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Transmission and reflection of a discontinuity wave through a characteristic shock in non linear optics (**)

1. - Introduction.

Recently, starting from some works by A. Jeffrey [5], G. Boillat and T. Ruggeri [2] developed a general theory in order to evaluate the reflected and transmitted amplitudes when a discontinuity wave comes in contact with a shock wave (including the case of characteristick shocks).

The aim of the present paper is to apply the theory to non linear electrodynamics in an isotropic dielectric medium. In particular we shall consider the case when incidence takes place on a characteristic shock, which can be developed, as we shall see, for quite general constitutive equations differently from the non characteristic shock case [1].

The paper is organized as follows: sec. **2** is devoted to the study of weak discontinuity propagation for non linear optics depending on one space variable; sec. **3** deals with shocks in one dimension; sec. **4** examines the Lax conditions for characteristic shocks; sec. **5** is concerned with reflection and transmission coefficients and finally in sec. **6** weak shocks and in sec. **7** particular strong shocks are examined.

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(1) Also the static shocks exhibit the said property as the characteristic ones but it will be shown they do not fulfil Lax conditions. An application of Jeffrey's theory to electromagnetic static shocks is performed in [**3**].

2. - Electromagnetic weak discontinuities in a non linear isotropic dielectric medium depending on one space variable.

The equations of non linear optics which we shall be concerned with have the form (²)

(2.1) $\boldsymbol{B}_t + \text{rot}(\boldsymbol{E}) = 0$, (2.2) $\text{div}(\boldsymbol{B}) = 0$,

(2.3) $\boldsymbol{D}_t - \text{rot}(\boldsymbol{H}) = 0$, (2.4) $\text{div}(\boldsymbol{D}) = 0$,

where

(2.5) $\boldsymbol{B} = \mu\boldsymbol{H}$, (2.6) $\boldsymbol{D} = \varepsilon\boldsymbol{E}$,

and the constitutive equations are

$$\mu = \text{const.}\,, \qquad \varepsilon = \varepsilon(E^2)\,. \tag{2.7}$$

When the fields depend only on the space variable x and the time t, it is immediate that eqs. (2.2) and (2.4) are fulfilled iff the components of the fields along the x axis B_1 and D_1 are constants respect to x and t. Then we can put

$$D_1 = 0\,, \qquad B_1 = 0\,,$$

without loss of generality.

So the system reduces to (2.1) and (2.3) and may be written in the compact form (³)

$$\boldsymbol{u}_t + A\boldsymbol{u}_x = 0\,, \tag{2.8}$$

where

$$\boldsymbol{u} \equiv \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{D} \end{pmatrix}, \qquad A \equiv \begin{pmatrix} 0 & \boldsymbol{n}\times \dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{D}} \\ -\dfrac{1}{\mu}\boldsymbol{n}\times & 0 \end{pmatrix}, \tag{2.9}$$

(²) $f_t = \partial f/\partial t$, $f_x = \partial f/\partial x$ for any differentiable function f.

(³) For a general approach to the problem of discontinuity waves in non linear electrodynamics and a detailed deduction of these results see [7]$_1$, [4].

and

$$\left(\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{D}}\right) = \frac{1}{\varepsilon^2}\{\varepsilon I - 2\varepsilon' \nu \boldsymbol{E} \otimes \boldsymbol{E}\}, \tag{2.10}$$

$$\nu = \frac{\varepsilon}{\varepsilon + 2\varepsilon' E^2}, \qquad \varepsilon' = \frac{\mathrm{d}\varepsilon}{\mathrm{d}E^2}, \tag{2.11}$$

$\boldsymbol{n} \equiv (1, 0, 0)$ being the direction of wave propagation.

