Riv. Mat. Univ. Parma (4) 5 (1979), 43-53

MARIO MARTELLI (*)

Non-repulsive fixed point theorems and applications to fixed point theory (**)

A GIORGIO SESTINI per il suo 70° compleanno

Introduction

The definition of *non-repulsive fixed point* goes back to F. Browder [**1**] (1965) who proved the following stronger version of Schauder's fixed point theorem.

« Let C be a compact, convex, infinite dimensional subset of a Banach space E and let $f: C \to C$ be continuous. Then f has a non-repulsive fixed point. »

Non-repulsive fixed points didn't get too much attention until Zabreiko-Krasnosel'skii [**13**] (1971) and Steinlein [**12**] (1972) proved independently the so called « mod p » theorem. This result captured immediately the interest of many mathematicians, the theory of non-repulsive fixed points, strictly related to that result, received a strong impulse and many applications of the theory have been given ever since (see for example [**9**]$_1$, [**10**]$_1$).

The purpose of the present paper is to present some new results on the existence of non-repulsive fixed points and to use non-repulsive fixed point theory for producing some new fixed point theorems and for presenting a different proof of some others that have been already proved by means of different techniques.

(*) Indirizzo: Department of Mathematics, University of California, Davis, California, 95616, U.S.A.; Istituto Matematico « U. Dini », Viale Morgagni 67/A, Università, 50134 Firenze, Italy.

(**) Ricevuto: 23-V-1978.

1. - Notations and definitions

Let Y be a nonempty subset of a topological space X and let $f: Y \to Y$ be continuous. A point $x_0 \in Y$ is said to be *a repulsive fixed point* for f (see [**1**]) if (i) $f(x_0) = x_0$; (ii) there exists a neighborhood U of x_0 such that for any neighborhood V of x_0 there exists an n_0 with the property that $\bigcup_{n \geq n_0} f^n(Y \setminus V) \subset Y \setminus U$.

A similar definition is given in the case when f is a multi-valued-map. We shall denote multi-valued maps by capital letters and we will use the symbol $F: Y \multimap Y$.

Let X be a metric space and $A \subset X$ be a bounded set. Following Kuratowski [**7**], define $\alpha(A)$ as the infimum of all $\varepsilon > 0$ such that A admits a covering with a finite number of sets with diameter less than ε.

The following properties of α will be used in this paper.

(1) $\alpha(A) = 0$ if and only if A is totally bounded.

(2) If $A \subset B$ and B is bounded, then $\alpha(A) \leqslant \alpha(B)$.

(3) If A, B are bounded subsets of X, then $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(4) If A is a bounded subset of a normed space E then $\alpha(A) = \alpha(\overline{\text{co}}\,(A))$ (see [**2**]).

(5) If A, B are bounded subsets of E then $\alpha(A+B) \leqslant \alpha(A) + \alpha(B)$.

(6) If $D = \{x \in E: \|x\| \leqslant r\}$ and $\dim E = +\infty$ then $\alpha(D) = 2r$ (see [**5**]).

A map $f: X \to X$ is said to be *α-Lipschitz with constant k* if $\alpha(f(A)) \leqslant k\alpha(A)$ for any bounded set $A \subset X$. Clearly an α-Lipschitz map sends bounded sets into bounded sets. If $k < 1$, then f is called an *α-contraction*. If $\alpha(f(A)) < \alpha(A)$, for any bounded set $A \subset X$ with $\alpha(A) > 0$, then f is said to be *condensing*. Similar definitions are given for multi-valued maps.

2. - Results

We begin with a very elementary lemma, which is stated for single-valued maps, but it holds as well for the multi-value ones.

