Riv. Mat. Univ. Parma (4) 5 (1979), 27-41

Antonio Romano (*)

A macroscopic theory of thermoelastic dielectrics (**)

A Giorgio Sestini per il suo 70° compleanno

1. - Introduction

Elastic dielectrics constitute a well studied class of materials owing to their related phenomenological bases (photoelasticity, piezoelectricity, electrostriction, ferroelectricity, etc.).

In 1956, R. A. Toupin $[\mathbf{15}]_1$ proposed a *statical theory* of non-linear elastic dielectrics in the absence of thermal phenomena. This Author, employing a variational principle in which the Lagrangian is fixed « a priori », deduced the local statical equation of dielectric, Maxwell's equations of electrostatics and finally boundary conditions. Moreover, in order to justify his choice of the Lagrangian, he showed that the local statical equations could be also deduced from a model in which the dielectric is regarded as a continuous distribution of dipoles. In a following paper, R. A. Toupin $[\mathbf{15}]_2$ proved the existence of infinite equivalent decompositions of the stress tensor in « mechanical » and « electrostatical » parts. In one of these decompositions, the electric stress tensor is identical to that produced by a continuous distribution of polarization charges. Starting from a different variational principle, A. C. Eringen $[\mathbf{3}]_1$ obtained results (see also $[\mathbf{3}]_2$) similar to those of Toupin.

A *dynamical theory* of non-linear elastic dielectrics (always in the absence of thermal phenomena) by R. A. Toupin $[\mathbf{15}]_3$ was developed employing the balance equations of momentum, energy and angular momentum. Once more, the equations of balance are written by using Lorentz's model for the inter-

(*) Indirizzo: Via Domenico Fontana, 80128 Napoli, Italy.

(**) Work made under the auspices of G.N.F.M. (C.N.R.). — Ricevuto: 23-V-1978.

action between the dielectric and the electromagnetic field. In this model, the dielectric is regarded as a fictitious continuous distribution of polarization charges and polarization and magnetization currents. However, it must be observed that in this theory Maxwell's equations, which are covariant with respect to Lorentzian transformations, are associated with the classical equations of motion which are covariant with respect to Galileian transformations.

In the aforsaid papers there are two fundamental restrictions: (a) the constitutive equations of the dielectric are dependent only on the deformation gradient $\boldsymbol{F}$ and the polarization $\boldsymbol{P}$; (b) there are no thermal effects. On the other hand, C. Mead [**8**] showed experimentally that the capacitance of a condenser does not go to infinity when the distance between the plates tends to zero but reaches a finite value. This effects has been deduced theoretically by R. D. Mindlin $[\mathbf{9}]_{1,2}$ using a linear theory of a dielectric based on a variational principle, in which the specific internal energy depends on $\boldsymbol{F}$, $\boldsymbol{P}$ and grad $\boldsymbol{P}$. E. S. Suhubi [**12**] supplied a variational principle for the statics of a non-linear dielectric whose specific internal energy depends on $\boldsymbol{F}$, $\boldsymbol{P}$ and grad $\boldsymbol{P}$.

Restriction (b) (but not (a)) was removed by H. F. Tiersten $[\mathbf{13}]_1$ who obtained the balance equations of momentum, angular momentum and energy, of a heat conducting dielectric by employing a suitable structural model (1). In this paper, the dielectric $\mathscr{S}$ is regarded as the whole of two distinct interacting continua: the lattice continuum $\mathscr{S}_1$ and the electronic continuum $\mathscr{S}_2$ which can move infinitesimally with respect to $\mathscr{S}_1$. Moreover, $\mathscr{S}_2$ interacts with $\mathscr{S}_1$ by a local electric field ${}_L\boldsymbol{E}$ that exerts a couple $\boldsymbol{P}\times{}_L\boldsymbol{E}$ per unit volume of $\mathscr{S}_1$. A similar couple is produced by the electric field $\boldsymbol{E}$. Finally, on $\mathscr{S}_1$ exterior body forces act as contact stress, the body couples $-\boldsymbol{P}\times{}_L\boldsymbol{E}$ (produced by $\mathscr{S}_2$) and the electric force $\boldsymbol{P}$. grad $\boldsymbol{E}$.

More recently, G. A. Maugin [**6**] obtained the general equations of the non-linear theory of interaction between thermoelastic and electric phenomena, employing the « virtual power principle » already applied by P. Germain $[\mathbf{5}]_{1,2}$ to thermomechanics of continua.

In concluding this brief review of the already proposed theories in $[\mathbf{15}]_1$, [**6**], we can observe that: (1) they derive results about the macroscopic behavior of a dielectric $\mathscr{S}$ by structural models; (2) each of them assumes a well-defined model of the interaction between matter and electric field. Similar considerations can also be made in regard to the paper [**6**] because the virtual power

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(1) This is essentially similar to the model employed in the study of ferromagnetic substances by H. F. Tiersten $[\mathbf{13}]_2$, H. F. Tiersten and C. Tsai [**14**], W. F. Brown $[\mathbf{1}]_1$, $[\mathbf{1}]_2$, G. A. Maugin and C. A. Eringen $[\mathbf{7}]_1$, $[\mathbf{7}]_2$.

principle leads to unkown multipliers whose physical meaning is again obtained employing « a posteriori » a particular structural model of the continuum and a particular interaction between matter and electric field.

In this paper, I propose a different point of view (2) for the analysis of dielectrics, namely that nothing need be presumed regarding the structure of matter as in the usual Continuum Mechanics of simple continua. In fact, it seems interesting to me to require if the usual procedures of Continuum Mechanics are able to lead to correct results also for more complicated materials and if it is possible to obtain them regardless of any model.

To this end, I postulate equations of balance of momentum, energy and angular momentum for the whole system of matter and electric field which exibit fluxes of energy and angular momentum more general than the corresponding ones adopted in the theory of simple continua. These variables are regarded as constitutive quantities because the theory is phenomenological and there is no model to give them a physical interpretation « a priori ». Moreover, I postulate the balance of angular momentum must be satisfied in every process. In other words, Maxwell's equations and the balance of momentum and energy being sufficient to determine the unknown processes when the constitutive equations are also given, the balance of angular momentum can be interpreted as a further restriction on these constitutive equations.

