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Cylindrical waves in an anisotropic plasma with generalized polytropic equations of state (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

B. Abraham-Shrauner in [1] has introduced a general theoretical model describing an anisotropic plasma with generalized polytropic equations of state. We refer to [1] for the physical and mathematical pecularities of the model, as well as for its range of applicability and for every detail.

For the plasma under consideration, in this paper solutions are obtained for cylindrical waves (1). The dispersion equation is given and discussed. In particular torsional oscillations are studied. Subject to certain conditions, we find two types of instability, which are related to the « fire-hose » instability and to the « mirror » instability.

The contents of the paper are indicated by the titles of the sections.

2. - Basic equations

The basic equations governing the plasma are [1] (2)

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(1) Jeans' gravitational instability has been examined in [7].
(2) Gaussian units are used.

$$\varrho\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\operatorname{div}\boldsymbol{P} + \frac{1}{4\pi\mu}(\operatorname{curl}\boldsymbol{B})\wedge\boldsymbol{B}\,, \tag{2.1}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \operatorname{curl}(\boldsymbol{v}\wedge\boldsymbol{B})\,, \tag{2.2}$$

$$\frac{\partial\varrho}{\partial t} = -\operatorname{div}(\varrho\boldsymbol{v})\,, \tag{2.3}$$

$$\frac{p_\parallel B^\alpha}{\varrho^\beta} = c_\parallel\,, \qquad \frac{p_\perp}{\varrho^\varepsilon B^\gamma} = c_\perp\,, \tag{2.4}$$

$$\boldsymbol{P} = p_\perp\boldsymbol{I} + (p_\parallel - p_\perp)\boldsymbol{n}\otimes\boldsymbol{n}\,. \tag{2.5}$$

In these: ϱ is the mass density; $\boldsymbol{v}$ is the velocity; t is the time; $\boldsymbol{P}$ is the pressure tensor; μ is the (constant) magnetic permeability; $\boldsymbol{B}$ is the magnetic induction vector; $p_\parallel$ and $p_\perp$ are respectively the pressure parallel and perpendicular to the direction of the magnetic field; α, β, γ and ε are the (constant) polytropic indices; $c_\parallel$ and $c_\perp$ are constants; $\boldsymbol{I}$ is the unit tensor; $\boldsymbol{n}$ is a unit vector along $\boldsymbol{B}$ and $\otimes$ denotes dyadic product.

The polytropic relations (2.4) are the generalization of well-known equations of state in plasma physics. For example: (i) if $\alpha = 2$, $\beta = 3$, $\gamma = \varepsilon = 1$, we recover the equations of state introduced in [2] by G. F. Chew, M. L. Goldberger and F. E. Low (CGL plasma); (ii) if $\alpha = \gamma = 0$, $\beta = \varepsilon = 1$, we have an isothermal equation of state for both pressures (this case should be of interest for the ion acoustic waves); (iii) if $\alpha = 0$, $\beta = \gamma = \varepsilon = 1$, we find an isothermal equation of state for the parallel pressure (this model seems to be of interest for the study of the solar wind); (iv) if $c_\perp = c_\parallel$, $\alpha = \gamma = 0$, $\beta = \varepsilon = c_p/c_v$, where c_p is the specific heat at constant pressure and c_v is the specific heat at constant volume, we recover the well-known model of an adiabatic plasma described by the magnetofluiddynamic equations (MFD plasma).

3. - Perturbation equations

We suppose that the unperturbed plasma is homogeneous, at rest and permeated by a uniform magnetic field $\boldsymbol{B}/\mu$.

