Riv. Mat. Univ. Parma (4) 4 (1978), 109-114

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Theorems on fixed points (**)

The following theorem is the well-known contraction mapping theorem.

Theorem 1. *If T is a mapping of the complete metric space X into itself satisfying the inequality $\varrho(Tx, Ty) \leqslant c\varrho(x, y)$, for all x, y in X, where $0 \leqslant c < 1$, then T has a unique fixed point.*

In a paper by Kannan [3] he proved the following theorem

Theorem 2. *If T is a mapping of the complete metric space X into itself satisfying the inequality $\varrho(Tx, Ty) \leqslant c\{\varrho(x, Tx) + \varrho(y, Ty)\}$, for all x, y in X, where $0 \leqslant c < \frac{1}{2}$, then T has a unique fixed point.*

Later, see $[2]_1$, the following theorem was proved

Theorem 3. *If T is a mapping of the complete metric space X into itself satisfying the inequality $\varrho(Tx, Ty) \leqslant c\{\varrho(x, Ty) + \varrho(y, Tx)\}$, for all x, y in X, where $0 \leqslant c < \frac{1}{2}$, then T has a unique fixed point.*

We will now prove the following theorem

Theorem 4. *If T is a mapping of the complete metric space X into itself satisfying the inequality*

$$\varrho(Tx, Ty) \leqslant \max\{2c_1\varrho(x, y),\ c_2[\varrho(x, Tx) + \varrho(y, Ty)],\ c_3[\varrho(x, Ty) + \varrho(y, Tx)]\}$$

for all x, y in X, where $0 \leqslant c_1, c_2, c_3 < \frac{1}{2}$, then T has a unique fixed point.

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Proof. Let x be an arbitrary point in X. Then

$$\varrho(T^n x, T^{n+1}x) \leqslant \max\{2c_1\varrho(T^{n-1}x, T^n x), c_2[\varrho(T^{n-1}x, T^n x) + \varrho(T^n x, T^{n+1}x)], c_3\varrho(T^{n-1}x, T^{n+1}x)\}$$

$$\leqslant \max\{2c\varrho(T^{n-1}x, T^n x), c[\varrho(T^{n-1}x, T^n x) + \varrho(T^n x, T^{n+1}x)]\},$$

where $c = \max\{c_1, c_2, c_3\} < \frac{1}{2}$. It follows that either

$$\varrho(T^n x, T^{n+1}x) \leqslant 2c\varrho(T^{n-1}x, T^n x) \quad \text{or} \quad \varrho(T^n x, T^{n+1}x) \leqslant \frac{c}{1-c}\varrho(T^{n-1}x, T^n x).$$

Since $0 \leqslant c < \frac{1}{2}$, we have in either case $\varrho(T^n x, T^{n+1}x) \leqslant b\varrho(T^{n-1}x, T^n x)$, where $b = \max\{2c, c/(1-c)\} < 1$. Thus $\varrho(T^n x, T^{n+1}x) \leqslant b^n\varrho(x, Tx)$, for $n = 1, 2, \ldots$, and it follows that

$$\varrho(T^n x, T^{n+r}x) \leqslant \varrho(T^n x, T^{n+1}x) + \ldots + \varrho(T^{n+r-1}x, T^{n+r}x)$$

$$\leqslant (b^n + \ldots + b^{n+r-1})\varrho(x, Tx) \leqslant \frac{b^n}{1-b}\varrho(x, Tx),$$

for $n, r = 1, 2, \ldots$. Since $b < 1$, it follows that $\{T^n x\}$ is a Cauchy sequence in X and so has a limit z in X, since X is complete.

We now have

$$\varrho(z, Tz) \leqslant \varrho(z, T^n x) + \varrho(T^n x, Tz)$$

$$\leqslant \varrho(z, T^n x) + \max\{2c\varrho(T^{n-1}x, z), c[\varrho(T^{n-1}x, T^n x) + \varrho(z, Tz)], c[\varrho(T^{n-1}x, Tz) + \varrho(z, T^n x)]\},$$

for $n = 1, 2, \ldots$. On letting n tend to infinity we see that $\varrho(z, Tz) \leqslant c\varrho(z, Tz)$. Since $c < \frac{1}{2}$, it follows that $\varrho(z, Tz) = 0$, so that z must be a fixed point.

Now suppose that T has a second fixed point z'. Then

$$\varrho(z, z') = \varrho(Tz, Tz')$$

$$\leqslant \max\{2c\varrho(z, z'), c[\varrho(z, Tz) + \varrho(z', Tz')], c[\varrho(z, Tz') + \varrho(z', Tz)]\} = 2c\varrho(z, z'),$$

and since $c < \frac{1}{2}$, it follows that $z = z'$. Hence the fixed point is unique. This completes the proof of the theorem.