Specifically, after giving equivalent formulations of Maxwell's equations and after recalling the equations of balance I proposed in $[\mathbf{11}]_{1,2}$ (sections **2**, **3**), I also adopt the following constitutive equations (section **4**)

$$\zeta = \zeta(\boldsymbol{F}, \theta, \boldsymbol{p}, \operatorname{grad}\boldsymbol{p}, \operatorname{grad}\theta)\,, \tag{1}$$

where ζ is a constitutive quantity, $\boldsymbol{F}$ the deformation gradient, θ the local temperature and $\boldsymbol{p}$ the specific polarization. In a phenomenological theory of dielectrics without memory, functions (1) represent the simplest form of the constitutive equations of dielectrics. Adopting them, we assume to be more interested in the electric effects than in the mechanical ones because, from the mechanical point of view, the dielectric is only thermoelastic. Restrictions imposed to the constitutive equations are deduced by the dissipation principle.

Finally, in section **6**, I show that in the absence of the electric dissipation the previous results coincide with those obtained by other authors. In particular, the local electric field and the local equilibrium condition are natural consequences.

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(2) This point of view has also been expressed in another paper of mine $[\mathbf{11}]_3$ regarding ferromagnetic substances.

2. - Preliminary considerations on the Maxwell-Minkowsky equations in quasi-static electricity

A continuous dielectric $\mathcal{S}$ satisfies the conditions of quasi-static electricity if: (1) The velocity of points of $\mathcal{S}$ are non-relativistic. (2) The electrothermodynamical processes of $\mathcal{S}$ occur at low frequencies. (3) The system $\mathcal{S}$ is non-magnetizable.

Introducing the characteristic quantities T, U, L, E, with the dimensions of time, velocity, lenght, electric field, we can easily prove that following relations hold

$$(2.1)\quad R_c \equiv \frac{U^2}{c^2} \ll 1\,, \qquad R_t \equiv \frac{TU}{L} \simeq 1\,, \qquad R_e \equiv \frac{H}{\varepsilon_0\, UE} \simeq 1\,, \qquad R_m \equiv \frac{H}{\varepsilon_0 UEL} \simeq \frac{1}{L}\,,$$

where c denotes the velocity of light in vacuum. Relations $(2.1)_{1,2}$ express quantitatively conditions (1), (2). The other ones follow from hypothesis (3), when we use the law of transformation

$$(2.2)\qquad \boldsymbol{H}^0 = \boldsymbol{H} - \dot{\boldsymbol{x}} \times \boldsymbol{D}\,,$$

which connect the magnetic field $\boldsymbol{H}$ in the frame I to the magnetic field $\boldsymbol{H}^0$ in the proper frame I^0, moving with velocity $\dot{\boldsymbol{x}}$ with respect to I. In (2.2), $\boldsymbol{D}$ denotes the electric induction.

A simple dimensionless analysis leads to the following Maxwell-Minkowky equations for the quasi-static electricity

$$(2.3)\qquad \operatorname{rot} \boldsymbol{E} = 0\,, \qquad \operatorname{div} \boldsymbol{D} = 0\,, \qquad \operatorname{rot} \boldsymbol{H} = \frac{\partial D}{\partial t}\,, \qquad \operatorname{div} \boldsymbol{B} = 0\,,$$

being $\boldsymbol{E}$ the electric field, $\boldsymbol{B}$ the magnetic induction. Moreover (2.3), are covariant with respect to Galileian transformations when we use, together with (2.2), the following other laws of transformation

$$(2.4)\qquad \boldsymbol{E}^0 = \boldsymbol{E}\,, \qquad \boldsymbol{D}^0 = \boldsymbol{D}\,, \qquad \boldsymbol{B}^0 = \boldsymbol{B} - \frac{\dot{\boldsymbol{x}}}{c^2} \times \boldsymbol{E}\,,$$

which are deduced by the relativistic laws when (2.1) hold. If $\boldsymbol{P}$ and $\boldsymbol{M}$ denote respectively the electric polarization and magnetic polarization

$$(2.5)\qquad \mu_0 \boldsymbol{M} = \boldsymbol{B} - \mu_0 \boldsymbol{H}\,, \qquad \boldsymbol{P} = \boldsymbol{D} - \varepsilon_0 \boldsymbol{E}\,,$$

where μ_0 and ε_0 are the magnetic and electric permeability of empty space, we can write (2.3) in two other equivalent forms that will be useful later on.
In the meanwhile, from (2.2), (2.4) and condition (4) (i.e. $\boldsymbol{M}^0 = \boldsymbol{0}$), there follows

$$\boldsymbol{P} = \boldsymbol{P}^0, \qquad \boldsymbol{M} = -\,\dot{\boldsymbol{x}} \times \boldsymbol{P}\,. \tag{2.6}$$

If we prefer to employ the fields $\{\boldsymbol{E}, \boldsymbol{P}, \boldsymbol{H}, \boldsymbol{M}\}$ instead of $\{\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}\}$ equations (2.3) become

$$\begin{aligned} &\operatorname{rot} \boldsymbol{E} = 0\,, \qquad \varepsilon_0 \operatorname{div} \boldsymbol{E} = -\operatorname{div} \boldsymbol{P} \equiv \varrho_P\,, \\ &\operatorname{rot} \boldsymbol{H} = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t}\,, \qquad \mu_0 \operatorname{div} \boldsymbol{H} = -\mu_0 \operatorname{div} \boldsymbol{M} \equiv \varrho_M\,. \end{aligned} \tag{2.7}$$

We can obtain these equations from Maxwell's ones for empty space *replacing $\mathscr{S}$ with continuous fictitious distributions of electric and magnetic dipoles.*