The eigenvalues of the matrix A representing the propagation velocities are given by

$$\lambda = \mp \frac{1}{\sqrt{\varepsilon\mu}}, \qquad \mp \sqrt{\frac{\nu}{\varepsilon\mu}} \tag{2.12}$$

and the corresponding right and left eigenvectors which may be obtained easily are

$$\boldsymbol{d}(\mp \lambda_0) = \begin{pmatrix} \pm \sqrt{\frac{\mu}{\varepsilon}}\boldsymbol{E} \\ \boldsymbol{n}\times\boldsymbol{E} \end{pmatrix}, \qquad \boldsymbol{d}(\mp \lambda_0\sqrt{\nu}) = \begin{pmatrix} \mp \sqrt{\frac{\mu\nu}{\varepsilon}}\boldsymbol{n}\times\boldsymbol{E} \\ \boldsymbol{E} \end{pmatrix}, \tag{2.13}$$

where the notation $\boldsymbol{d}(\lambda)$ means the right eigenvector corresponding to the eigenvalue λ.

$$\begin{aligned} \boldsymbol{l}(\mp \lambda_0) &= \frac{1}{2E^2}\left(\pm \sqrt{\frac{\varepsilon}{\mu}}\boldsymbol{E}, \boldsymbol{n}\times\boldsymbol{E}\right), \\ \boldsymbol{l}(\mp \lambda_0\sqrt{\nu}) &= \frac{1}{2E^2}\left(\mp \sqrt{\frac{\varepsilon}{\mu\nu}}\,\boldsymbol{n}\times\boldsymbol{E}, \boldsymbol{E}\right), \quad \lambda_0 = \sqrt{\frac{1}{\varepsilon\mu}}. \end{aligned} \tag{2.14}$$

We remark that A being a real matrix we have chosen the left eigenvectors orthonormalized to the right ones according to the relation

$$\boldsymbol{l}(\lambda)\cdot\boldsymbol{d}(\lambda') = \delta_{\lambda\lambda'}, \tag{2.15}$$

$\delta_{\lambda\lambda'}$ being the Kronecker symbol.

These information on discontinuities are enough for our problem of determining reflection and transmission coefficients.

3. - Electromagnetic shocks.

The hyperbolic system (2.1), (2.3) is equivalent to the conservative form $[7]_2$, $[7]_3$

$$u_t + F_x = 0 \,, \tag{3.1}$$

where

$$F = \begin{pmatrix} n\times E \\ -n\times H \end{pmatrix}, \tag{3.2}$$

and this enables us to write the Rankine-Hugoniot equations, for shock waves, by employing the correspondence

$$\partial_t \to -s[\;]\,, \qquad \partial_x \to [\;]. \tag{3.3}$$

We have

$$-s[B] + n\times[E] = 0\,, \tag{3.4}$$ $$s[D] + n\times[H] = 0\,. \tag{3.5}$$

As usual in literature the notation, $[w] = w - w_*$, $\forall w$ indicates the value of the strong discontinuity (jump) through the shock wave-front and s is the speed of the shock propagation.

It is immediate to verify that $s = 0$ (static shock) implies

$$[E] = [E\cdot n]n = 0\,, \tag{3.6}$$ $$[H] = [H\cdot n]n = 0\,, \tag{3.7}$$

i.e. a null shock, since we have chosen the longitudinal components of the field equal zero (4).

Then on considering $s \neq 0$ from (3.4), (2.5) and (2.7) we have

$$[H] = \frac{1}{s\mu}\, n\times[E]\,, \tag{3.8}$$

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(4) In any case it is not difficult to see that $s = 0$ does not satisfy the Lax conditions [6] and so even a different choice of the longitudinal field would not give rise to static shocks.

that we introduce into (3.5) to gain

$$s^2\mu[\boldsymbol{D}]-[\boldsymbol{E}]=0\,. \tag{3.9}$$

Now from (2.6) and remembering that

$$\forall a, b:\ [ab]=\tilde{a}[b]+[a]\tilde{b}\,, \tag{3.10}$$

where the tilde stands for the average, we have

$$(s^2\tilde{\varepsilon}\mu-1)[\boldsymbol{E}]+s^2\mu\tilde{\boldsymbol{E}}[\varepsilon]=0\,. \tag{3.11}$$