Many other similar theorems can be proved. For example, we have

Theorem 5. *If T is a mapping of the complete metric space X into itself satisfying the inequality*

$$[\varrho(Tx, Ty)]^2 \leqslant \max\{2c_1\varrho(x, y)[\varrho(x, Tx) + \varrho(y, Ty)],$$
$$2c_2\varrho(x, y)[\varrho(x, Ty) + \varrho(y, Tx)],\ c_3[\varrho(x, Tx) + \varrho(y, Ty)][\varrho(x, Ty) + \varrho(y, Tx)]\},$$

for all x, y in X, where $0 \leqslant c_1, c_2, c_3 < \frac{1}{4}$, then T has a unique fixed point.

Proof. Let x be an arbitrary point in X. Then

$$[\varrho(T^n x, T^{n+1}x)]^2 \leqslant \max\{2c_1\varrho(T^{n-1}x, T^n x)[\varrho(T^{n-1}x, T^n x) + \varrho(T^n x, T^{n+1}x)], 2c_2$$
$$\varrho(T^{n-1}x, T^n x)\varrho[T^{n-1}x, T^{n+1}x),\ c_3[\varrho(T^{n-1}x, T^n x) + \varrho(T^n x, T^{n+1}x)]\varrho(T^{n-1}x, T^{n+1}x)\}$$
$$\leqslant \max\{2c\varrho(T^{n-1}x, T^n x)[\varrho(T^{n-1}x, T^n x) + \varrho(T^n x, T^{n+1}x)],$$
$$c[\varrho(T^{n-1}x, T^n x) + \varrho(T^n x, T^{n+1}x)]^2\},$$

where $c = \max\{c_1, c_2, c_3\} < \frac{1}{4}$. It follows that either

$$[\varrho(T^n x, T^{n+1}x)]^2 \leqslant 2c\varrho(T^{n-1}x, T^n x)[\varrho(T^{n-1}x, T^n x) + \varrho(T^n x, T^{n+1}x)],$$

from which it follows that

$$\varrho(T^n x, T^{n+1}x) \leqslant [c + (c^2 + 2c)^{\frac{1}{2}}]\varrho(T^{n-1}x, T^n x),$$

or

$$\varrho(T^n x, T^{n+1}x) \leqslant \frac{c^{\frac{1}{2}}}{1 - c^{\frac{1}{2}}}\varrho(T^{n-1}x, T^n x).$$

Since $0 \leqslant c < \frac{1}{4}$, we have in either case $\varrho(T^n x, T^{n+1}x) \leqslant b\varrho(T^{n-1}x, T^n x)$, where $b = \max\{c + (c^2 + 2c)^{\frac{1}{2}}, c^{\frac{1}{2}}(1 - c)^{-\frac{1}{2}}\} < 1$. Thus $\varrho(T^n x, T^{n+1}x) \leqslant b^n\varrho(x, Tx)$, for $n = 1, 2, \dots$ and it again follows that $\{T^n x\}$ is a Cauchy sequence in X with a limit z in X.

We now have $\varrho(z, Tz) \leqslant \varrho(z, T^n x) + \varrho(T^n x, Tz)$, and

$$[\varrho(T^n x, Tz)]^2 \leqslant \max\{2c\varrho(T^{n-1}x, z)[\varrho(T^{n-1}x, T^n x) + \varrho(z, Tz)], 2c\varrho(T^{n-1}x, z)$$
$$[\varrho(T^{n-1}x, Tz) + \varrho(z, T^n x)],\ c[\varrho(T^{n-1}x, T^n x) + \varrho(z, Tz)][\varrho(T^{n-1}x, Tz) + \varrho(z, T^n x)]\},$$

for $n = 1, 2, \ldots$. On letting n tend to infinity it follows that $\varrho(z, Tz) \leqslant c^{\frac{1}{2}}\varrho\cdot(z, Tz)$. Since $c < \frac{1}{4}$, it follows that $\varrho(z, Tz) = 0$ so that z must be a fixed point.

Now suppose that T has a second fixed point z'. Then

$$\varrho(z, z') = \varrho(Tz, Tz') \leqslant 2c^{\frac{1}{2}}\varrho(z, z')\,,$$

and since $c < \frac{1}{4}$, it follows that $z = z'$. Hence the fixed point is unique. This completes the proof of the theorem.

We will now consider similar theorems on compact metric spaces. The following theorems were given in [**1**], $[\mathbf{2}]_2$ and $[\mathbf{2}]_3$ respectively.

Theorem 6. *If T is a mapping of the compact metric space X into itself satisfying the inequality $\varrho(Tx, Ty) < \varrho(x, y)$, for all distinct x, y in X, then T has a unique fixed point.*

Theorem 7. *It T is a continuous mapping of the compact metric space X into itself satisfying the inequality $\varrho(Tx, Ty) < \frac{1}{2}\{\varrho(x, Tx) + \varrho(y, Ty)\}$, for all distinct x, y in X, then T has a unique fixed point.*

Theorem 8. *If T is a continuous mapping of the compact metric space X into itself satisfying the inequality $\varrho(Tx, Ty) < \frac{1}{2}\{\varrho(x, Ty) + \varrho(y, Tx)\}$, for all distinct x, y in X, then T has a unique fixed point.*

We will now prove the following theorem

Theorem 9. *If T is a continuous mapping of the compact metric space X into itself satisfying the inequality*

$$\varrho(Tx, Ty) < \max\left\{\varrho(x, y), \tfrac{1}{2}[\varrho(x, Tx) + \varrho(y, Ty)], \tfrac{1}{2}[\varrho(x, Ty) + \varrho(y, Tx)]\right\},$$

for all distinct x, y in X, then T has a unique fixed point.