Likewise, if we prefer to employ $\{\boldsymbol{E}, \boldsymbol{P}, \boldsymbol{B}, \boldsymbol{M}\}$ instead of $\{\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}\}$ equations (2.3), owing to $(2.5)_1$, $(2.6)_2$, become

$$\begin{aligned} &\operatorname{rot} \boldsymbol{E} = 0\,, \qquad \varepsilon_0 \operatorname{div} \boldsymbol{E} = -\operatorname{div} \boldsymbol{P} \equiv \varrho_P\,, \\ &\operatorname{rot} \Big(\frac{\boldsymbol{B}}{\mu_0} + \dot{\boldsymbol{x}} \times \boldsymbol{P}\Big) = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t}\,, \qquad \operatorname{div} \boldsymbol{B} = 0\,. \end{aligned} \tag{2.8}$$

We conclude observing that $(2.8)_3$ can also be written

$$\operatorname{rot} \frac{\boldsymbol{B}}{\mu_0} = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \overset{*}{\boldsymbol{P}} - (\operatorname{div} \boldsymbol{P})\, \dot{\boldsymbol{x}}\,, \tag{2.9}$$

where

$$\overset{*}{\boldsymbol{P}} = \frac{\partial \boldsymbol{P}}{\partial t} + (\operatorname{div} \boldsymbol{P})\, \dot{\boldsymbol{x}} + \operatorname{rot} (\boldsymbol{P} \times \dot{\boldsymbol{x}})$$

is the convected time derivative of $\boldsymbol{P}$.

3. - Equations of balance of momentum and energy for a polarizable continuous system in quasi-static electricity. Reduced dissipation inequality

Let $\mathscr{S}$ be a continuous polarizable system moving with respect to Galileian frame I. Accepting approximations (1), ..., (4), we shall ignore all relativistic contributions to momentum and energy. So, if $\mathscr{C}$ denotes a material volume of $\mathscr{S}$ and $\boldsymbol{n}$ the exterior unit normal vector to surface element $\mathrm{d}\sigma$ of boundary $\partial\mathscr{C}$ of $\mathscr{C}$, the equations of balance of momentum and energy, can be written as

$$
(3.1)\qquad
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{C}} \varrho\dot{\boldsymbol{x}}\,\mathrm{d}\mathscr{C} = \int_{\partial\mathscr{C}} \boldsymbol{T}\cdot\boldsymbol{n}\,\mathrm{d}\sigma + \int_{\mathscr{C}} \varrho\boldsymbol{b}\,\mathrm{d}\mathscr{C}\,,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{C}} \varrho\Big(\frac{1}{2}\dot{x}^2 + e\Big)\,\mathrm{d}\mathscr{C} = \int_{\partial\mathscr{C}} (\dot{\boldsymbol{x}}\cdot\boldsymbol{T} + \boldsymbol{\Phi})\cdot\boldsymbol{n}\,\mathrm{d}\sigma + \int_{\mathscr{C}} \varrho(\boldsymbol{b}\cdot\dot{\boldsymbol{x}} + r)\,\mathrm{d}\mathscr{C}\,,
\end{aligned}
$$

where ϱ = mass density, $\boldsymbol{T}$ = total stress tensor, e = specific internal energy, $\boldsymbol{\Phi}$ = energy flux vector, $\boldsymbol{b}$ = specific external body force, r = external energy supply.

These equations are invariant with respect to Galileian transformations provided $\boldsymbol{T}$, $\boldsymbol{b}$, e, $\boldsymbol{\Phi}$, r are invariant.

As previously $[\mathbf{11}]_{1,2}$ I assume

$$
(3.2)\qquad \boldsymbol{\Phi} = -\boldsymbol{E}^0\times\boldsymbol{H}^0 + \boldsymbol{h} + \boldsymbol{\tau}\cdot\dot{\boldsymbol{p}}\,, \qquad \Big(\boldsymbol{p}\equiv\frac{\boldsymbol{P}}{\varrho}\Big)
$$

being $-\boldsymbol{E}^0\times\boldsymbol{H}^0$ the *electromagnetic energy flux vector*, $\boldsymbol{h}$ the *heat flux vector* and $\boldsymbol{\tau}$ is an *extra flux of energy* due to the presence of polarization. When $\boldsymbol{\tau} = \boldsymbol{0}$, we obtain again the expression for $\boldsymbol{\Phi}$ adopted by B. Coleman and E. Dill $[\mathbf{2}]_{1,2}$ in their thermodynamics of rigid conductors in electromagnetic fields. On the contrary, if $\boldsymbol{\tau}\neq\boldsymbol{0}$, beside $-\boldsymbol{E}^0\times\boldsymbol{H}^0$ and $\boldsymbol{h}$, (3.2) presents another flux of energy due to the variations of polarization. Quantity $\boldsymbol{\tau}$ has to be considered as a constitutive quantity because we have no model to assigne any expression to it. I shall prove that $\boldsymbol{\tau} = \boldsymbol{0}$ when constitutive equations do not depend on grad $\boldsymbol{p}$.

Under suitable smoothness assumptions, (3.1) yields the following local forms

$$
(3.3)\qquad
\begin{aligned}
\varrho\ddot{\boldsymbol{x}} &= \operatorname{div}\boldsymbol{T} + \varrho\boldsymbol{b}\,,\\
\varrho\dot{e} &= \boldsymbol{T}\colon \operatorname{grad}\dot{\boldsymbol{x}} - \operatorname{div}(\boldsymbol{E}^0\times\boldsymbol{H}^0) + \operatorname{div}(\boldsymbol{h} + \boldsymbol{\tau}\cdot\dot{\boldsymbol{p}}) + \varrho r\,.
\end{aligned}
$$

In order to put (3.3) in a more convenient form, let us observe that (2.2) and (2.4) imply

$$
\begin{aligned}
\boldsymbol{E}^0\times\boldsymbol{H}^0 &= \boldsymbol{E}\times(\boldsymbol{H} - \dot{\boldsymbol{x}}\times\boldsymbol{D}) = \boldsymbol{E}\times\boldsymbol{H} - \boldsymbol{E}\times(\dot{\boldsymbol{x}}\times\boldsymbol{D})\\
&= \boldsymbol{E}\times\boldsymbol{H} + (\boldsymbol{E}\cdot\dot{\boldsymbol{x}})\,\boldsymbol{D} - (\boldsymbol{E}\cdot\boldsymbol{D})\,\dot{\boldsymbol{x}}\,.
\end{aligned}
$$