We observe that when the shocks with speeds

$$s=\mp\frac{1}{\sqrt{\tilde{\varepsilon}\mu}} \tag{3.12}$$

occur, we have

$$[\varepsilon]=0\,,\qquad [E^2]=0\,. \tag{3.13}$$

In fact from (3.11) we get $\tilde{\boldsymbol{E}}[\varepsilon]=0$, that holds either when $[\varepsilon]=0$ or $\tilde{\boldsymbol{E}}=0$. In the first case, since ε is supposed to be a continuous monotonic function of E^2, it must be $[E^2]=0$ to ensure: $[\varepsilon]=\varepsilon(E^2)-\varepsilon(E_*^2)=0$. In the second case: $\boldsymbol{E}=-\boldsymbol{E}_*$ and so: $[\boldsymbol{E}^2]=[\varepsilon]=0$. Then from (3.13) we have: $\tilde{\varepsilon}=\varepsilon=\varepsilon_*$ and (3.12) becomes (5) $s=\mp\lambda_0$ (characteristic shocks).

For $[\boldsymbol{H}]$ we have from (3.8)

$$[\boldsymbol{H}]=\mp\sqrt{\frac{\varepsilon}{\mu}}\,\boldsymbol{n}\times[\boldsymbol{E}]\,. \tag{3.14}$$

It is a remarkable feature of non linear electrodynamics the fact that when a characteristic shock occurs all the eigenvalues $\lambda^{(i)}$ become continuous through the shock (Riemann invariants) since they depend only on E^2

$$[\lambda^{(i)}]=0 \qquad (i=1,2,3,4)\,. \tag{3.15}$$

(5) For a general theory of characteristic shocks see [**1**]$_1$.

4. - Lax conditions.

Lax conditions of evolution [**6**], [**2**] must be satisfied to pick up physical shocks among all mathematical solutions of the Rankine-Hugoniot equations. They form the following system of inequalities ($k = 1, 2, \dots, N$)

$$\begin{aligned} &\lambda_*^{(1)} \leqslant \lambda_*^{(2)} \leqslant \dots \leqslant \lambda_*^{(k)} \leqslant s \leqslant \lambda_*^{(k+1)} \leqslant \dots \leqslant \lambda_*^{(N)}, \\ &\lambda^{(1)} \leqslant \lambda^{(2)} \leqslant \dots \leqslant \lambda^{(k-1)} \leqslant s \leqslant \lambda^{(k)} \leqslant \dots \leqslant \lambda^{(N)}, \end{aligned} \tag{4.1}$$

where N is the dimensionality of the vector field $\boldsymbol{u}$ appearing in the system (3.1) or (2.8) and $\lambda_*^{(i)}$, $\lambda^{(i)}$ are the characteristic speeds (eigenvalues of the matrix A) computed respectively in correspondence of the values of the field $\boldsymbol{u}_*$ (region on the right of the shock) and $\boldsymbol{u}$ (region on the left of the shock), increasingly ordered taking account of multiplicity.

For our system $N = 4$. To order the eigenvalues we have to distinguish two possibilities: (a) $\varepsilon' > 0$, from which $\nu < 1$ and the ordering $\lambda^{(1)} = -\lambda_0$, $\lambda^{(2)} = - \leqslant \lambda_0\sqrt{\nu}$, $\lambda^{(3)} = +\lambda_0\sqrt{\nu}$, $\lambda^{(4)} = +\lambda_0$; (b) $-\varepsilon/2E^2 < \varepsilon' < 0$, $\varepsilon > 0$ from which $\nu > 1$ and the ordering $\lambda^{(1)} = -\lambda_0\sqrt{\nu}$, $\lambda^{(2)} = -\lambda_0$, $\lambda^{(3)} = +\lambda_0$, $\lambda^{(4)} = +\lambda_0\sqrt{\nu}$.

Linear electrodynamics is obtained when $\varepsilon' = 0$ so that $\nu = 1$ and the distinct eigenvalues are only two with double multiplicity.

In this paper we shall consider the characteristic shocks, which occur in correspondence of the exceptional waves [**4**], [**7**]$_2$, i.e. for $s = \mp\lambda_0$. In this case eq. (3.15) holds and the Lax conditions give the following results.

(a) ε' *positive*: the shock $s = -\lambda_0$ occurs for $k = 1$, the shock $s = +\lambda_0$ occurs for $k = 4$.

(b) ε' *negative*: the shock $s = -\lambda_0$ occurs for $k = 2$, the shock $s = +\lambda_0$ occurs for $k = 3$.