Lemma 1. *Let X be a topological space and let $f: X \to X$ be a continuous map. Let $Y \subset X$ be invariant under f (i.e., $f(Y) \subset Y$) and such that $f|_Y$ has a non-repulsive fixed point in Y. Then f has a non-repulsive fixed point in X.*

Proof. Let $x_0 = f(x_0)$ be non-repulsive for $f|_Y$. Assume that x_0 is repulsive for f. Then there exists a neighborhood U of x_0 such that for any neighborhood V of x_0 there exists an n_0, depending on V, such that $f^n(X\setminus V) \subset X\setminus U$ for any $n \geqslant n_0$. Since $Y\setminus V = (X\setminus V) \cap Y$, $Y\setminus U = (X\setminus U) \cap Y$, $f(Y) \subset Y$, and $f(A \cap B) \subset f(A) \cap f(B)$ we have

$$f^n(Y\setminus V) = f^n((X\setminus V) \cap Y) \subset f^n(X\setminus V) \cap f^n(Y) \subset f^n(X\setminus V) \cap Y \subset (X\setminus U) \cap Y$$
$$= Y\setminus U .$$

Therefore, x_0 is repulsive for $f|_Y$. This contradiction shows that x_0 is non-repulsive for f. Q.E.D.

Browder's result deals with compact infinite dimensional convex sets C of a Banach space E and continuous maps $f\colon C \to C$. It is natural to ask whether or not Browder's theorem holds in the finite dimensional case. We may answer this question in the following way.

(a) Given a finite dimensional convex and compact set C it is always possible to produce a continuous map $f\colon C \to C$ without non-repulsive fixed points;

(b) nevertheless, a continuous map $g\colon C \to C$ has a non-repulsive fixed point if it lowers the dimension (in a sense that will be made precise later).

To prove part (a) of this statement we may assume, without loss of generality, that the convex closed and bounded set C is a subset of $\boldsymbol{R}^n$ with non-empty interior.

Let $x_0 \in C/\partial C$ and let $\varepsilon > 0$ be such that $B(x_0, \varepsilon) \subset C$. We may also assume that $x_0 = 0$. Define $f\colon C \to C$ in the following way

$$f(x) = \begin{cases} -2x & \text{if } \|x\| \leqslant \dfrac{1}{2}\varepsilon , \\ -\varepsilon \dfrac{x}{\|x\|} & \text{if } \|x\| \geqslant \dfrac{1}{2}\varepsilon . \end{cases}$$

Clearly f is continuous and 0, the only fixed point of f, is a repulsive fixed point.

In the above example, one can easily see that the image of f has the same dimension of C. It is therefore natural to ask if it is possible to produce an example of a map $f\colon C \to C$ which lowers the dimension but it does have only repulsive fixed points. The answer is negative as we will show in a moment, proving the second part of the statement. We need to recall first the following theorem due to H. O. Peitgen $[\mathbf{10}]_2$.

« Let X be a metric ANR, $x_0 \in X$, $f\colon X \to X$ a compact map such that (i) x_0 is a repulsive fixed point of f relative to a neighborhood U of x_0 and $f(x) \neq x$ for all $x \in \partial U$, (ii) there exists a neighborhood V of x_0 such that $\overline{V} \subset U$ and the inclusion $i\colon X \diagdown V \to X$ induces isomorphisms in H^*. Then $i(X, f, U) = 0$ and $i(X, f, X \diagdown \overline{U}) = \Lambda(f, X)$. »

In the above result H^* stands for the singular homology functor with rational coefficients, $\Lambda(f, X)$ denotes the Lefschetz number in the generalized sense as given by J. Leray [**8**] and $i(X, f, U)$ is the fixed point index for metric ANR's developed by A. Granas [**6**].

Theorem 1 below, as it is stated, is a consequence of Theorem 1.1 of R. D. Nussbaum in [**9**]$_1$. But the proof given here can be repeated almost verbatim for the larger class of upper-semicontinuous admissible maps (the definition of which is given later) by means of a result of C. C. Fenske - H. O. Peitgen [**3**] analogous to the one recalled above (H. O. Peitgen [**10**]$_2$). Therefore the theorem is stated and proved for single-valued continuous maps and its natural extension to the class of upper-semicontinuous multi-valued and admissible maps is mentioned later without any proof.