On the other hand, in view of (2.3), we have

$$\mathrm{div}\,(\boldsymbol{E}\times\boldsymbol{H}) = \boldsymbol{H}\cdot\mathrm{rot}\boldsymbol{E} - \boldsymbol{E}\cdot\mathrm{rot}\,\boldsymbol{H} = -\boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t} = -\boldsymbol{E}\cdot\dot{\boldsymbol{D}} + \dot{\boldsymbol{x}}\cdot\mathrm{grad}\,\boldsymbol{D}\cdot\boldsymbol{E}\ (^3)\,,$$

$$\mathrm{div}\,[(\boldsymbol{E}\cdot\dot{\boldsymbol{x}})\,\boldsymbol{D} - (\boldsymbol{E}\cdot\boldsymbol{D})\,\dot{\boldsymbol{x}}] = \boldsymbol{D}\cdot\mathrm{grad}\,\boldsymbol{E}\cdot\dot{\boldsymbol{x}} + \boldsymbol{D}\cdot\mathrm{grad}\,\dot{\boldsymbol{x}}\cdot\boldsymbol{E}$$

$$= \dot{\boldsymbol{x}}\cdot\mathrm{grad}\,\boldsymbol{E}\cdot\boldsymbol{D} - \dot{\boldsymbol{x}}\cdot\mathrm{grad}\,\boldsymbol{D}\cdot\boldsymbol{E} - \boldsymbol{E}\cdot\boldsymbol{D}\,\mathrm{div}\,\dot{\boldsymbol{x}}\,,$$

and so we can say that

$$\mathrm{div}\,(\boldsymbol{E}^0\times\boldsymbol{H}^0) = -\boldsymbol{E}\cdot\dot{\boldsymbol{D}} + [\boldsymbol{E}\otimes\boldsymbol{D} - (\boldsymbol{E}\cdot\boldsymbol{D})\,\boldsymbol{I}]:\mathrm{grad}\,\dot{\boldsymbol{x}} + 2\boldsymbol{D}\cdot(\mathrm{grad}\,\boldsymbol{E})^A\cdot\dot{\boldsymbol{x}}\ (^4)\,, \tag{3.4}$$

where $(\mathrm{grad}\,\boldsymbol{E})^A$ is the skew-symmetric part of grad $\boldsymbol{E}$. Moreover, by $(2.3)_1$ it results $(\mathrm{grad}\,\boldsymbol{E})^A_{ij} = \boldsymbol{E}_{[i,j]} = -\frac{1}{2}\varepsilon_{lij}(\mathrm{rot}\,\boldsymbol{E})^l = 0$, so that we can give (3.4) the following form

$$-\mathrm{div}\,(\boldsymbol{E}^0\times\boldsymbol{H}^0) = \boldsymbol{E}\cdot\dot{\boldsymbol{D}} + [(\boldsymbol{E}\cdot\dot{\boldsymbol{D}})\,\boldsymbol{I} - \boldsymbol{E}\otimes\boldsymbol{D}]:\mathrm{grad}\,\dot{\boldsymbol{x}}\,. \tag{3.5}$$

Taking into account this last relation, (3.2) can be written as follows

$$\begin{aligned} \varrho\ddot{\boldsymbol{x}} &= \mathrm{div}\,\boldsymbol{T} + \varrho\dot{\boldsymbol{b}}\,,\\ \varrho\dot{e} &= [T + (\boldsymbol{E}\cdot\boldsymbol{D})\,\boldsymbol{I} - \boldsymbol{E}\otimes\boldsymbol{D}]:\mathrm{grad}\,\dot{\boldsymbol{x}} + \boldsymbol{E}\cdot\dot{\boldsymbol{p}} + \mathrm{div}\,(\boldsymbol{h} + \boldsymbol{\tau}\cdot\dot{\boldsymbol{p}}) + \varrho r\,. \end{aligned} \tag{3.6}$$

These equations, together with (2.3) represent the fundamental system of quasi-static electricity of moving bodies, when the electromagnetic fields are decribed by the vectors $\{\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}\}$. However, to facilitate the comparison between the theory here developed and the others already proposed, it will be suitable express (3.6) in terms of either the fields $\{\boldsymbol{E}, \boldsymbol{P}, \boldsymbol{H}, \boldsymbol{M}\}$ or $\{\boldsymbol{E}, \boldsymbol{P}, \boldsymbol{B}, \boldsymbol{M}\}$. We shall begin with the first choice.

We have the identities

$$\begin{aligned} \boldsymbol{E}\cdot\dot{\boldsymbol{D}} &= \frac{1}{2}\varrho\frac{\mathrm{d}}{\mathrm{d}t}\Big(\varepsilon_0\frac{E^2}{\varrho}\Big) - \frac{1}{2}\varepsilon_0 E^2\boldsymbol{I}:\mathrm{grad}\,\dot{\boldsymbol{x}} + \varrho\boldsymbol{E}\cdot\dot{\boldsymbol{p}} - (\boldsymbol{E}\cdot\boldsymbol{P})\,\boldsymbol{I}:\mathrm{grad}\,\dot{\boldsymbol{x}}\,,\\ -\mathrm{div}\,[\tfrac{1}{2}\varepsilon_0 E^2\boldsymbol{I} - \boldsymbol{E}\otimes\boldsymbol{D}] &= -\varepsilon_0\,\mathrm{grad}\,\boldsymbol{E}\cdot\boldsymbol{E} + \boldsymbol{E}\,\mathrm{div}\,\boldsymbol{D} + \boldsymbol{D}\cdot\mathrm{grad}\,\boldsymbol{E} = \boldsymbol{P}\cdot\mathrm{grad}\,\boldsymbol{E}\,. \end{aligned} \tag{3.7}$$

(3) Here, $\dot{\boldsymbol{x}}\cdot\mathrm{grad}\,\boldsymbol{D}\cdot\boldsymbol{E} = H^i D_{i,j}\dot{x}^j$.

