2\alpha)\beta z^n}{1-\beta z^n}$$

for $n\geqslant 1$ and $z\in E$.

3. - A sufficient condition for a function to be in $P(\alpha,\beta)$.

Theorem 2. *Let $p(z)=1+\sum_{n=1}^{\infty}b_n z^n$ be analytic in the unit disc E. If for some α,β $(0\leqslant\alpha<1,\ 0<\beta\leqslant 1)$*

$$\sum_{n=1}^{\infty}(1+\beta)|b_n|\leqslant 2(1-\alpha)\beta \,, \tag{3.1}$$

then $p(z)$ belongs to $P(\alpha,\beta)$.

Proof. We employ the same technique as used by Clunie and Keogh [2]. Thus suppose that (3.1) holds and that $p(z)=1+\sum_{n=1}^{\infty}b_n z^n$, then for $|z|<1$,

$$|p(z)-1|-\beta|p(z)+(1-2\alpha)|=|\sum_{n=1}^{\infty}b_n z^n|-\beta|2(1-\alpha)+\sum_{n=1}^{\infty}b_n z^n|$$

$$\leqslant\sum_{n=1}^{\infty}|b_n|r^n-\beta\{2(1-\alpha)-\sum_{n=1}^{\infty}|b_n|r^n\}<\sum_{n=1}^{\infty}|b_n|-2(1-\alpha)\beta+\beta\sum_{n=1}^{\infty}|b_n|=$$

$$=\sum_{n=1}^{\infty}(1+\beta)|b_n|-2(1-\alpha)\beta\leqslant 0 .$$

Hence it follows that for $z\in E$ $|(p(z)-1)/(p(z)+(1-2\alpha))|<\beta$, therefore $p\in P(\alpha,\beta)$.

We note that $p(z)=1-\{2(1-\alpha)\beta/(1+\beta)\}z^n$ is an estremal function with respect to the above theorem since $|(p(z)-1)/(p(z)+(1-2\alpha))|=\beta$ for $z=1$, $0\leqslant\alpha<1$, $0<\beta\leqslant 1$ and $n=1,2,\ldots$. We also observe that the converse to the above theorem is false in that $p(z)=(1+(1-2\alpha)\beta z)/(1-\beta z)\in P(\alpha,\beta)$ but

$$\sum_{n=1}^{\infty}\frac{(1+\beta)}{2(1-\alpha)\beta}|b_n|=\sum_{n=1}^{\infty}\frac{1+\beta}{2(1-\alpha)\beta}\cdot 2(1-\alpha)\beta^n=\sum_{n=1}^{\infty}(1+\beta)\beta^{n-1}>1\,,$$

for α, β satisfying $0\leqslant\alpha<1$, $0<\beta\leqslant 1$.

4. - The radius of starlikeness for the functions in the class $P_{2a}(\alpha,\beta)$.

To determine the radius of starlikeness for the class $P_{2a}(\alpha,\beta)$, we require the following lemmas.

Lemma 1. *If $f(z)$ is regular in E and $|f(z)|\leqslant 1$ there, then*

$$|f'(z)|\leqslant\frac{|z|^2-|f(z)|^2}{1-|z|^2}\,.$$

Lemma 2. *If $f(z)$ is regular in E and $|f(z)|\leqslant 1$ there, then*

$$\frac{|f(0)|-|z|}{1-|f(0)|\,|z|}\leqslant|f(z)|\leqslant\frac{|z|+|f(0)|}{1+|f(0)|\,|z|}\,.$$

The proof of the above lemmas can be found in ([**5**], p. 167).

Theorem 3. *Let $p\in P_{2a}(\alpha,\beta)$ and $f(z)=p(z)-1=2az+a_2z^2+\ldots$, then $f(z)$ is starlike for $|z|<r_0(\alpha,\beta)$, where*

$$r_0(\alpha,\beta)=\frac{a}{(1-\alpha)\beta+\sqrt{(1-\alpha)^2\beta^2-|a|^2}}\,.$$

The above result is sharp.

Proof. Since $p\in P_{2a}(\alpha,\beta)$, we have

$$p(z)=\frac{1+(2\alpha-1)\beta z\varphi(z)}{1+\beta z\varphi(z)}\,, \tag{4.1}$$

where $\varphi(z)$ is regular in Δ and $|\varphi(z)| \leq 1$ there. (4.1) gives

$$z\varphi(z) = \frac{1-p(z)}{\beta[p(z)+(1-2\alpha)]} = -\frac{a}{(1-\alpha)\beta}z + \dots .$$

Applying Lemma 2 to the above equation, we obtain

$$\frac{|z|(|a|-(1-\alpha)\beta|z|)}{(1-\alpha)\beta-|a|\,|z|} \leq |z\varphi(z)| \leq \frac{|z|(|a|+(1-\alpha)\beta|z|)}{(1-\alpha)\beta+|a|\,|z|} . \tag{4.2}$$

From the relation $f(z) = p(z)-1$ and (4.1), we get

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} = \operatorname{Re}\left\{\frac{zp'(z)}{p(z)-1}\right\} = \operatorname{Re}\left\{\frac{z^2\varphi'(z)+z\varphi(z)}{z\varphi(z)(1+\beta z\varphi(z))}\right\} . \tag{4.3}$$

Using Lemma 1 to (4.3), we obtain

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > \operatorname{Re}\left\{\frac{1}{1+\beta z\varphi(z)}\right\} - \frac{|z|^2-|\varphi(z)|^2}{(1-|z|^2)\,|z\varphi(z)|\,|1+\beta z\varphi(z)|} .$$