Theorem 1. *Let C be an n-dimensional, convex, closed and bounded subset of a Banach space E and let $f\colon C \to C$ be continuous and such that* $\operatorname{Im} f$ *is contained in a linear variety V^m with $m < n$. Then f has a non-repulsive fixed point.*

Proof. We may assume, without loss of generality, that $C \subset \boldsymbol{R}^n$ and $0 \in C \diagdown \partial C$. Let V^{n-1} be a linear variety such that $\operatorname{Im} f \subset V^{n-1}$. Denote by H^n one of the two closed halfspaces of $\boldsymbol{R}^n$ which have V^{n-1} as boundary and such that $0 \in \mathring{H}_n$. Put $C_0 = C \cap H^n$. Clearly C_0 is closed, convex, invariant under f, and $C_0 \diagdown \partial C_0 \neq \emptyset$. Moreover, $\operatorname{Im} f \subset \partial C_0$. Therefore all repulsive fixed points of f are boundary points of C_0. Since repulsive fixed points are isolated fixed points, they can be only finitely many: $x_1, x_2, \ldots, x_p$ and we can find neighborhoods $V_1, V_2, \ldots, V_p$ of $x_1, x_2, \ldots, x_p$ respectively, such that $i\colon C_0 \diagdown V \to C_0$, $V = \bigcup V_i$, induces isomorphisms in H^*, and $\overline{V} \subset U$ where $U = \bigcup U_i$, U_i being a suitable neighborhood of x_i such that x_i is repulsive relative to U_i. Therefore, $i(C_0, f, U) = 0$ and $i(C_0, f, C_0 \diagdown \overline{U}) = \lambda(f, C_0) = 1$, where $\lambda(f, C_0)$ is the Lefschetz number of $f|_{C_0}$. Hence f has a fixed point x in $C_0 \diagdown \overline{U}$ which is clearly non-repulsive for $f|_{C_0}$ and, in view of Lemma 1, it is also non-repulsive for f. Q.E.D.

The above theorem clarifies, in some sense, what we mean by the statement: « f lowers the dimension ». We can give a more general version of the theorem, which, in turn, enlarges the class of « lowering dimension maps », namely.

Let $f\colon C \to C$ be a continuous map of a compact n-dimensional convex subset *of a Banach space E into itself. Assume that there is a homeomorphism $\varphi\colon C \to C$ such that* $\operatorname{Im} \varphi \circ f$ *is contained in a linear variety of dimension at most $n-1$. Then f has a non-repulsive fixed point.*

It is well known that there are three versions of Schauder's fixed point theorem.

(1) « Let C be a compact convex subset of a Banach space E and $f\colon C \to C$ be continuous. Then f has a fixed point. »

(2) « Let C be a convex, closed and bounded subset of a Banach space E and $f\colon C \to C$ be continuous and compact. Then f has a fixed point. »

(3) « Let C be a convex closed subset of a Banach space E and f: $C \to C$ be continuous and such that $\overline{f(C)}$ is compact. Then f has a fixed point. »

Clearly $3 \to 2 \to 1$. On the other hand it is not difficult to show that $1 \to 3$. Therefore the three formulations are equivalent.

Browder's theorem is stated in terms of the first formulation and we may ask if it is possible to state it in terms of the other two. In other words we may ask if the following hold.

(2′) « Let C be a convex, closed, bounded and infinite-dimensional subset of a Banach space E and let $f\colon C \to C$ be continuous and compact. Then f has a non-repulsive fixed point. »

(3′) « Let C be a closed, convex, infinite dimensional subset of a Banach space E and let $f\colon C \to C$ be continuous. Assume that $\overline{f(C)}$ is compact. Then f has a non-repulsive fixed point. »

Clearly $3' \to 2' \to 1'$ (= Browder's theorem). Therefore we will be allowed to say that they are equivalent if we show that $1' \to 3'$. This is our aim in the following theorem, which is a consequence of Browder result.

Theorem 2. *Let C be a closed, convex, infinite dimensional subset of a Banach space E and let $f\colon C \to C$ be continuous and such that $\overline{f(C)}$ is compact. Then f has a non-repulsive fixed point.*

Proof. Let $P \subset C$ be infinite dimensional and compact. By Mazur's theorem $\overline{\text{co}}\,(f(C) \cup P) = C_0$ is compact. Moreover it is clearly infinite dimensional, and $f(C_0) \subset C_0$. Therefore, by Browder's theorem, $f|_{C_0}$ has a non-repulsive fixed point, which is also non-repulsive for f in view of Lemma 1. Q.E.D.

It is known that Schauder's fixed point theorem in its 2-formulation was extended by Darbo [**2**] to the class of continuous α-contractions. More precisely Darbo proved the following result.