(4) $\boldsymbol{E}\otimes\boldsymbol{D}:\mathrm{grad}\,\dot{\boldsymbol{x}} = E^i D^j \dot{x}_{i,j}$.

If we use these relations and the following definitions

$$\varepsilon \equiv e - \varepsilon_0 \frac{E^2}{2\varrho}, \qquad \boldsymbol{t} \equiv \boldsymbol{T} + \frac{1}{2}\varepsilon_0 E^2 \boldsymbol{I} - \boldsymbol{E} \otimes \boldsymbol{D}, \tag{3.8}$$

the previous equations of balance (3.6) assume the form

$$\begin{aligned} \varrho\ddot{\boldsymbol{x}} &= \operatorname{div} \boldsymbol{t} + \boldsymbol{P}\cdot \operatorname{grad} \boldsymbol{E} + \varrho \boldsymbol{b}, \\ \varrho\dot{\varepsilon} &= \boldsymbol{t} : \operatorname{grad} \dot{\boldsymbol{x}} + \varrho \boldsymbol{E}\cdot\dot{\boldsymbol{p}} + \operatorname{div}(\boldsymbol{h} + \boldsymbol{\tau}\cdot\dot{\boldsymbol{p}}) + \varrho r . \end{aligned} \tag{3.9}$$

In deriving (3.9), we have found the electric force $\boldsymbol{P}\cdot\operatorname{grad}\boldsymbol{E}$ in $(3.9)_1$ and the energetic term $\boldsymbol{E}\cdot\dot{\boldsymbol{p}}$ in $(3.9)_2$ by adopting the fields $\{\boldsymbol{E}, \boldsymbol{P}, \boldsymbol{H}, \boldsymbol{M}\}$ instead of $\{\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}\}$, without assuming any model of interaction between matter and electric field.

Besides the equations of balance (3.9), we shall adopt the principle of entropy in the form of Clausiu-Duhem inequality

$$\varrho\theta\dot{\eta} \geqslant \operatorname{div} \boldsymbol{h} - \frac{\boldsymbol{h}\cdot\operatorname{grad}\theta}{\theta} + \varrho r, \tag{3.10}$$

being η the *specific entropy*. Consequently, the reduced dissipation inequality, obtained by eliminating $\operatorname{div}\boldsymbol{h} + \varrho r$ between $(3.9)_2$ and (3.10), results in

$$-\varrho[\dot{\psi} + \eta\dot{\theta}] + \boldsymbol{t} : \operatorname{grad} \dot{\boldsymbol{x}} + \varrho\boldsymbol{E}\cdot\dot{\boldsymbol{p}} + \operatorname{div}(\boldsymbol{\tau}\cdot\dot{\boldsymbol{p}}) + \frac{\boldsymbol{h}\cdot\operatorname{grad}\theta}{\theta} \geqslant 0, \tag{3.11}$$

where $\psi = \varepsilon - \theta\eta$ is the *thermodynamical potential* or the *specific free energy*.

In order to write (3.11) in the coordinate (X^L) of a reference configuration $\mathscr{C}_*$, we remember the identity (5) $(\partial/\partial X^L)(\mathscr{I}(\partial X^L/\partial x^i)) = 0$ and the equation of continuity $\varrho_* = \mathscr{I}\varrho$, where ϱ_* is the mass density in $\mathscr{C}_*$, $\mathscr{I}$ is the determinant of the gradient of deformation $\boldsymbol{F} = \|\partial x^i/\partial X^L\|$. Employing the previous relations, (3.9), (3.10) can be written as

$$\begin{aligned} \varrho_*\ddot{\boldsymbol{x}} &= \operatorname{Div} \boldsymbol{t}_* + \varrho_* \operatorname{Grad} \boldsymbol{E}(\boldsymbol{F}^{-1})\cdot\dot{\boldsymbol{p}} + \varrho_* \boldsymbol{b}, \\ \varrho_*\dot{\varepsilon} &= \boldsymbol{t}_* : \dot{\boldsymbol{F}}^T + \varrho_* \boldsymbol{E}\cdot\dot{\boldsymbol{p}} + \operatorname{Div}(\boldsymbol{h}_* + \boldsymbol{\tau}_*\cdot\dot{\boldsymbol{p}}) + \varrho_* r, \end{aligned} \tag{3.12}$$

$$-\varrho_*[\dot{\psi} + \eta\dot{\theta}] + \boldsymbol{t}_* : \dot{\boldsymbol{F}}^T + \varrho_* \boldsymbol{E}\cdot\dot{\boldsymbol{p}} + \operatorname{Div}(\boldsymbol{\tau}\cdot\dot{\boldsymbol{p}}) + \frac{\boldsymbol{h}_*\cdot\operatorname{Grad}\theta}{\theta} \geqslant 0 \tag{3.13}$$

(5) See, for instance, $[\mathbf{3}]_2$.

being

$$(3.14)\qquad \boldsymbol{t}_* = \mathscr{I}\boldsymbol{t}(\boldsymbol{F}^{-1})^T, \qquad \boldsymbol{h}_* = \mathscr{I}\boldsymbol{F}^{-1}\boldsymbol{h}\,, \qquad \boldsymbol{\tau}_* = \mathscr{I}\boldsymbol{F}^{-1}\boldsymbol{\tau} \quad \left(\tau^L_{*i} = \mathscr{I}(F^{-1})^L_j\,\tau^j_i\right)\,.$$

The operators Grad and Div express derivations with respect to coordinate (X^L).

4. - Constitutive equations and their restrictions derived by the dissipation principle

$(2.7)_1$, (3.12) represent a system of ten differential equations whose unknown functions are given by the fields

$$(4.1)\qquad \varepsilon,\ t^L_{*i},\ E^i,\ h^L_*,\ \tau^L_*,\ p^i,\ \theta$$

and by the equations of motion

$$(4.2)\qquad x^i = x^i(\boldsymbol{X}, t)\,.$$

System (4.2), $(2.7)_1$, (3.12) will lead to the determination of the fields $\boldsymbol{p}$, θ, $\boldsymbol{x}(\boldsymbol{X}, t)$ when the constitutive equations are given expressing the fields $\boldsymbol{t}_*$, $\boldsymbol{E}$, $\boldsymbol{h}_*$, $\boldsymbol{\tau}_*$ as functions of $\boldsymbol{p}$, θ, $\boldsymbol{F}$ and their derivatives.

