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Clifford formulation of classical field equations (**)

1. - Introduction.

It often happens that, because of historical reasons or real convenience, different mathematical instruments are employed in different physical theories. This natural state of affairs, however, hides the possible analogies, that could make easier the study of more complicated, or less known theories. Clifford algebra, for instance, is employed in the theory of elementary particles. However, in classical physics, its use seems to be confined to the electromagnetic field theory in vacuum or in a homogeneous and isotropic medium. In this case Clifford algebra allows to synthetize Maxwell's equations into a single Lorentz-invariant equation ([7], p. 179), ([4], p. 29), the electromagnetic field being described by an appropriate Clifford number [7]. The economy in writing in this form is considerable, while no gain in physical content is added.

Two are the purposes of this paper: i) showing that the basic equations of many classical field theories with linear and isotropic constitutive equations can be put into a form analogous to that of Maxwell's equations in vacuum; ii) formulating some extensions, valid also when the constitutive equations are general.

The use of Clifford algebra automatically assures the invariance with respect to rotations of the reference system, and, in particular, the relativistic invariance in the four-dimensional case [6].

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2. - Mathematical preliminaries.

To every n-dimensional vector space V^n with a scalar product corresponds a unique Clifford algebra of dimension 2^n ([4], p. 7) (1), whose elements, « Clifford numbers », are aggregates generally formed by a scalar, a vector, a bivector, ..., an n-vector.

Clifford product of two vectors $a, b \in V^n$ decomposes into a commutative part (scalar) and an anticommutative part (bivector), as follows

$$ab = a\cdot b + a \wedge b \,. \tag{1}$$

In the following we shall mainly employ two Clifford algebras: the real Clifford algebra of the Euclidean three-dimensional space E^3 with the usual scalar product, and the real Clifford algebra of the Minkowski space-time M^4 with pseudoeuclidean metric of signature $+\,---$. These algebras are known also as « Pauli algebra » and « Dirac algebra », respectively, and their elements as p-numbers and d-numbers, respectively ([4], p. 20 and 24).

Let us consider Pauli algebra. The unit basis vectors $\boldsymbol{e}_k \in E^3$ $(k = 1, 2, 3)$ satisfy the same multiplication rules as the Pauli matrices

$$\boldsymbol{e}_j \cdot \boldsymbol{e}_k = \tfrac{1}{2}(\boldsymbol{e}_j \boldsymbol{e}_k + \boldsymbol{e}_k \boldsymbol{e}_j) = \delta_{jk} \,. \tag{2}$$

By taking all the $2^3 = 8$ products of the $\boldsymbol{e}_k$, one obtains a tensor basis for Pauli algebra.

The unit three-vector (or pseudoscalar)

$$e \equiv \boldsymbol{e}_1 \boldsymbol{e}_2 \boldsymbol{e}_3 \tag{3}$$

is such that $e^2 = -1$, and commutes with every p-number. Formally it behaves as the imaginary unit and it is sometimes denoted by i ([4], p. 20).

If one introduces the differential operator (2)

$$\nabla \equiv \boldsymbol{e}_k \partial_k \tag{4}$$

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(1) A concise exposition of Clifford algebras particulary direct to physical applications is contined in [4].

(2) Here and in the following the usual notations of tensor calculus, as the summation convention over repeated indices, are adopted.

that formally behaves as a vector, from (1) it follows

$$
\left\{
\begin{aligned}
\nabla \boldsymbol{u} &= \nabla \cdot \boldsymbol{u} + \nabla \wedge \boldsymbol{u} = \operatorname{div} \boldsymbol{u} + e \operatorname{rot} \boldsymbol{u} \\
\boldsymbol{u} \nabla &= \nabla \cdot \boldsymbol{u} - \nabla \wedge \boldsymbol{u} = \operatorname{div} \boldsymbol{u} - e \operatorname{rot} \boldsymbol{u} ,
\end{aligned}
\right. \tag{5}
$$

where $\boldsymbol{u}$ is a general vector $\in E^3$ ([7], p. 36).

In an analogous way, in Dirac algebra, the unit basis vectors $\eta_\mu \in M^4$ ($\mu = 0, 1, 2, 3$) satisfy the same multiplication rules as the Dirac matrices

$$
\eta_\mu \cdot \eta_\nu = \tfrac{1}{2} (\eta_\mu \eta_\nu + \eta_\nu \eta_\mu) = g_{\mu\nu} \tag{6}
$$

and, by taking all the $2^4 = 16$ products of the η_μ, one obtains a tensor basis for Dirac algebra.

The unit four-vector (or pseudoscalar)

$$
\eta = \eta_0 \eta_1 \eta_2 \eta_3 \tag{7}
$$

is such that $\eta^2 = -1$, but it anticommutes with any η_μ. Some caution is necessary in replacing it with the imaginary unit i ([4], p. 26).