5. - Reflected and transmitted amplitudes.

As an application of the theory developed in [**2**] we calculate the transmission and reflection coefficients for a discontinuity wave coming in contact with a characteristic shock for our electrodynamic problem. Following [**2**] the algebraic system of the coefficients has the general form

$$\underset{\smile}{\bar{s}}[\boldsymbol{u}] + \sum_{j=1}^{k-1}\beta^{(j)}(s-\lambda^{(j)})^2\boldsymbol{d}^{(j)} - \sum_{j=k+1}^{N}\alpha^{(j)}(s-\lambda_*^{(j)})^2\boldsymbol{d}_*^{(j)} = -\Pi(s-\lambda^{(N)})^2\boldsymbol{d}^{(N)}\,, \tag{5.1}$$

in which $\underline{\overline{\dot{s}}}$ is the jump of the shock acceleration, i.e. the difference of the right and left limits of

$$\dot{s} = \frac{ds}{dt} = \frac{\partial s}{\partial t} + s\,\frac{\partial s}{\partial x}$$

calculated along the shock world-line in correspondence of the point of impact of the discontinuity wave

$$\underline{\overline{\dot{s}}} = \dot{s}_{t_{p+}} - \dot{s}_{t_{p-}}\,,$$

t_p being the impact time (see fig. 1 in ref. [2]), $\beta^{(j)}$ and $\alpha^{(j)}$ are respectively the reflection and transmission coefficients and Π is the amplitude of the incident wave.

(a) ε' *positive.* In this case the incident discontinuity wave is exceptional $\lambda^{(4)} = +\lambda_0$ and there are no problems of critical time since the discontinuity will never evolve into a shock $[\mathbf{1}]_2$. The only possible incidence involves the characteristic shock with speed $s = -\lambda_0$, $(k = 1)$. In fact if $s = +\lambda_0$ no incidence takes place because the discontinuity and the shock run the former after the latter with the same velocity.

When $s = -\lambda_0$, $(k = 1)$ the system (5.1) becomes

$$\underline{\overline{\dot{s}}}\,[\boldsymbol{u}] - \alpha^{(2)}(s-\lambda^{(2)})^2\boldsymbol{d}_*^{(2)} - \alpha^{(3)}(s-\lambda^{(3)})^2\boldsymbol{d}_*^{(3)} - \\ - \alpha^{(4)}(s-\lambda^{(4)})^2\boldsymbol{d}_*^{(4)} = -\Pi(s-\lambda^{(4)})^2\boldsymbol{d}^{(4)}\,, \tag{5.2}$$

where $\boldsymbol{d}_*^{(j)}$, $\boldsymbol{d}^{(j)}$ are the eigenvectors corresponding to ordered eigenvalues $\lambda_*^{(j)} = \lambda^{(j)}$. Note that $\boldsymbol{d}_*^{(j)}$ and $\boldsymbol{d}^{(j)}$ do not identify since they depend directly on the field $\boldsymbol{E}_*$ and $\boldsymbol{E}$ and not only on the continuous square modulus.

No $\beta^{(j)}$ coefficients appears in (5.2) since $k = 1$ implies total transmission (no reflected waves). Thanks to condition (2.15) the calculation of the unknown variables $\underline{\overline{\dot{s}}}$, $\alpha^{(j)}$ is not difficult. In fact the dot product of (5.2) and $\boldsymbol{l}_*^{(1)}$ yields

$$\underline{\overline{\dot{s}}}\,\boldsymbol{l}_*^{(1)}\cdot[\boldsymbol{u}] = -\Pi(s-\lambda^{(4)})^2\,\boldsymbol{l}_*^{(1)}\cdot\boldsymbol{d}^{(4)}\,. \tag{5.3}$$

Direct computation gives

$$\boldsymbol{l}_*^{(1)}\cdot[\boldsymbol{u}] = \frac{\varepsilon}{E^2}\boldsymbol{E}_*\times\boldsymbol{E}\cdot\boldsymbol{n}\,, \qquad \boldsymbol{l}_*^{(1)}\cdot\boldsymbol{d}^{(4)} = 0\,.$$