The equations which the small perturbations of the field variables $\boldsymbol{v}$, $\delta\boldsymbol{P}$, $\delta\boldsymbol{B}$ and $\delta\varrho$ satisfy are

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\varrho}\,\mathrm{div}\,\delta\boldsymbol{P} + \frac{1}{4\pi\mu\varrho}(\mathrm{curl}\,\delta\boldsymbol{B})\wedge\boldsymbol{B}\,, \tag{3.1}$$

$$\frac{\partial}{\partial t}\delta\boldsymbol{B} = \mathrm{curl}\,(\boldsymbol{v}\wedge\boldsymbol{B})\,, \tag{3.2}$$

$$\frac{\partial}{\partial t}\delta\varrho = -\varrho\,\mathrm{div}\,\boldsymbol{v}\,, \tag{3.3}$$

$$\frac{\delta p_\parallel}{p_\parallel} + \alpha\frac{\delta B}{B} = \beta\frac{\delta\varrho}{\varrho}\,, \qquad \frac{\delta p_\perp}{p_\perp} = \varepsilon\frac{\delta\varrho}{\varrho} + \gamma\frac{\delta B}{B}\,. \tag{3.4}$$

Using (3.4) we obtain from (2.5) the perturbation in the pressure tensor

$$\delta\boldsymbol{P} = \Big(\frac{\varepsilon p_\perp}{\varrho}\,\delta\varrho + \frac{\gamma p_\perp}{B}\,\delta B\Big)\,\boldsymbol{I} + \Big(\frac{\beta p_\parallel - \varepsilon p_\perp}{\varrho}\,\delta\varrho - \frac{\alpha p_\parallel + \gamma p_\perp}{B}\,\delta B\Big)\,\boldsymbol{n}\otimes\boldsymbol{n} + \tag{3.5}$$
$$+ (p_\parallel - p_\perp)(\boldsymbol{n}\otimes\delta\boldsymbol{n} + \delta\boldsymbol{n}\otimes\boldsymbol{n})\,,$$

in which $\delta\boldsymbol{n}$ can be derived from

$$\delta\boldsymbol{B} = \delta B\,\boldsymbol{n} + B\,\delta\boldsymbol{n}\,. \tag{3.6}$$

The system (3.1)-(3.3), in which $\delta\boldsymbol{P}$ is specified by (3.5) and (3.6), is a linear system of seven scalar partial differential equations with seven scalar unknowns ($\delta\varrho$ and six from $\boldsymbol{v}$ and $\delta\boldsymbol{B}$).

4. - Perturbation equations in cylindrical coordinates

Introducing a frame of reference $\mathscr{T}(0; r, \varphi, z)$, where r, φ, z are cylindrical coordinates with the z-axis parallel to $\boldsymbol{B}$, we suppose that the perturbations are endowed with cylindrical symmetry about such an axis.

The resolutes of $\delta\boldsymbol{B}$ relative to $\mathscr{T}$, denoted by b_r, b_φ and b_z ($= \delta B$), are expressed in terms of the covariant components b_1, b_2, b_3 of $\delta\boldsymbol{B}$ by (see [3], Chap. II, Sect. 8, n. 5)

$$b_r = b_1\,, \qquad b_\varphi = b_2/r\,, \qquad b_z = b_3\,. \tag{4.1}$$

To calculate div $\delta\boldsymbol{P}$, we note that if $\boldsymbol{T}$ is a second order symmetrical tensor, then in an orthogonal coordinate system (x^1, x^2, x^3) with metric coefficients h_k (see [**8**], Eq. (12.9))

$$(\operatorname{div}\boldsymbol{T})_i = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}\,T^k{}_i) - T^k{}_k\frac{\partial \log h_k}{\partial x^i}, \quad (\sqrt{g} = h_1 h_2 h_3)\,. \tag{4.2}$$

In our case we have $h_1 = h_3 = 1$, $h_2 = r$, $\sqrt{g} = r$.

Using (4.2), (4.1) and (3.6), we obtain from (3.5) the resolutes of the vector div $\delta\boldsymbol{P}$

$$\frac{\varepsilon p_\perp}{\varrho}\frac{\partial}{\partial r}\delta\varrho + \frac{\gamma p_\perp}{B}\frac{\partial b_z}{\partial r} + \frac{p_\parallel - p_\perp}{B}\frac{\partial b_r}{\partial z}, \qquad \frac{p_\parallel - p_\perp}{B}\frac{\partial b_\varphi}{\partial z},$$

$$\frac{p_\parallel - p_\perp}{B}\frac{1}{r}\frac{\partial(rb_r)}{\partial r} + \frac{\beta p_\parallel}{\varrho}\frac{\partial}{\partial z}\delta\varrho - \frac{\alpha p_\parallel}{B}\frac{\partial b_z}{\partial z}.$$