If ζ denotes any of the quantities ψ, η, $\boldsymbol{t}_*$, $\boldsymbol{E}$, $\boldsymbol{h}_*$, $\boldsymbol{\tau}_*$, I assume the following constitutive equations

$$(4.3)\qquad \zeta = \hat{\zeta}(\boldsymbol{F}, \theta, \boldsymbol{p}, \operatorname{Grad}\boldsymbol{p}, \operatorname{Grad}\theta)\,,$$

which trivially verify the principles of determinism, local action and equipresence.

By employing these hypotheses, we can write (3.13) in the form

$$(4.4)\qquad -\varrho_*[\dot{\psi} + \eta\dot{\theta}] + \boldsymbol{t}_* : \dot{\boldsymbol{F}}^T + \varrho_*[\boldsymbol{E} + \operatorname{div}\boldsymbol{\tau}_*]\cdot\dot{\boldsymbol{p}} + \boldsymbol{\tau}_*\cdot\operatorname{Grad}\dot{\boldsymbol{p}} + \frac{\boldsymbol{h}_*\cdot\operatorname{Grad}\theta}{\theta} \geqslant 0\,.$$

On the other hand, we have

$$(4.5)\qquad \dot{\psi} = \frac{\partial\hat{\psi}}{\partial F^i_L}\dot{F}^i_L + \frac{\partial\hat{\psi}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\psi}}{\partial p^i}\dot{p}^i + \frac{\partial\hat{\psi}}{\partial p^i_{,L}}\dot{p}^i_{,L} + \frac{\partial\psi}{\partial\theta_{,L}}\dot{\theta}_L\,,$$

so that (4.4) can be written as

$$(4.6)\quad \begin{aligned} &-\varrho_* \frac{\partial \hat\psi}{\partial \theta_{,L}} \dot\theta_{,L} - \varrho_* [\eta + \frac{\partial \hat\psi}{\partial \theta}] \dot\theta + [t^L_{*i} - \varrho_* \frac{\partial \hat\psi}{\partial F^i_L}] \dot F^i_L + \\ &+ \varrho_* [E_i - \frac{\partial \hat\psi}{\partial p^i} + \frac{1}{\varrho_*} (\tau^L_{*i})_{,L}] \dot p^i + [\tau^L_{*i} - \varrho_* \frac{\partial \hat\psi}{\partial p^i_{,L}}] \dot p^i_{,L} + \frac{h^L_*}{\theta} \theta_{,L} \geqslant 0 \,. \end{aligned}$$

This inequality must be identically satisfied in every process satisfying equations $(2.7)_1$ and (3.12). Owing to the possibility of selecting arbitrarily the external sources $\boldsymbol{b}$, r it is immediately seen that $(2.7)_1$ and (3.12) are satisfied in every process. Then (4.6) is equivalent to the following relations

$$(4.7)\quad \begin{aligned} &\psi = \hat\psi(\boldsymbol{F}, \theta, \boldsymbol{p}, \operatorname{Grad} \boldsymbol{p}) \,, \qquad \eta = -\frac{\partial \hat\psi}{\partial \theta} \,, \qquad t^L_{*i} = \varrho_* \frac{\partial \hat\psi}{\partial F^i_L} \,, \\ &\tau^{*L}_{i} = \varrho_* \frac{\partial \hat\psi}{\partial p^i_{,L}} \,, \qquad E_i = \frac{\partial \hat\psi}{\partial p^i} - \frac{1}{\varrho_*} \tau^L_{*i,L} \,, \qquad h^L_* \theta_{,L} \geqslant 0 \,. \end{aligned}$$

We can say that ψ, which depends only on $\boldsymbol{F}$, θ, $\boldsymbol{p}$, $\operatorname{Grad} \boldsymbol{p}$, is a thermodynamical potential for η, $\boldsymbol{t}_*$, $\boldsymbol{\tau}_*$ and $\boldsymbol{E}$. Moreover, there is no heat conduction when $\operatorname{Grad} \theta = \boldsymbol{0}$.

In concluding this section, we observe that (4.7) can be written in the following eulerian form

$$(4.8)\quad \begin{aligned} &\psi = \tilde\psi(\boldsymbol{F}, \theta, \boldsymbol{p}, \operatorname{Grad} \boldsymbol{p}) \,, \qquad \eta = -\frac{\partial \tilde\psi}{\partial \theta} \,, \quad t^i_j = \varrho F^i_L \frac{\partial \tilde\psi}{\partial F^j_L} \,, \\ &\tau^L_{*i} F^j_L = \varrho \mathscr{I} \frac{\partial \tilde\psi}{\partial p^i_{,j}} \equiv \mathscr{I} \tau^j_i \,, \quad E_i = \frac{\partial \tilde\psi}{\partial p^i} - \frac{1}{\varrho} (\varrho \frac{\partial \psi}{\partial p^i_{,h}})_{,h} \equiv \frac{\partial \tilde\psi}{\partial p^i} - \frac{1}{\varrho} \tau^h_{i,h} \,. \end{aligned}$$

5. - Restrictions on constitutive equations derived by the balance of angular momentum

In the previous discussion, we employed only the principles of balance of momentum and energy (3.12), Maxwell's equations $(2.7)_1$ and the principle of entropy. Giving constitutive equations compatible with (4.7), system (3.12), (2.7) permits us to determine the fundamental fields $\boldsymbol{x}(\boldsymbol{X}, t)$, $\theta(\boldsymbol{X}, t)$, $\boldsymbol{p}(\boldsymbol{X}, t)$.