« Let C be a closed bounded, convex subset of a Banach space E and $f\colon C \to C$ be a continuous α-contraction. Then f has a fixed point. »

It is natural to ask if Browder's theorem can be extended to α-contractions, or, more generally, to condensing maps. The answer is positive, as the following theorem shows. In proving it, some ideas of Sadovskii [**11**] and Fenske-Peitgen [**3**] are used.

Theorem 3. *Let C be a closed, convex, bounded and infinite dimensional subset of a Banach space E and let $f\colon C \to C$ be a continuous condensing map. Then f has a non-repulsive fixed point.*

Proof. Let $P \subset C$ be compact and infinite dimensional. Construct a transfinite sequence of sets $\{C_\alpha\}$, following Sadovskii [**11**], with a slight modification

$$C_0 = C \cup P = C\,,$$

$$C_\beta = \begin{cases} \overline{\mathrm{co}}\,(f(C_{\beta-1}) \cup P) & \text{if } \beta \text{ is an ordinal number of the first kind}\,,\\ \bigcap\limits_{\gamma<\beta} C_\gamma & \text{if } \beta \text{ is an ordinal number of the second kind}\,. \end{cases}$$

It is easy to see that, for every β,

(1) C_β is convex, closed and bounded;

(2) C_β is invariant under f, i.e., $f(C_\beta) \subset C_\beta$;

(3) $P \subset C_\beta$.

Moreover, there exists an ordinal number δ, whose power does not exceed the power of the set of all subset of C such that $C_\delta = C_{\delta+1}$.

This implies $\alpha(C_\delta) = 0$, i.e., C_δ is compact. In fact for that δ we have $\overline{\mathrm{co}}\,(f(C_\delta) \cup P) \equiv C_{\delta+1} = C_\delta$; since f is condensing, $\alpha\big(\overline{\mathrm{co}}\,(f(C_\delta) \cup P)\big) = \alpha(f(C_\delta)) < \alpha(C_\delta)$ if $\alpha(C_\delta) > 0$. In addition, C_δ is infinite-dimensional. Hence $f|_{C_\delta}$ has a non-repulsive fixed point x_0, by Browder's theorem [**1**]. This point is non-repulsive for f, by Lemma 1. Q.E.D.

Our aim now is to prove that a continuous α-contraction $f\colon S \to S$, where S is the unit sphere in an infinite dimensional Banach space E, has a fixed point.

The above result was first proved by Nussbaum [**9**]$_2$ by means of other arguments. Here we give a proof using non-repulsive fixed point theory.

We need the following proposition which was stated and proved in Furi-Martelli [**4**]$_2$ for single-valued maps. Here it is stated and proved for the class of upper-semicontinuous multi-valued maps.

We recall first that a multi-valued map $F\colon X \multimap X$, where X is a topological space, is said to be upper-semicontinuous at $x_0 \in X$, if for any neighborhood U of $F(x_0)$ there exists a neighborhood V of x_0 such that $F(V) = \bigcup\{F(x)\colon x \in V\} \subset U$. If for any $x \in X$ we have that F is upper semicontinuous at x and, moreover, $F(x)$ is compact then F is said to be upper-semicontinuous (on X). It is known that upper-semicontinuous multi-valued maps send compact sets into compact sets.

Proposition 1. *Let X be a complete metric space and let $\mathscr{F}$ be a non-empty family of subsets of X such that for every $\varepsilon > 0$ there exists a finite sub-family $\{G_1, G_2, \ldots, G_n\}$ of $\mathscr{F}$ with the property that $\alpha(X \setminus \bigcup G_i) < \varepsilon$. Assume that the restrictions $F|_G$, $G \in \mathscr{F}$, of an upper-semicontinuous multi-valued map $F\colon X \multimap X$ are α-Lipschitz with constant k. Then F is α-Lipschitz with the same constant.*

Proof. Assume first that $\mathscr{F}$ admits a finite subfamily $\{G_i\colon i = 1, 2, \ldots, n\}$ which is a covering of X. If $A \subset X$ is bounded we have (see item 3 in the notation section)

$$\begin{aligned}\alpha[F(A)] &= \max\{\alpha[F((A \cap G_i))]\colon i = 1, 2, \ldots, n\}\\ &\leqslant \max\{k\alpha(A \cap G_i)\colon i = 1, 2, \ldots, n\} = k\alpha(A).\end{aligned}$$

