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BRIAN FISHER (*)

On a theorem of Khan (**)

In a recent paper, see [1], M. S. Khan gives the following

Theorem. Let S and T be mappings of the complete metric space (X, d) into itself. Suppose that there exists a non-negative real number α such that $\alpha < 1$ and

$$(1) \qquad d(Tx, Sy) \leqslant \alpha \left\{ \frac{d(x, Tx)\, d(x, Sy) + d(y, Sy)\, d(y, Tx)}{d(x, Sy) + d(y, Tx)} \right\}$$

for all x, y in X. Then S and T have a unique common fixed point.

The conditions of this theorem are insufficient since it is not explained what happens if

$$d(x, Sy) + d(y, Tx) = 0 .$$

The numerator and the denominator in the rational term in inequality (1) are then both zero. If inequality (1) is replaced by the apparently equivalent inequality

$$(2) \qquad d(Tx, Sy)\{d(x, Sy) + d(y, Tx)\} \leqslant \alpha\{d(x, Tx)\, d(x, Sy) + d(y, Sy)\, d(y, Tx)\}$$

then the theorem is false.

(*) Indirizzo: Dept. of Math., Univ. of Leicester, England.
(**) Ricevuto: 16-VI-1977.

This is easily seen by letting $X = \{x, y\}$ with the discrete metric. Define mappings S and T on X by

$$Sx = Tx = y\,, \qquad Sy = Ty = x\,.$$

Inequality (2) is satisfied with $\alpha = (1/2)$ but $S = T$ does not have a fixed point, although $ST = TS$ has two distinct fixed points.

The theorem does in fact hold if we add the extra condition that

$$d(x, Sy) + d(y, Tx) = 0$$

implies that

$$d(Tx, Sy) = 0\,.$$

For if x is an arbitrary point in X, then

$$d\big(T(ST)^{n-1}x,\ (ST)^n x\big) \leqslant \alpha \left\{\frac{d\big((ST)^{n-1}x,\ T(ST)^{n-1}x\big)\, d\big((ST)^{n-1}x,\ (ST)^n x\big)}{d\big((ST)^{n-1}x,\ (ST)^n x\big)}\right\}$$

$$= \alpha d\big((ST)^{n-1}x,\ T(ST)^{n-1}x\big)\,,$$

provided that $d\big((ST)^{n-1}x, (ST)^n x\big) \neq 0$.

Khan assumes that this is always the case, but in fact if $d\big((ST)^{n-1}x, (ST)^n x\big) = 0$, then the extra condition given above implies that

$$d\big(T(ST)^{n-1}x,\ (ST)^n x\big) = 0$$

and so $z = T(ST)^{n-1}x$ is a fixed point of S.

If we then assume that $Tz \neq z$, we have

$$d(Tz, z) = d(Tz, Sz) \leqslant \frac{2\alpha d(z, Tz)\, d(z, Sz)}{d(z, Sz) + d(z, Tz)} = 0\,,$$

giving a contradiction and it follows that z is a common fixed point of S and T.

Similarly

$$d\big((ST)^n x,\ T(ST)^n x\big) \leqslant \alpha d\big(T(ST)^{n-1}x,\ (ST)^n x\big)\,,$$

provided that $d\big(T(ST)^{n-1}x, T(ST)^n x\big) \neq 0$. If $d\big(T(ST)^{n-1}x, T(ST)^n x\big) = 0$, then it follows that $z' = (ST)^n x$ is a common fixed point of S and T.

If

$$d\big((ST)^{n-1}x,\,(ST)^n x\big),\; d\big(T(ST)^{n-1}x,\,T(ST)^n x\big) \neq 0 \qquad (n = 1, 2, \ldots)\,,$$

then it follows, as Khan shows, that the sequence $\{x, Tx, STx, \ldots, (ST)^n x, T(ST)^n x, \ldots\}$ is a Cauchy sequence with limit z''. This limit point z'' is then a common fixed point of S and T.

It is easily seen that the common fixed point, however it is obtained, is unique.

We note that without the extra condition, then if for example $d\big((ST)^{n-1}x, (ST)^n x\big) = 0$, all we can prove is that $w = (ST)^{n-1}x$ is a fixed point of ST and then that $w' = Tw$ is a fixed point of TS. The example given above shows that w is not necessarily equal to w'. If $w = w'$, then w will be a unique common fixed point of S and T.

The above remarks also apply to the other theorems given in [**1**].

Reference

[1] M. S. Khan, *A fixed point theorem for metric spaces*, Riv. Mat. Univ. Parma (4) **3** (1977), 53-57.

Sommario

Si dimostra che, date due applicazioni S e T di uno spazio metrico completo $\langle X, d\rangle$ in sè, se $(0 \leqslant \alpha < 1)$

$$d(Tx, Sy) \leqslant \alpha \left\{ \frac{d(x, Tx)\,d(x, Sy) + d(y, Sy)\,d(y, Tx)}{d(x, Sy) + d(y, Tx)} \right\} \qquad \textit{per} \quad d(x, Sy) + d(y, Tx) \neq 0\,,$$

$$d(Tx, Sy) = 0 \qquad \textit{per} \quad d(x, Sy) + d(y, Tx) = 0\,,$$

allora S e T hanno un unico punto unito, il medesimo per entrambi.

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