The former has been obtained through eq. (3.13) holding for the characteristic shocks and the Rankine-Hugoniot eqs. (3.4), (3.5) which for $s = -\lambda_0$ specialize as

$$[\boldsymbol{D}] = \varepsilon[\boldsymbol{E}]\,, \qquad [\boldsymbol{B}] = -\sqrt{\varepsilon\mu}\,\boldsymbol{n}\times[\boldsymbol{E}]\,.$$

Now it is immediate that if we exclude the shock: $\boldsymbol{E} = -\boldsymbol{E}_*$ which gives $\boldsymbol{E}_*\times\boldsymbol{E}\cdot\boldsymbol{n} = 0$, $[E^2] = 0$ we arrive at

$$\overset{\sqcap}{\underset{\sqcup}{s}} = 0\,. \tag{5.4}$$

The result (5.4) holds in general, even for the shock $E = -E_*$, as one is able to verify after computing the coefficients $\alpha^{(j)}$ as functions of $\overset{\sqcap}{\underset{\sqcup}{s}}$ and introducing them into the original system (5.2).

Then on taking the dot product of eq. (5.2) and $\boldsymbol{l}_*^{(j)}$, with $j = 2, 3, 4$ we arrive at the corresponding coefficients $\alpha^{(j)}$. After some calculations one has

$$\alpha^{(2)} = \frac{2\boldsymbol{E}_*\times\boldsymbol{E}\cdot\boldsymbol{n}}{(1-\sqrt{\nu})\sqrt{\nu}E^2}\,\Pi\,, \tag{5.5}$$

$$\alpha^{(3)} = \frac{-2\boldsymbol{E}_*\times\boldsymbol{E}\cdot\boldsymbol{n}}{(1+\sqrt{\nu})\sqrt{\nu}E^2}\,\Pi\,, \tag{5.6}$$

$$\alpha^{(4)} = \frac{\boldsymbol{E}_*\cdot\boldsymbol{E}}{E^2}\,\Pi\,. \tag{5.7}$$

Summarizing: *when an exceptional discontinuity wave is completely transmitted through a characteristic shock the jump of the acceleration of the shock vanishes as in a linear theory. The effect of non linearity appears only for the presence of the coefficients* $\alpha^{(2)}$, $\alpha^{(3)}$.

It would be an interesting problem to investigate if this result is a peculiarity of the structure of electrodynamics or it may be extended.

(b) ε' *negative* (6). *Shock* $s = -\lambda_0$, $(k = 2)$.

The algebraic system is

$$\overset{\sqcap}{\underset{\sqcup}{s}}\,[u] + \beta^{(1)}(s-\lambda^{(1)})^2\boldsymbol{d}^{(1)} - \alpha^{(3)}(s-\lambda^{(3)})^2\boldsymbol{d}_*^{(3)} - \\ - \alpha^{(4)}(s-\lambda^{(4)})^2\boldsymbol{d}_*^{(4)} = -\Pi(s-\lambda^{(4)})^2\boldsymbol{d}^{(4)} \tag{5.8}$$

(6) The incident wave $\lambda^{(4)} = +\lambda_0\sqrt{\nu}$ is not exceptional and it is intended that incidence must occur before the weak discontinuity becomes a shock (there is a critical time). Furthermore the caustic case does not occur since we are in one space dimension.

where the eigenvalues and the eigenvectors have been opportunely labelled. Now the orthonormality condition (2.15) is not enough to gain the coefficients directly as in the previous case, but calculations do not involve conceptual troubles. A simplification arises from the fact that

$$\boldsymbol{l}_*^{(1)}\cdot\boldsymbol{d}^{(4)}=0\,,\qquad \boldsymbol{l}^{(2)}\cdot\boldsymbol{d}_*^{(3)}=0\,,\qquad \boldsymbol{l}^{(3)}\cdot[\boldsymbol{u}]=0\,.$$