If the four-dimensional differential operator

$$
\square \equiv \eta^\mu \partial_\mu \tag{8}
$$

is introduced, the following generalizations of (5) are derived.

If a is a vector of M^4, i.e.

$$
a = a^\mu \eta_\mu , \tag{9}
$$

$$
\square a = \square \cdot a + \square \wedge a , \qquad a \square = \square \cdot a - \square \wedge a . \tag{10}
$$

If F is a bivector of M^4, i.e.

$$
F \equiv \frac{1}{2!} F^{\mu\nu} \eta_\mu \eta_\nu , \tag{11}
$$

$$
\square F = \square \cdot F + \square \wedge F , \qquad F \square = - \square \cdot F + \square \wedge F , \tag{12}
$$

where $\square \cdot a$, $\square \wedge a$, $\square \cdot F$, $\square \wedge F$ are the four-dimensional generalizations for a vector and a bivector fields of the usual divergence and rotor of a vector field of E^3, i.e.

(13) $\square\cdot a = \partial_\mu a^\mu$ (scalar), $\square\wedge a = \eta \frac{1}{2!}\varepsilon^{\alpha\beta\gamma\delta}\partial_\gamma\, a_\delta\, \eta_\alpha\, \eta_\beta$ (bivector),

(14) $\square\cdot F = \partial_\mu F^{\mu\nu}\eta_\nu$ (vector), $\square\wedge F = \eta \frac{1}{2!}\varepsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta}\,\eta_\alpha$ (threevector),

where $\varepsilon^{\alpha\beta\gamma\delta}$ denotes the Ricci tensor of M^4.

3. - Basic equations.

a) Let us start from the electromagnetic field theory. Its basic equations are, of course, Maxwell's equations. To stress the geometrical character of the involved entities, let us write them as follows

$$\partial_\nu G^{\mu\nu} = j^\mu\,, \qquad \frac{1}{2!}\varepsilon^{\alpha\beta\gamma\delta}\,\partial_\beta F_{\gamma\delta} = k^\alpha\,, \tag{15}$$

where, of course, k^α is usually zero. The electromagnetic field is described by two skew-symmetrical tensors $F_{\gamma\delta}$, $G^{\mu\nu}$ linked by a set of generally nonlinear constitutive equations

$$G^{\mu\nu} = G^{\mu\nu}(F_{\gamma\delta})\,. \tag{16}$$

Equations (15) furnish the four-dimensional « divergence » of $G^{\mu\nu}$ and the four-dimensional « rotor » of $F_{\gamma\delta}$, respectively.

b) In an analogous way we consider as basic equations of many stationary classical field theories the following (see table)

$$\operatorname{div}\boldsymbol{v} = \varrho\,, \qquad \operatorname{rot}\boldsymbol{u} = \boldsymbol{w}\,. \tag{17}$$

The field is described now by two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, linked by a set of generally nonlinear constitutive equations

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{u})\,, \tag{18}$$

Field	Basic equations		Const. link	References
gravitational	$\operatorname{div}\boldsymbol{h} = -\varrho$	$\operatorname{rot}\boldsymbol{g} = 0$	$\boldsymbol{h} = \frac{1}{4\pi k}\boldsymbol{g}$	([**1**], p. 155), ([**9**], p. 142)
electrostatic	$\operatorname{div}\boldsymbol{D} = \varrho$	$\operatorname{rot}\boldsymbol{E} = 0$	$\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E})$	[**8**], ([**9**], p. 154)
magnetostatic	$\operatorname{div}\boldsymbol{B} = 0$	$\operatorname{rot}\boldsymbol{H} = \boldsymbol{J}$	$\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{B})$	[**8**], ([**9**], p. 156)
electric (stationary)	$\operatorname{div}\boldsymbol{J} = 0$	$\operatorname{rot}\boldsymbol{E} = 0$	$\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{E})$	([**5**]$_2$ p. 92), ([**9**], p. 160)
thermic (stationary)	$\operatorname{div}\boldsymbol{q} = \sigma$	$\operatorname{rot}\boldsymbol{p} = 0$	$\boldsymbol{q} = k\boldsymbol{p}$	([**5**]$_3$ p. 119), ([**9**], p. 151)
perfect fluid (irrotational stationary motion)	$\operatorname{div}\boldsymbol{j} = 0$	$\operatorname{rot}\boldsymbol{v} = 0$	$\boldsymbol{j} = \varrho\boldsymbol{v}$	([**5**]$_1$, p. 2 and 16), ([**9**], p. 144)
geometric optics	$\operatorname{div}\boldsymbol{I} = 0$	$\operatorname{rot}\boldsymbol{p} = 0$	$\boldsymbol{I} = a^2\boldsymbol{p}$	([**2**], p. 113), ([**9**], p. 163)