Proof. Define a real-valued function f on X by $f(x) = \varrho(x, Tx)$, for all x in X. Since ϱ and T are continuous functions it follows that f is a continuous function on X. Since X is compact there exists a point z in X such that $f(z) = \inf\{f(x) \colon x \in X\}$. Assuming that $Tz \neq z$, we have

$$\begin{aligned} f(Tz) = \varrho(Tz, T^2z) &< \max\left\{\varrho(z, Tz), \tfrac{1}{2}[\varrho(z, Tz) + \varrho(Tz, T^2z)], \tfrac{1}{2}\varrho(z, T^2z)\right\} \\ &< \max\left\{\varrho(z, Tz), \tfrac{1}{2}[\varrho(z, Tz) + \varrho(Tz, T^2z)]\right\} = \max\left\{f(z), \tfrac{1}{2}[f(z) + f(Tz)]\right\}. \end{aligned}$$

It follows that either $f(Tz) < f(z)$, which gives a contradiction, or $f(Tz) <$

$< \frac{1}{2}[f(z) + f(Tz)]$, which also gives a contradiction. It follows that our assumption was false and so we must have $Tz = z$. Thus z is a fixed point.

Now suppose that T has a second distinct fixed point z'. Then

$$\varrho(z, z') = \varrho(Tz, Tz') < \max\{\varrho(z, z'), \tfrac{1}{2}[\varrho(z, Tz) + \varrho(z', Tz')], \tfrac{1}{2}[\varrho(z, Tz') + \varrho(z', Tz)]\} = \varrho(z, z'),$$

giving a contradiction. It follows that our assumption was false and so the fixed point must be unique. This completes the proof of the theorem.

We finally prove the following theorem

Theorem 10. *If T is a continuous mapping of the compact metric space X into itself satisfying the inequality*

$$[\varrho(Tx, Ty)]^2 < \tfrac{1}{2}\max\{\varrho(x, y)[\varrho(x, Tx) + \varrho(y, Ty)], \varrho(x, y)[\varrho(x, Ty) + \varrho(y, Tx)], \tfrac{1}{2}[\varrho(x, Tx) + \varrho(y, Ty)][\varrho(x, Ty) + \varrho(y, Tx)\},$$

for all distinct x, y in X, then T has a unique fixed point.

Proof. Define a real-valued function f on X by $f(x) = \varrho(x, Tx)$ for all x in X. Since ϱ and T are continuous functions it follows that f is a continuous function on X. Since X is compact, there exists a point z in X such that $f(z) = \inf\{f(x): x \in X\}$. Assuming that $Tz \neq z$, we have

$$\begin{aligned}[f(Tz)]^2 &= [\varrho(Tz, T^2z)]^2 \\ &< \tfrac{1}{2}\max\{\varrho(z, Tz)[\varrho(z, Tz) + \varrho(Tz, T^2z)], \varrho(z, Tz)\varrho(z, T^2z), \tfrac{1}{2}[\varrho(z, Tz) + \varrho(Tz, T^2z)]\varrho(z, T^2z)\} \\ &\leqslant \tfrac{1}{2}\max\{\varrho(z, Tz)[\varrho(z, Tz) + \varrho(Tz, T^2z)], \tfrac{1}{2}[\varrho(z, Tz) + \varrho(Tz, T^2z)]^2\} = \\ &= \tfrac{1}{2}\max\{f(z)[f(z) + f(Tz)], \tfrac{1}{2}[f(z) + f(Tz)]^2\}.\end{aligned}$$

It follows that either

$$[f(Tz)]^2 < \tfrac{1}{2}f(z)[f(z) + f(Tz)] \qquad \text{or} \qquad f(Tz) < \tfrac{1}{2}[f(z) + f(Tz)].$$

Both of these inequalities imply that $f(Tz) < f(z)$, giving a contradiction. It follows that our assumption was false and so we must have $Tz = z$. Thus z is a fixed point.

Now suppose that T has a second distinct fixed point z'. Then

$$[\varrho(z, z')]^2 = [\varrho(Tz, Tz')]^2 < \tfrac{1}{2}\varrho(z, z')[\varrho(z, z') + \varrho(z', z)] = [\varrho(z, z')]^2$$

giving a contradiction. It follows that assumption was false and so the fixed point must be unique. This completes the proof of the theorem.

Again, many other similar theorems can be proved for compact metric spaces.

References

[1] M. Edelstein, *On fixed and periodic points under contractive mappings,* J. London Math. Soc. **37** (1962), 74-79.

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[3] R. Kannan, *Some results on fixed points,* Bull. Calcutta Math. Soc. **60** (1968), 71-76.

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