The results are

$$\underset{\llcorner}{\overset{\ulcorner}{s}}=\frac{(\nu-1)(1+\sqrt{\nu})}{2\varepsilon^2\mu(1-\chi)}\Pi\,,\tag{5.9}$$

$$\beta^{(1)}=\frac{1+\sqrt{\nu}}{1-\sqrt{\nu}}\cdot\frac{\chi}{1-\chi}\Pi\,,\tag{5.10}$$

$$\alpha^{(3)}=\frac{(1+\sqrt{\nu})^2\boldsymbol{E}_*\times\boldsymbol{E}\cdot\boldsymbol{n}}{8(1-\chi)\boldsymbol{E}_*\cdot\boldsymbol{E}}\Pi\,,\tag{5.11}$$

$$\alpha^{(4)}=\frac{\Pi}{1-\chi}\,,\qquad \chi=(1+\sqrt{\nu})^2\frac{\boldsymbol{E}_*\cdot[\boldsymbol{E}]}{4\sqrt{\nu}\boldsymbol{E}_*\cdot\boldsymbol{E}}\,.\tag{5.12}$$

In this case: *the incident wave is not exceptional, the effect of non linearity appears in the presence of a non vanishing jump of the shock acceleration and one reflected and two transmitted wave characterstics.*

Shock $s=+\lambda_0$, $(k=3)$. In this case the incident wave can reach the shock because the weak discontinuity is faster than s. The coefficients obey the system

$$\underset{\llcorner}{\overset{\ulcorner}{s}}[\boldsymbol{u}]+\beta^{(1)}(s-\lambda^{(1)})^2\boldsymbol{d}^{(1)}+\beta^{(2)}(s-\lambda^{(2)})^2\boldsymbol{d}^{(2)}-\alpha^{(4)}(s-\lambda^{(4)})^2\boldsymbol{d}_*^{(4)}=-\Pi(s-\lambda^{(4)})^2\boldsymbol{d}^{(4)}\,.\tag{5.13}$$

Solving the system is analogous to the previous case, with the remark that now the Rankine-Hugoniot equations yield

$$[\boldsymbol{D}]=\varepsilon[\boldsymbol{E}]\,,\qquad [\boldsymbol{B}]=+\sqrt{\varepsilon\mu}\,\boldsymbol{n}\times[\boldsymbol{E}]\,,$$

for the signature of s now is positive. It is useful to observe that

$$\boldsymbol{l}_*^{(2)}\cdot[\boldsymbol{u}]=0\,,\qquad \boldsymbol{l}^{(2)}\cdot[\boldsymbol{u}]=0\,.$$

The solutions are

$$(5.14)\qquad [\![\dot{s}]\!] = \frac{(\nu-1)(1-\sqrt{\nu})}{2\varepsilon^2\mu(1-\chi)} \cdot \frac{E^2}{\boldsymbol{E}_*\cdot\boldsymbol{E}} \Pi ,$$

$$(5.15)\qquad \beta^{(1)} = \frac{(1-\sqrt{\nu})^3}{(1+\sqrt{\nu})^3} \cdot \frac{\chi}{1-\chi} \Pi ,$$

$$(5.16)\qquad \beta^{(2)} = -\frac{(1-\sqrt{\nu})^3}{8(1-\chi)} \cdot \frac{\boldsymbol{E}_*\times\boldsymbol{E}\cdot\boldsymbol{n}}{\boldsymbol{E}_*\cdot\boldsymbol{E}} \Pi ,$$

$$(5.17)\qquad \alpha^{(4)} = \frac{E^2}{\boldsymbol{E}_*\cdot\boldsymbol{E}} \cdot \frac{\Pi}{1-\chi} ,$$

χ is still defined by (5.12).

Now: *the incident wave is not exceptional, non linearity implies a non vanishing jump of the shock acceleration and the presence of two reflected and one transmitted wave characteristics.*

On concluding this section we emphasise that the system (5.1) being a linear algebraic non homogeneous system in the unknown variables $[\![\dot{s}]\!]$, $\beta^{(j)}$, $\alpha^{(j)}$ with non vanishing determinant, we have the unicity of the solution and absence of reflected and transmitted waves (trivial solution) when the incident amplitude Π is zero. Furthermore one notes that the linear case ($\nu = 1$ corresponding to $\varepsilon' = 0$) cannot be obtained directly from the previous coefficients, but one has to come back to system (5.1) that must be written again taking account of the multiplicity of the eigenvalues.