Let $|z| = r$ and $1/(1+\beta z\varphi(z)) = u+iv$, then we have

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \geq u - \frac{\beta^2r^2(u^2+v^2)-(1-u)^2-v^2}{\beta(1-r^2)\sqrt{(1-u)^2+v^2}} \equiv H(u, v, r) . \tag{4.4}$$

Differentiating H partially w.r.t. v, we obtain

$$\frac{\partial H}{\partial v} = v\left[\frac{2(1-\beta^2r^2)}{\beta(1-r^2)\sqrt{(1-u)^2+v^2}} + \frac{\beta^2r^2(u^2+v^2)-(1-u)^2-v^2}{(1-r^2)\{(1-u)^2+v^2\}^{3/2}}\right] . \tag{4.5}$$

It can be easily shown that the quantity within the square bracket on the right hand side of (4.5) is strictly positive. Therefore the minimum of H w.r.t. v occurs at $v = 0$. Putting $v = 0$ in (4.4), we have

$$h(u, r) \equiv H(u, 0, r) = u - \frac{\beta^2r^2u^2-(1-u)^2}{\beta(1-r^2)|1-u|} . \tag{4.6}$$

Also putting $v = 0$ in $1/(1+\beta z\varphi(z)) = u+iv$, we get

$$|z\varphi(z)| = \frac{|1-u|}{\beta u} . \tag{4.7}$$

From (4.2) and (4.7) we have

$$(4.8)\quad \frac{(1-\alpha)\beta+|a|r}{(1-\alpha)\beta+(1+\beta)|a|r+(1-\alpha)\beta^2r^2}\leqslant u\leqslant \frac{(1-\alpha)\beta-|a|r}{(1-\alpha)\beta-(1-\beta)|a|r-(1-\alpha)\beta^2r^2}$$

if $1-u\geqslant 0$, and

$$(4.9)\quad \frac{(1-\alpha)\beta-|a|r}{(1-\alpha)\beta-(1+\beta)|a|r+(1-\alpha)\beta^2r^2}\leqslant u\leqslant \frac{(1-\alpha)\beta+|a|r}{(1-\alpha)\beta+(1-\beta)|a|r-(1-\alpha)\beta^2r^2}$$

if $u-1\geqslant 0$.

It is easy to check that h is a monotone decreasing function of u if $1-u\geqslant 0$ and it is monotone increasing function of u if $u-1\geqslant 0$. Therefore if $1-u\geqslant 0$, minimum of h occurs at

$$u=u_1=\frac{(1-\alpha)\beta-|a|r}{(1-\alpha)\beta-(1-\beta)|a|r-(1-\alpha)\beta^2r^2}$$

and is equal to

$$h(u_1,r)=\frac{(1-\alpha)\beta(|a|-2(1-\alpha)\beta r+|a|r^2)}{(|a|-(1-\alpha)\beta r)((1-\alpha)\beta-(1-\beta)|a|r-(1-\alpha)\beta^2r^2)},$$

and if $u-1\geqslant 0$, minimum of h occurs at

$$u=u_2=\frac{(1-\alpha)\beta-|a|r}{(1-\alpha)\beta-(1+\beta)|a|r+(1-\alpha)\beta^2r^2}$$

and is equal to

$$h(u_2,r)=\frac{(1-\alpha)\beta(|a|-2(1-\alpha)\beta r+|a|r^2)}{(|a|-(1-\alpha)\beta r)((1-\alpha)\beta-(1+\beta)|a|r+(1-\alpha)\beta^2r^2)}.$$

Now it is easy to check that

$$h(u_1,r)\leqslant h(u_2,r)\quad \text{for}\quad r<\frac{|a|}{(1-\alpha)\beta}.$$

Thus the absolute minimum of h in $(0,\infty)$ will occur at $u=u_1$. Hence

$$(4.10)\quad \operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\}=$$

$$=\operatorname{Re}\left\{\frac{zp'(z)}{p(z)-1}\right\}\geqslant\frac{\beta(1-\alpha)(|a|-2(1-\alpha)\beta r+|a|r^2)}{(|a|-(1-\alpha)\beta r)((1-\alpha)\beta-(1-\beta)|a|r-(1-\alpha)\beta^2r^2)},$$

provided $r\leqslant |a|/(1-\alpha)\,\beta$.

Therefore

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} = \operatorname{Re}\left\{\frac{zp'(z)}{p(z)-1}\right\} \geqslant 0\ ,$$

for

$$|z| < r_0(\alpha, \beta) \equiv \frac{|a|}{(1-\alpha)\beta + \sqrt{(1-\alpha)^2\beta^2 - |a|^2}} \leqslant \frac{|a|}{(1-\alpha)\beta}\,.$$

The equality sign in (4.10) is attained for the function

$$p(z) = \frac{(1-\alpha)\beta - \big(1 + (1-2\alpha)\beta\big)az + (1-\alpha)(1-2\alpha)\beta^2 z^2}{(1-\alpha)\beta - (1-\beta)az - (1-\alpha)\beta^2 z^2}\,.$$

Remarks. (i) For $\beta = 1$, we get the corresponding result recently obtained by Bajpai [**1**].

(ii) $(\alpha, \beta) = (0, 1)$ leads to the corresponding results due to Gupta [**3**] and Singh and Goel [**10**].

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Summary

See Introduction.

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