We have now to consider the balance of angular momentum beside (3.1), i.e. the equations

$$(5.1)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{C}} \varrho x^{[i}\dot{x}^{j]}\,\mathrm{d}\mathscr{C} = \int_{\delta\mathscr{C}} (x^{[i}T^{j]h} + A^{ijh})\, n_h\, \mathrm{d}\sigma + \int_{\mathscr{C}} \varrho x^{[i} b^{j]}\,\mathrm{d}\mathscr{C}\,,$$

where $A^{ijh} = - A^{jih} =$ extra-flux of angular momentum. On account of $(3.3)_1$, (5.1) can be written in the following local form

$$(5.2)\qquad T^{[ji]} + A^{ijh}{}_{,h} = 0$$

which, when we use $\{\boldsymbol{E}, \boldsymbol{P}, \boldsymbol{H}, \boldsymbol{M}\}$ to describe the electromagnetic field, becomes (see (3.8))

$$(5.3)\qquad t^{[ji]} + \varrho E^{[j} p^{i]} + A^{ijh}{}_{,h} = 0\,.$$

This last equation is now regarded as a further restriction on the constitutive equations, already subjected to the restrictions (4.7), because we do not need it to determine the motion of the system $\mathscr{S}$.

In order to derive the implications of such a restriction, we recall that the objectivity of ψ implies

$$(5.4)\qquad \frac{\partial\tilde{\psi}}{\partial F^{[j}_{L}} F^{i]}_{L} + \frac{\partial\tilde{\psi}}{\partial p^{[j}} p^{i]} + \frac{\partial\tilde{\psi}}{\partial p^{[j}_{.L}} p^{i]}_{.L} = 0\,,$$

which, by $(4.8)_{3,4,5}$, can be written as

$$t_{[j}{}^{i]} + \varrho E_{[j} p^{i]} + (\tau^{h}_{[j} p^{i]})_{,h} = 0\,.$$

By comparing this last equation with (5.3), we obtain the conditions

$$(A^{ijh} - \tau^{[jL} p^{i]})_{,h} = 0$$

which in material coordinates assume the form (see (4.8))

$$(5.5)\qquad (A^{ijL}_{*} - \tau^{[jL}_{*} p^{i]})_{,L} = 0\,,$$

with

$$(5.6)\qquad A^{ijL}_{*} = \mathscr{I}(F^{-1})^{L}_{h} A^{ijh}.$$

Next, let us consider the constitutive equation for A_* such as (4.3). If we use, for the sake of simplicity, the notation $f^{ijL} = A^{ijL}_{*} - \tau^{[jL}_{*} p^{i]}$, (5.5) can assume the form

$$\frac{\partial f^{ij(L}}{\partial F^h_{M)}} x^h{}_{,LM} + \frac{\partial f^{ijL}}{\partial \theta} \theta_{,L} + \frac{\partial f^{ijL}}{\partial p^h} p^h{}_{,L} + \frac{\partial f^{ij(L}}{\partial p^h{}_{,M)}} p^h_{,LM} = 0 , \tag{5.7}$$

which are valid in every process. So (5.7) is equivalent to the following restriction on the constitutive equation

$$\frac{\partial f^{ij(L}}{\partial F^h_{M)}} = \frac{\partial f^{ij(L}}{\partial p^h_{,M)}} = 0 , \qquad \frac{\partial f^{ijL}}{\partial \theta} \theta_{,L} + \frac{\partial f^{ijL}}{\partial p^h} p^h{}_{,L} = 0 . \tag{5.8}$$

It is well known that $(5.8)_1$ determines the dependence of f^{ijL} on F^h_L and $p^h{}_{,L}$ and the objectivity principle implies f^{ijL} is a function only of θ, $\boldsymbol{p}$, $\operatorname{Grad} \boldsymbol{p}$ (see [**4**]).

We can conclude, supposing

$$A^{ijL}_{*} = \tau^{[iL}_{*} p^{j]} \tag{5.9}$$

i.e. $f^{ijL} = 0$, that the balance of angular momentum is trivially satisfied.

6. - Comparison with the results of other authors

In order to compare the obtained results with the ones proposed by other authors, we observe the balance of energy $(3.9)_2$ assumes the form

$$\varrho \dot{\varepsilon} = \boldsymbol{t} \colon \operatorname{grad} \dot{\boldsymbol{x}} + \varrho \, [\boldsymbol{E} + \frac{1}{\varrho} \operatorname{div} \boldsymbol{\tau}] \cdot \dot{\boldsymbol{p}} + \boldsymbol{\tau} \colon \operatorname{grad} \dot{\boldsymbol{p}} + \operatorname{div} \boldsymbol{h} + \varrho r . \tag{6.1}$$

On the other hand, by introducing the *local electric field*

$$- {}_L\boldsymbol{E} \equiv \boldsymbol{E} + \frac{1}{\varrho} \operatorname{div} \boldsymbol{\tau} , \tag{6.2}$$

equations (3.9) of balance can be written as

$$\begin{aligned} \varrho \ddot{\boldsymbol{x}} &= \operatorname{div} \boldsymbol{t} + \boldsymbol{P} \cdot \operatorname{grad} \boldsymbol{E} + \varrho \boldsymbol{b} , \\ \varrho \dot{\varepsilon} &= \boldsymbol{t} \colon \operatorname{grad} \dot{\boldsymbol{x}} - \varrho \, {}_L\boldsymbol{E} \cdot \dot{\boldsymbol{p}} + \boldsymbol{\tau} \colon \operatorname{grad} \dot{\boldsymbol{p}} + \operatorname{div} \boldsymbol{h} + \varrho r , \end{aligned} \tag{6.3}$$

where, in view of (4.7) and (6.2), it results

$$t^i_j = \varrho \, F^i_L \frac{\partial \bar{\psi}}{\partial F^j_L} , \qquad \tau^i_j = \varrho \frac{\partial \bar{\psi}}{\partial p^j{}_{,i}} , \qquad {}_LE_i = - \frac{\partial \bar{\psi}}{\partial p^i} . \tag{6.4}$$