Assume hereafter that the finite covering property fails for $\mathscr{F}$. Given any finite union of elements of $\mathscr{F}$, $G_{j_1} \cup G_{j_2} \cup \ldots \cup G_{j_n} = W$, the subset $\overline{X \setminus W}$ is a closed nonvoid subset of X. Denote by $\mathscr{B}$ the family of all sets $\overline{X \setminus W}$. The family $\mathscr{B}$ of nonvoid closed sets has the finite intersection property, and $\inf\{\alpha(B)\colon B \in \mathscr{B}\} = 0$. Therefore, by a result proved in [**4**]$_1$, we have that $K = \bigcap\{B\colon B \in \mathscr{B}\}$ is nonempty and compact. Moreover, for any neighborhood U of K, there exists an element $B \in \mathscr{B}$ such that $K \subset B \subset U$; (see Furi-Martelli [**4**]$_1$).

Let $A \subset X$ be bounded, $\alpha(A) > 0$, and let $V = F(K) + B(0, \varepsilon)$, where $0 < 2\varepsilon < k\alpha(A)$. The upper-semicontinuity of F implies $\alpha(F(K)) = 0$, so by items 5 and 6 of the notation section, $\alpha(V) \leqslant \alpha(F(K)) + \alpha(B(0, \varepsilon)) < k\alpha(A)$. Let U be a neighborhood of K such that $F(U) \subset V$ and let $B \in \mathscr{B}$ be such that $K \subset B \subset U$. Denote by $\{G_i\colon i = 1, 2, \ldots, n\}$ the finite subfamily of $\mathscr{F}$ such that $B = \overline{X \setminus (G_1 \cup \ldots \cup G_p)}$. Therefore

$$\alpha(F(A)) = \max\{\alpha(F(A \cap B)), \alpha(F(A \cap G_i)) \colon i = 1, 2, \ldots, p\}$$

$$\leqslant \max\{k\alpha(A), k\alpha(A \cap G_i) \colon i = 1, 2, \ldots, p\} \leqslant k\alpha(A) .$$

We are now in a position of giving a different proof of Nussbaum's result [**9**]$_2$.

Theorem 4. *Let $f \colon S \to S$ be a continuous α-contraction of the unit sphere S of an infinite dimensional Banach space E into itself. Then f has a fixed point.*

Proof. Let $\varphi \colon [0, 1] \to \boldsymbol{R}$ be continuous and such that $\varphi(0) = \varphi(1) = 1$, $1 < \varphi(t) < 1/k$ for any $t \in (0, 1)$ where k is the α-Lipschitz constant of f. Define $f_0 \colon D \to D$, $D = \{x \in E \colon \|x\| \leqslant 1\}$, by

$$f_0(x) = \begin{cases} \|x\| \, f\left(\dfrac{x}{\|x\|}\right), & x \neq 0 , \\ 0 , & x = 0 , \end{cases}$$

and consider the map $\bar{f} \colon D \to D$; $\bar{f}(x) = \varphi(\|x\|) f_0(x)$. We want to show that $\bar{f}$ is an α-contraction with constant $r < 1$, where $r = k\|\varphi\|$, $\|\varphi\| = \max\{\varphi(r) \colon r \in [0, 1]\}$.

Let $0 < q < 1$ and define $R_n = \{x \in D \colon q^{n+1} \leqslant \|x\| \leqslant q^n\}$. The family $\{R_n \colon n \in \boldsymbol{N}\}$ satisfies the assumption of Proposition 1. Therefore, it is enough to show that $\bar{f}|_{R_n}$ is an α-contraction with constant r/q.

Let $A \subset R_n$ and denote by A_0 the set $A_0 = \{x/\|x\|, \ x \in A\}$. It is easily shown that $\alpha(A_0) \leqslant (1/q^{n+1})\alpha(A)$; in fact, A_0 is contained in the convex closure of the set $A_1 = \{0\} \cup (1/q^{n+1}) A$, so $\alpha(A_0) \leqslant \alpha(\overline{\mathrm{co}}\,(A_1)) = \alpha(A_1) = (1/q^{n+1})\,\alpha(A)$.