Finally we observe that the theory developed in [**2**] requires for characteristic shocks that the following two equations are identically fulfilled

$$(5.18)\qquad [\![\dot{s}]\!] = [\![\dot{\lambda}_*^{(k)}]\!] = -\nabla_*\lambda_*^{(k)}\cdot\sum_{j=k+1}^{N}\alpha^{(j)}(\lambda_*^{(j)}-\lambda_*^{(k)})\boldsymbol{d}_*^{(j)} ,$$

$$(5.19)\qquad [\![\dot{s}]\!] = [\![\dot{\lambda}^{(k)}]\!] = -\nabla\lambda^{(k)}\cdot\Big\{\sum_{j=1}^{k-1}\beta^{(j)}(\lambda^{(j)}-\lambda^{(k)})\boldsymbol{d}^{(j)} + \Pi(\lambda^{(N)}-\lambda^{(k)})\boldsymbol{d}^{(N)}\Big\} ,$$

where ∇ is the gradient operator respect to the field $\boldsymbol{u}$. Calculations are straightforward.

6. - Weak shocks.

When the characteristic shock is a weak shock the theory [2] says

$$[\boldsymbol{u}] = h\boldsymbol{d}_*^{(k)} + \boldsymbol{O}(h^2)\,, \tag{6.1}$$

$$\beta^{(j)} = O(h) \qquad (j = 1, 2, \dots, k-1)\,, \tag{6.2}$$

$$\alpha^{(j)} = O(h) \qquad (j = k+1, \dots, N-2, N-1)\,, \tag{6.3}$$

$$\alpha^{(N)} = \Pi + O(h)\,, \tag{6.4}$$

$$[\dot{s}] = (\lambda^{(k)} - \lambda^{(N)})\nabla\lambda^{(k)}\cdot\boldsymbol{d}^{(N)}\Pi + O(h)\,. \tag{6.5}$$

We observe that in any case (6.1) specializes for the electric field in

$$[\boldsymbol{E}] = \boldsymbol{E} - \boldsymbol{E}_* = \frac{h}{\varepsilon}\,\boldsymbol{n}\times\boldsymbol{E} + \boldsymbol{O}(h^2)\,, \tag{6.6}$$

from which we have

$$\boldsymbol{E}_*\times\boldsymbol{E}\cdot\boldsymbol{n} = \frac{h}{\varepsilon}\,E^2 + O(h^2)\,, \tag{6.7}$$

$$\boldsymbol{E}_*\cdot\boldsymbol{E} = E^2 + O(h^2)\,. \tag{6.8}$$

Now we introduce these expressions into the coefficients formulae and consider first order approximation for weak shocks

(a) ε' *positive. Shock* $s = -\lambda_0$, $[\dot{s}] = 0$,

$$\alpha^{(2)} = \frac{2\Pi}{(1-\sqrt{\nu})\sqrt{\nu}}\cdot\frac{h}{\varepsilon} + O(h^2)\,, \tag{6.9}$$

$$\alpha^{(3)} = \frac{-2\Pi}{(1+\sqrt{\nu})\sqrt{\nu}}\cdot\frac{h}{\varepsilon} + O(h^2)\,, \tag{6.10}$$

$$\alpha^{(4)} = \Pi + O(h^2)\,. \tag{6.11}$$

(b) ε' *negative. Shock* $s = -\lambda_0$,

$$[\dot{s}] = \frac{(\nu-1)(1+\sqrt{\nu})}{2\varepsilon^2\mu}\,\Pi + O(h^2)\,, \tag{6.12}$$

(6.13) $$\beta^{(1)} = O(h^2)\,,$$

(6.14) $$\alpha^{(3)} = \frac{(1+\sqrt{\nu})^3 \Pi}{8\varepsilon}\, h + O(h^2)\,,$$

(6.15) $$\alpha^{(4)} = \Pi + O(h^2)\,, \qquad \chi = O(h^2)\,.$$

Shock $s = +\lambda_0$,

(6.16) $$\underset{\llcorner\!\lrcorner}{\overset{\ulcorner\!\urcorner}{\dot{s}}} = \frac{(\nu-1)(1-\sqrt{\nu})}{2\varepsilon^2\mu}\,\Pi + O(h^2)\,,$$