c) Finally we consider linear acoustics. Its basic equations are continuity and Euler equations, plus the irrotationality condition [**5**]$_1$ ([**9**], p. 146). In three-dimensional form they are

$$\frac{\partial \varrho'}{\partial t} + \operatorname{div}(\varrho_0 \boldsymbol{v}) = 0\,, \qquad \frac{\partial \boldsymbol{v}}{\partial t} + \frac{\nabla p'}{\varrho_0} = 0\,, \qquad \operatorname{rot} \boldsymbol{v} = 0. \tag{19}$$

By introducing the four-dimensional vectors

$$b^\mu = (\varrho' c, \varrho_0 \boldsymbol{v})\,, \qquad a^\mu = \left(\frac{p'}{\varrho_0 c}, \boldsymbol{v}\right)$$

the equations (19) can be put into the four-dimensional form

$$\partial_\mu b^\mu = m\,, \qquad \varepsilon^{\alpha\beta\gamma\delta}\partial_\gamma a_\delta = n^{\alpha\beta}\,, \tag{20}$$

where m, $n^{\alpha\beta}$, denoting a scalar and a skew symmetric tensor, respectively, are usually, zero. Constitutive equations

$$b^\mu = b^\mu(a^\nu) \tag{21}$$

reduce to $b^\mu = \varrho^0 a^\mu$.

The Klein-Gordon field can be described by the same equations (20). In this case the constitutive link reduces to the identity of b^μ and a^μ.

In the next section the equations (17) are unified by means of Clifford algebra, independently from the nature of constitutive link. The same is done for the equations (20) and (15).

If the constitutive links are linear and isotropic, some simplifications can be made. This case is considered in sect. **5**.

4. - Clifford formulation.

In order to unify the equations (17), it is enough to add them formally. The result may be interpreted in terms of Clifford algebra, as follows

$$\operatorname{div} \boldsymbol{v} + \operatorname{rot} \boldsymbol{u} = \varrho + \boldsymbol{w}\,. \tag{22}$$

Both members of the preceding equation may be thinked as p-numbers formed by a scalar and a vector. Since, from eq. (5)

$$\operatorname{div} \boldsymbol{v} = \tfrac{1}{2}(\nabla \boldsymbol{v} + \boldsymbol{v}\nabla)\,, \qquad \operatorname{rot} \boldsymbol{u} = -\tfrac{1}{2}e(\nabla \boldsymbol{u} - \boldsymbol{u}\nabla)\,, \tag{23}$$

putting

$$w \equiv v + eu\,, \qquad \overline{w} \equiv v - eu\,, \qquad p \equiv \varrho + w\,, \tag{24}$$

eq. (22) yields the Clifford formulation:

$$\tfrac{1}{2}(\nabla \overline{w} + w\nabla) = p\,. \tag{25}$$

So the basic equations of every field theory of the table can be converted into the synthetic form (25), independently of the nature of the constitutive equations. The field is now described by a single entity, i.e. the p-number w formed of a vector and a bivector (or formally complex vector), and its complex conjugate $\overline{w}$. The constitutive link is contained in it as an inner coupling.

An analogous process can be applied to equations (20). From them by formal addition, we get

$$\partial_\mu b^\mu + \frac{1}{2!}\varepsilon^{\alpha\beta\gamma\delta}\partial_\gamma a_\delta \eta_\alpha \eta_\beta = m + \frac{1}{2!}n^{\alpha\beta}\eta_\alpha\eta_\beta\,. \tag{26}$$

The members on both sides of preceding equation can be interpreted as d-numbers formed of a scalar and a bivector. Since from (10) and (13)

$$\partial_\mu b^\mu = \tfrac{1}{2}(\Box b + b\Box)\,, \qquad \tfrac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\partial_\gamma a_\delta\eta_\alpha\eta_\beta = -\tfrac{1}{2}\eta(\Box a - a\Box)\,, \tag{27}$$

putting

$$c \equiv b + \eta a\,, \qquad q \equiv m + \frac{1}{2!}n^{\alpha\beta}\eta_\alpha\eta_\beta\,, \tag{28}$$

one obtains from (26) the Clifford formulation ([3])

$$\tfrac{1}{2}(\Box c + c\Box) = q\,. \tag{29}$$

The field is described by the d-number c formed of a vector and a threevector (or formally complex vector). As before, the constitutive link is an inner coupling.