(6.3) and (6.4) constitute the equations obtained in $[\mathbf{7}]_{1,2}$, [**6**] adopting structural models or variational principles whereas they have been obtained simply by adopting the variable $\{\boldsymbol{E}, \boldsymbol{P}, \boldsymbol{H}, \boldsymbol{M}\}$ to describe the electromagnetic fields. Moreover (6.2), written in the form

$$\boldsymbol{E} + {}_L\boldsymbol{E} + \frac{1}{\varrho}\operatorname{div}\boldsymbol{\tau} = \boldsymbol{0}\,, \tag{6.5}$$

coincides with the *local equilibrium condition* (11.15) in [**4**] or (4.8) in [**6**]. Finally, the balance of angular momentum, in view of (5.3), (6.2), results

$$t^{[ji]} = \varrho\, {}_L E^{[j} p^{i]} - \tau^{[jh} p^{i]}{}_{,h}\,. \tag{6.6}$$

It is obvious from (6.4) that if ψ does not depend on $\operatorname{grad}\boldsymbol{p}$, it follows $\boldsymbol{\tau} = \boldsymbol{0}$ and (6.3), (6.4), (6.5), (6.6) coincide with the result of $[\mathbf{15}]_1, [\mathbf{13}]_1, [\mathbf{3}]_1$. We observe explicitly that the equations (11.14) in $[\mathbf{11}]_3$ are not compatible with the dependence of ψ on $\operatorname{grad}\boldsymbol{p}$ because it is assumed $\boldsymbol{\tau} = \boldsymbol{0}$ and in the same time ψ depending on $\operatorname{grad}\boldsymbol{p}$.

In order to derive the results of $[\mathbf{15}]_3$ from the developed theory, we observe that $(3.6)_2$ can be written as

$$\begin{aligned}\varrho\dot{\varepsilon} = {} & [\boldsymbol{T} + \varepsilon_0 \boldsymbol{E}\cdot\boldsymbol{E}\boldsymbol{I} - \varepsilon_0 \boldsymbol{E}\otimes\boldsymbol{E}]:\operatorname{grad}\dot{\boldsymbol{x}} + (\boldsymbol{E}\cdot\boldsymbol{P}\boldsymbol{I} - \boldsymbol{E}\otimes\boldsymbol{P}):\operatorname{grad}\dot{\boldsymbol{x}} \\ & + \varepsilon_0 \boldsymbol{E}\cdot\dot{\boldsymbol{E}} + \boldsymbol{E}\cdot\dot{\boldsymbol{P}} + \operatorname{div}(\boldsymbol{h} + \boldsymbol{\Phi}) + \varrho r\,.\end{aligned} \tag{6.7}$$

On the other hand, it results

$$\varepsilon_0 \boldsymbol{E}\cdot\dot{\boldsymbol{E}} = \varrho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\varepsilon_0 E^2}{2\varrho}\right) - \frac{1}{2}\varepsilon_0 E^2 \boldsymbol{I}:\operatorname{grad}\dot{\boldsymbol{x}}\,,$$

$$\overset{*}{\boldsymbol{P}} = \frac{\partial\boldsymbol{P}}{\partial t} + \dot{\boldsymbol{x}}\operatorname{div}\boldsymbol{P} + \operatorname{rot}(\boldsymbol{P}\times\dot{\boldsymbol{x}}) = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} + \boldsymbol{P}\operatorname{div}\dot{\boldsymbol{x}} - \boldsymbol{P}\cdot\operatorname{grad}\dot{\boldsymbol{x}}\,,$$

so that (6.7) assumes the form

$$\varrho\dot{\varepsilon}' = \boldsymbol{t}':\operatorname{grad}\dot{\boldsymbol{x}} + \boldsymbol{E}\cdot\overset{*}{\boldsymbol{P}} + \operatorname{div}(\boldsymbol{h} + \boldsymbol{\Phi}) + \varrho r\,, \tag{6.8}$$

where $\varepsilon' = \varepsilon - (\varepsilon_0 E^2)/2\varrho$, (6.9) $$\boldsymbol{t}' = \boldsymbol{T} + \tfrac{1}{2}\varepsilon_0 E^2 \boldsymbol{I} - \varepsilon_0 \boldsymbol{E} \otimes \boldsymbol{E}\,,$$

while $(3.6)_1$ becomes

(6.10) $$\varrho\ddot{\boldsymbol{x}} = \operatorname{div}\boldsymbol{t}' - (\operatorname{div}\boldsymbol{P})\boldsymbol{E} + \varrho\boldsymbol{b}\,.$$

In the approximations of quasi-static electricity and when $\boldsymbol{\tau} = \boldsymbol{0}$, (6.8) and (6.10) coincide with the equations proposed by R. Toupin $[\mathbf{15}]_3$. In fact, they differ only by a term $\boldsymbol{P}^* \times \boldsymbol{B}$ which is present in the balance of momentum proposed in $[\mathbf{15}]_3$ but non in (6.10). The term $\boldsymbol{P}^* \times \boldsymbol{B}$ is of the 2-th order in U^2/c^2 compared with $-$ (div $\boldsymbol{P})\boldsymbol{E}$ in the quasi-static electricity, as it is displayed by the following adimensional analysis (see, sect. **2**)

$$(\operatorname{div}\boldsymbol{P})\boldsymbol{E} \quad \sim PE/L\,,$$

$$(\overset{*}{\boldsymbol{P}} \times \boldsymbol{B}) \quad \sim \frac{PU}{L}\mu_0 H \sim \frac{PE}{L}\,\frac{U^2 H}{c^2\,\varepsilon_0\, UE} \sim \frac{PE}{L}\,\frac{U^2}{c^2}\,.$$

Therefore, (6.10) coincides with (5.2) in $[\mathbf{15}]_3$.

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Sunto

In questo lavoro, senza adottare alcun modello di interazione tra materia e campo elettromagnetico ed impiegando una forma generale per le equazioni del bilancio, deduco le restrizioni termodinamiche per le equazioni costitutive che conseguono dal principio di dissipazione e dal bilancio del momento angolare. In tal modo pervengo ad una teoria dei dielettrici deformabili che include come casi particolari i risultati già noti.

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