(6.17) $$\beta^{(1)} = O(h^2)\,,$$

(6.18) $$\beta^{(2)} = -\frac{(1-\sqrt{\nu})^3 \Pi}{8\varepsilon}\, h + O(h^2)\,,$$

(6.19) $$\alpha^{(4)} = \Pi + O(h^2)\,.$$

Observation: *in any case the only relevant coefficient is the one relative to the fastest transmitted wave and its value equals the incident amplitude as in the linear theory. When the incident discontinuity is not exceptional the effect of non linearity appears in the value of* $\underset{\llcorner\!\lrcorner}{\overset{\ulcorner\!\urcorner}{\dot{s}}}$ *that is of the same order as* $\alpha^{(4)}$.

7. - Particular strong shocks.

An interesting situation arises when

(7.1) $$\boldsymbol{E}_* \cdot \boldsymbol{E} = 0\,.$$

For characteristic shocks (7.1) is compatible with the Rankine-Hugoniot equations (3.13) iff

(7.2) $$\boldsymbol{E} = \pm\, \boldsymbol{n} \times \boldsymbol{E}_*\,.$$

When introduced into the coefficient expressions these equations provide the following limit values

(a) ε' *positive. Shock* $s = -\lambda_0$, $\underset{\llcorner\!\lrcorner}{\overset{\ulcorner\!\urcorner}{\dot{s}}} = 0$,

(7.3) $$\alpha^{(2)} = \pm \frac{2\Pi}{(1-\sqrt{\nu})\sqrt{\nu}}\,,$$

$$\alpha^{(3)} = \mp \frac{2\Pi}{(1+\sqrt{\nu})\sqrt{\nu}}, \tag{7.4}$$

$$\alpha^{(4)} = 0 . \tag{7.5}$$

(b) *ε′ negative. Shock* $s = -\lambda_0$,

$$[\dot{s}] = 0 , \tag{7.6}$$

$$\beta^{(1)} = -\frac{1+\sqrt{\nu}}{1-\sqrt{\nu}}\Pi, \tag{7.7}$$

$$\alpha^{(3)} = \pm \frac{(1+\sqrt{\nu})\sqrt{\nu}}{2}\Pi, \tag{7.8}$$

$$\alpha^{(4)} = 0 , \qquad \chi \to \infty . \tag{7.9}$$

Shock $s = +\lambda_0$,

$$[\dot{s}] = -\frac{2(1-\sqrt{\nu})^2\sqrt{\nu}}{\varepsilon^2\mu(1+\sqrt{\nu})}\Pi, \tag{7.10}$$

$$\beta^{(1)} = -\frac{(1-\sqrt{\nu})^3}{(1+\sqrt{\nu})^3}\Pi, \tag{7.11}$$

$$\beta^{(2)} = \mp \frac{(1-\sqrt{\nu})^3\sqrt{\nu}}{2(1+\sqrt{\nu})^2}\Pi, \tag{7.12}$$

$$\alpha^{(4)} = \frac{4\sqrt{\nu}}{(1+\sqrt{\nu})^2}\Pi. \tag{7.13}$$

Concluding: *the characteristic shocks* $\boldsymbol{E} = \pm \boldsymbol{n}\times\boldsymbol{E}_*$ *give the peculiar result that when* $[\dot{s}] = 0$ *also the coefficient* $\alpha^{(4)}$ *vanishes. This fact is a clear consequence of non linearity since the linear theory always yields* $\alpha^{(4)} = \Pi$.

Finally it is important to observe that condition (7.1) is compatible with the law of evolution of the characteristic shock $[7]_2$ which ensures mutual orthogonality of $\boldsymbol{E}$ and $\boldsymbol{E}_*$ to be conserved in time when imposed as initial condition.

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Sommario

Viene presentata un'applicazione di una teoria sviluppata in un lavoro di G. Boillat e T. Ruggeri all'ottica non lineare per l'interazione di un'onda di discontinuità e di una onda d'urto in una dimensione spaziale. Si ricavano le ampiezze di riflessione e di trasmissione corrispondenti ad un segnale incidente su un urto caratteristico. Si analizzano gli urti deboli ed un caso particolare.

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