Further, $f_0(A) \subset \bigcup\,\{\lambda f(A_0) \colon q^{n+1} \leqslant \lambda \leqslant q^n\} \subset \overline{\mathrm{co}}\,(\{0\} \cup q^n f(A_0))$, and $\bar{f}(A) \subset \bigcup\,\{\lambda f_0(A) \colon 1 \leqslant \lambda \leqslant \|\varphi\|\} \subset \overline{\mathrm{co}}\,(\{0\} \cup \|\varphi\| f_0(A))$. Hence, by the properties of α in the notation section, and the preceding inequality

$$\alpha(\bar{f}(A)) \leqslant \|\varphi\|\alpha(f_0(A)) \leqslant \|\varphi\| q^n \alpha(f(A_0)) \leqslant \frac{1}{q}\, k \|\varphi\| \alpha(A) .$$

Therefore, by Proposition 1, $\bar{f}$ is α-Lipschitz with constant r/q. Since this is true for any $0 < q < 1$, it follows that $\bar{f}$ is an α-contraction with constant $r = k\|\varphi\|$. On the basis of Theorem 3, $\bar{f}$ has a non-repulsive fixed point $x_0 \in D$. Clearly 0 is a repulsive fixed point for $\bar{f}$, and no fixed points of $\bar{f}$ belong to $D \setminus (S \cup \{0\})$. Hence $\bar{f}$ has a non-repulsive fixed point x_0 on S, which is clearly a fixed point for f. Q.E.D.

We want to derive now the above result as a consequence of the following more general theorem.

Theorem 6. *Let $F\colon S \multimap S$ be an upper-semicontinuous, admissible α-contraction of the unit sphere of an infinite dimensional Banach space E into itself. Then F has a fixed point.*

We first recall some facts about *admissible maps*. Let X be a metric space and let $G\colon X \multimap X$ be an upper-semicontinuous multivalued map. If for any $x \in X$ we have that $G(x)$ is acyclic in the Čech cohomology with rational coefficients, then we will say that G is acyclic.

Let $X_0, X_1, \ldots, X_{n+1}$ be metric spaces and $G_i\colon X_i \multimap X_{i+1}$ $(i=0, 1, 2, \ldots, n)$ be upper-semicontinuous and acyclic. Then the composite map $F\colon X_0 \to X_{n+1}$ $F = G_n \circ G_{n-1} \circ \ldots \circ G_0$ is called *admissible* (see [**3**]). Let Y be a subset of a Banach space E, $F\colon Y \multimap E$ be an admissible map and $\varphi\colon Y \to \boldsymbol{R}$ be continuous. Then $\tilde{F}\colon Y \to E$ defined by $\tilde{F}(x) = \varphi(x)\bar{F}(x)$ is clearly admissible.

In [**3**] the following theorem is proved.

« Let C be a convex, closed, bounded and infinite-dimensional subset of a Banach space E and let $F\colon C \multimap C$ be an upper-semicontinuous, compact admissible map. Then F has a non-repulsive fixed point. »

By means of arguments similar to the ones used in the proof of Theorem 3 and applying the above result one can prove the following.

Theorem 7. *Let C be a convex, closed, bounded and infinite dimensional subset of a Banach space E and let $F\colon C \multimap C$ be an upper-semicontinuous admissible map. Assume that F is condensing. Then F has a non-repulsive fixed point.*

We are now in a position of proving Theorem 6.

Proof of Theorem 6. Let $\varphi\colon [0, 1] \to \boldsymbol{R}$ be as in Theorem 3 and define $F_0\colon D \multimap D$, $D = \{x \in E\colon \|x\| \leqslant 1\}$, by

$$F_0(x) = \begin{cases} \|x\|\, F\left(\dfrac{x}{\|x\|}\right), & \text{if } x \neq 0\,, \\ 0\,, & \text{if } x = 0\,. \end{cases}$$

Consider the map $\tilde{F}\colon D \multimap D$; $\tilde{F}(x) = \varphi(\|x\|)F_0(x)$. Clearly $\tilde{F}$ is admissible, upper-semicontinuous and it is an α-contraction. Thus by Theorem 7, $\tilde{F}$ has a non-repulsive fixed point. Since 0 is repulsive and $x \notin \tilde{F}(x)$ for any $0 < \|x\| < 1$ we have $x_0 \in \tilde{F}(x_0)$ for some $x_0 \in S$. Thus $x_0 \in F(x_0)$. Q.E.D.