Denoting v_r, v_φ, v_z the resolutes of $\boldsymbol{v}$ in $\mathscr{T}$ we find the resolutes of (3.1) to be

$$\varrho\frac{\partial v_r}{\partial t} + \left(\frac{\gamma p_\perp}{B} + \frac{B}{4\pi\mu}\right)\frac{\partial b_z}{\partial r} + \left(\frac{p_\parallel - p_\perp}{B} - \frac{B}{4\pi\mu}\right)\frac{\partial b_r}{\partial z} + \frac{\varepsilon p_\perp}{\varrho}\frac{\partial}{\partial r}\delta\varrho = 0\,, \tag{4.3}$$

$$\varrho\frac{\partial v_\varphi}{\partial t} + \left(\frac{p_\parallel - p_\perp}{B} - \frac{B}{4\pi\mu}\right)\frac{\partial b_\varphi}{\partial z} = 0\,, \tag{4.4}$$

$$\varrho\frac{\partial v_z}{\partial t} + \frac{p_\parallel - p_\perp}{B}\frac{1}{r}\frac{\partial(rb_r)}{\partial r} + \frac{\beta p_\parallel}{\varrho}\frac{\partial}{\partial z}\delta\varrho - \frac{\alpha p_\parallel}{B}\frac{\partial b_z}{\partial z} = 0\,. \tag{4.5}$$

From (3.2) we have

$$\frac{\partial b_r}{\partial t} - B\frac{\partial v_r}{\partial z} = 0\,, \tag{4.6}$$

$$\frac{\partial b_\varphi}{\partial t} - B\frac{\partial v_\varphi}{\partial z} = 0\,, \tag{4.7}$$

$$\frac{\partial b_z}{\partial t} + \frac{B}{r}\frac{\partial(rv_r)}{\partial r} = 0\,, \tag{4.8}$$

whilst from (3.3)

$$\frac{\partial}{\partial t}\delta\varrho + \varrho\left[\frac{\partial v_z}{\partial z} + \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right] = 0\,. \tag{4.9}$$

Equations (4.3)-(4.9) are the perturbation equations in cylindrical coordinates.

5. - Torsional oscillations. Cylindrical waves. Instabilities

Equations (4.4) and (4.7) are independent of the remaining ones (uncoupled) and from these we deduce that v_φ and b_φ satisfy the same equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{h}{\varrho}\frac{\partial^2}{\partial z^2}\right)(v_\varphi, b_\varphi) = 0\,, \tag{5.1}$$

with

$$h = 2p_m + p_\perp - p_\parallel\,, \tag{5.2}$$

where $p_m = B^2/8\pi\mu$ denotes the magnetic pressure. Equations (5.1) have the solutions

$$v_\varphi = f(r)\exp[i(\omega t - kz)]\,, \qquad b_\varphi = g(r)\exp[i(\omega t - kz)]\,, \tag{5.3}$$

where f and g are arbitrary functions of r and ω (the pulsation) and k (the wave-number) satisfy the dispersion equation

$$\omega^2 = \frac{h}{\varrho}k^2\,. \tag{5.4}$$

If $h > 0$, the perturbations v_φ and b_φ are propagated along the axis of z (torsional oscillations) with the (real) phase velocity

$$u = \pm\left(\frac{p_\perp - p_\parallel}{\varrho} + A^2\right)^{1/2} \tag{5.5}$$

(where $A^2 = B^2/4\pi\mu\varrho$ is the square of the Alfvén velocity) and with an amplitude which is an arbitrary function of r. When the pressure is isotropic we have $u = \pm A$.

If $h < 0$, u becomes imaginary and this fact is related to a well-known instability phenomenon in the anisotropic plasma: the «fire-hose» instability. (For this type of instability see, for example, [**1**] and [**5**], pp. 228-229).

We may note that the propagation of waves ($h > 0$) or the fire-hose instability ($h < 0$) are independent of polytropic indices.