([3]) If the identification $e_k = \eta_k\eta_0$ is made, Pauli algebra can be considered as a subalgebra of Dirac algebra ([4], p. 25). Eq. (25) can, then, assume the same form (29). In this case only the formally complex field vector is needed, not its complex conjugate.

Finally, let us consider Maxwell's equations (15). Since both are vector equations, it is not possible to add them without loosing their identity. We will proceed as follows: putting

$$G \equiv \tfrac{1}{2} G^{\mu\nu}\eta_\mu\eta_\nu\,, \qquad F \equiv \tfrac{1}{2} F^{\mu\nu}\eta_\mu\eta_\nu\,, \tag{30}$$

from (12) and (14) we get

$$\partial_\mu G^{\mu\nu}\eta_\nu = \tfrac{1}{2}(\Box G - G\Box)\,, \qquad \tfrac{1}{2!}\varepsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta}\eta_\alpha = \tfrac{1}{2}\eta(\Box F + F\Box). \tag{31}$$

Then (15) assume the form

$$\tfrac{1}{2}(\Box G - G\Box) = -J\,, \qquad \eta\tfrac{1}{2}(\Box F + F\Box) = -K\,, \tag{32}$$

where

$$J = j^\mu\eta_\mu\,, \qquad K = k^\alpha\eta_\alpha\,. \tag{33}$$

By multiplying on the left the second of the preceding equations (32) by $-\eta$, both members become three-vectors. We perform now the addition of the result just obtained with the first equation of (32) and obtain the following Clifford form for Maxwell's equations

$$\tfrac{1}{2}[\Box(F + G) + (F - G)\Box] = -J + \eta K\,. \tag{34}$$

Obviously, the bivectors F, G must have the same physical dimensions. The field is described by the two bivectors $F + G$ and $F - G$.

Another possibility involves the complex Clifford algebra of Minkowski space-time. We multiply the second of (32) by the imaginary unit i and, then, add the result to the first equation. Putting

$$H = G - i\eta F\,, \qquad h = -(J + iK)\,, \tag{35}$$

we get

$$\tfrac{1}{2}(\Box H - H\Box) = h\,. \tag{36}$$

Since both G and ηF are bivectors, the field is a true complex entity: a complex bivector. Likewise a complex entity is h also, the second member of (36): a complex vector.

5. - Linear and isotropic constitutive links.

Equations (26), (31), (34) have a validity also when the constitutive links are linear and isotropic, i.e. when (18), (16), (21) reduce to simple proportionality between the quantities involved. In particular, (34) may be written

$$\Box F = -J . \tag{37}$$

This is the Riesz formulation of Maxwell's equations in a homogeneous and isotropic medium.

Equations (17) and (20) may get a simpler form in this case. Because of the proportionality between $\boldsymbol{u}$ and $\boldsymbol{v}$, (17) can be expressed as follows

$$\operatorname{div} \boldsymbol{u} = \varrho , \qquad \operatorname{rot} \boldsymbol{u} = \boldsymbol{w} . \tag{38}$$

Then, from the first of (5) we have

$$\nabla \boldsymbol{u} = \boldsymbol{r} , \tag{39}$$

where

$$\boldsymbol{r} = \varrho + e\boldsymbol{w} \tag{40}$$

is a p-number constituted of a scalar and a bivector.

Analogously (20) may be rewritten

$$\partial_\mu a^\mu = l , \qquad \varepsilon^{\alpha\beta\gamma\delta}\, \partial_\gamma a_\delta = n^{\alpha\beta} . \tag{41}$$

By virtue of the first of (10)

$$\Box a = s , \tag{42}$$

where

$$s = l + \eta \left(\frac{1}{2\,!}\, n^{\alpha\beta}\, \eta_\alpha\, \eta_\beta \right) \tag{43}$$

is a d-number consisting of a scalar and a bivector.

6. - Conclusive remarks.

The leading inspiration in developing the preceeding sections has been the desire of arriving at a unitarian formulation of as many physical theories as possible. We have shown few fundamental equations of classical field theories expressed by means of Clifford formulation. Apparently their synthetic pattern contains something more than a formal simplicity. It represents a possibility of binding together the intimate logical processes on which the different theories are based.

A further step, which looks promising, is the extension of our analysis to quantum mechanics. In this case, since Clifford algebra may be represented with a matrix algebra, and spinors are vectors in a spin space on which the matrices operate ([**6**], p. 38), the use of Clifford algebra seems particularly well fit. Hopefully, in this way, unclear aspects of quantum mechanics, such as observables which are operators, or in general the correspondence principle, might receive some illuminating insight.

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Riassunto

Attraverso l'algebra di Clifford le equazioni fondamentali di molte teorie classiche di campo vengono riformulate in maniera sintetica, indipendentemente dalla natura del legame costitutivo.

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