We recall that in the case when C is a convex, closed, bounded and finite dimensional subset of a Banach space E we can always construct a continuous map $f\colon C \to C$ without non-repulsive fixed points. Such an f can be regarded as a multi-valued, upper-semicontinuous and admissible map.

Using the same argument of Theorem 1 one can prove that if an upper-semicontinuous admissible map $F\colon C \multimap C$, where C is a finite dimensional closed and convex subset of a Banach space E « lowers the dimension », then it has a non-repulsive fixed point.

To obtain this result it is enough to repeat the construction used in the case of a single-valued map and to apply the following theorem, proved by Fenske-Peitgen [**3**].

« Let X be a compact metric ANR, $F\colon X \multimap X$ be an admissible map and x_0 be a repulsive fixed point of F with respect to a neighborhood U of x_0. Assume that there exists a neighborhood V of x_0 such that $\overline{V} \subset U$ and $i\colon X \setminus V \to X$ induces isomorphisms $i^*\colon H^*(X\setminus V) \to H^*(X)$. Then there are arbitrarily fine coverings $\mathscr{V}$ of X such that $\operatorname{ind}_{\mathscr{V}}(X, f, U) = 0$. »

We end this paper by pointing out that our arguments in the proofs of Theorem 3 and Theorem 7 break down in the case where f or F are condensing instead of α-contractions, leaving open an old question on whether or not a condensing single-valued continuous (multi-valued upper-semicontinuous and acyclic) map of the unit sphere S of an infinite dimensional Banach space into itself has a fixed point.

I thank G. Gustafson for stimulating conversations on the subject matter of this paper.

References

[**1**] F. Browder, *Another generalization of the Schauder fixed point theorem*, Duke Math. J. **32** (1965), 399-406.

[**2**] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, Rend. Sem. Mat. Univ. Padova **24** (1955), 84-92.

[**3**] C. C. Fenske and H. O. Peitgen, *Repulsive fixed points of multivalued trasformations and the fixed point index*, Math. Ann. **218** (1975), 9-18.

[**4**] M. Furi and M. Martelli: [•]$_1$ *A characterization of compact filter basis in complete metric spaces*, Rend. Ist. Mat. Univ. Trieste **2** (1970), 109-113; [•]$_2$ *On α-Lipschitz retractions of the unit closed ball onto its boundary*, Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur. **47** (1974), 61-65.

[**5**] M. Furi and A. Vignoli, *On a property of the unit sphere in a linear normed space*, Bull. Acad. Pol. Sci. **18** (1970), 333-336.

[**6**] A. Granas, *The Leray-Schauder index and the fixed point theory for arbitrary ANR's*, Bull. Soc. Mat. France **100** (1972), 209-228.

[7] K. Kuratowski, *Topologie*, Warsawa 1958.

[8] J. Leray, *Theorie des point fixes: indice totale et nombre de Lefschetz*, Bull. Soc. Mat. France **87** (1959), 221-233.

[9] R. D. Nussbaum: [•]$_1$ *Periodic solutions of some non linear autonomous functional differential equations*, Ann. Mat. Pura Appl. **101** (1974), 263-306; [•]$_2$ *Some fixed point theorems*, Bull. Amer. Mat. Soc. **77** (1971), 360-365.

[10] H. O. Peitgen: [•]$_1$ *Asymptotic fixed point theorems and stability*, J. Math. Anal. Appl. **47** (1974), 32-42; [•]$_2$ *Some applications of the fixed point index in asymptotic fixed point theory* (preprint).

[11] B. N. Sadovskii, *A fixed point principle*, Funct. Anal. Appl. **1** (1967), 151-153.

[12] H. Steinlein, *Über die verallgemeinerten Fixpunktindizes von Iterierten verdichtender Abbildungen*, Manus. Math. **8** (1972), 251-266.

[13] P. O. Zabreiko and M. A. Krasnosel'skii, *Iterations of operators and the fixed point index*, Doklady Akad. Nauk SSSR. **196** (1971), 1006-1009.

Abstract

Some existence theorems in non-repulsive fixed point theory are given and they are used to prove fixed point theorems for single-valued (admissible multi-valued) α-contractions defined in a sphere S of a infinite dimensional Banach space.

* * *