It can be shown that the system of the remaining equations (4.3), (4.5), (4.6), (4.8) and (4.9) admits, subject to the dispersion equation (5.8) below, the following solution, corresponding to the propagation along the z-axis of cylindrical waves

$$\begin{aligned} v_r &= \bar{v}_r J_1(ar) \exp[i(\omega t - kz)]\,, \qquad & v_z &= \bar{v}_z J_0(ar) \exp[i(\omega t - kz)]\,, \\ b_r &= \bar{b}_r J_1(ar) \exp[i(\omega t - kz)]\,, \qquad & b_z &= \bar{b}_z J_0(ar) \exp[i(\omega t - kz)]\,, \end{aligned} \tag{5.6}$$

$$\delta\varrho = \overline{\delta\varrho}\, J_0(ar) \exp[i(\omega t - kz)]\,,$$

where a is a real constant and J_1 and J_0 are Bessel functions of the first kind of order unity and zero respectively and $\bar{v}_r$, $\bar{v}_z$, $\bar{b}_r$, $\bar{b}_z$ and $\overline{\delta\varrho}$ are small constants in our linearized approximation. The constant a will be determined by the boundary conditions. If, for example, the boundary conditions to be specified on a cylindrical surface of radius R are

$$\boldsymbol{v}\cdot\boldsymbol{N} = 0\,, \qquad (\boldsymbol{B} + \delta\boldsymbol{B})\cdot\boldsymbol{N} = 0 \tag{5.7}$$

(for the discussion of these conditions see, for example, [**6**] Chap. II), where $\boldsymbol{N}$ is a unit vector normal to the surface, we see that these conditions are satisfied by (5.6) provided $a = \xi_n/R$ $(n = 1, 2, 3, \dots)$, where ξ_n is the n-th zero of $J_1(\xi)$; in fact, the above conditions are equivalent in our case to $b_r = 0$ and $v_r = 0$. The solution (5.6) is regular and finite in the whole of the field.

The algebraic system deduced from equations (4.3), (4.5), (4.6), (4.8) and (4.9) in correspondence to the solution (5.6), yelds the dispersion equation

$$\omega^4 - C\omega^2 + D = 0\,, \tag{5.8}$$

where the coefficients (real) are given by the expressions

$$\varrho C = (h + \beta p_\parallel)k^2 + a^2\,[(\gamma + \varepsilon)p_\perp + 2p_m]\,, \quad \frac{\varrho^2 D}{k^2} = \beta h p_\parallel k^2 + M a^2\,,$$

with

$$M = [\beta\gamma + \varepsilon(\alpha + 1)]p_\parallel p_\perp + 2\beta p_\parallel p_m - \varepsilon p_\perp^2\,. \tag{5.9}$$

We may therefore conclude that the linearized equations describing an anisotropic plasma with generalized polytropic equations of state are satisfied by the solutions (5.3) and (5.6), where ω and k satisfy the dispersion equations (5.4) and (5.8).

Considering k to be real, the equations (5.4) and (5.8) show that there are in general three distinct roots for ω^2. We can therefore say in general that there are three modes of propagation. Besides the fire-hose instability, which is related to the azimuthal component of $\boldsymbol{v}$ and $\delta\boldsymbol{B}$, equation (5.8) also indicates the possibility of an instability (i.e. the existence of roots of ω for which $\exp[i\omega t]$ diverges with respect to the time). In fact, for particular values of the wave-number and of the parameters which characterize the physical properties of the plasma, we can have $D < 0$. This instability is of « mirror » type (for this type of instability see, for example, [**1**]) and it is affected by the polytropic indices.

From (5.8) it follows that if $h > 0$ and $M > 0$, none of the three modes of propagation will be diffusive, whatever be the value of the wave-number. (The equations of dispersion have no purely imaginary roots ω in this case).

For the CGL plasma, the condition $M > 0$ becomes (from (5.9))

$$6p_{\parallel}(p_m + p_{\perp}) - p_{\perp}^2 > 0 \,. \tag{5.10}$$

The condition (5.10) is well-known in the theory of the CGL plasma (see, for example, [**4**], p. 768).

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Abstract

Solutions are obtained for cylindrical waves of the equations governing an anisotropic plasma with generalized polytropic equations of state. The dispersion equation is given and discussed. In particular torsional oscillations are studied. Subject to certain conditions, we find two types of instability, which are related to the « fire-hose » instability and to the « mirror » instability.

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