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Particle transport
in a slab bounded by capturing and reflecting planes (**)
\( k(y, y') \) are nonnegative functions, such that
\[
\int_0^1 h(y, y') \, dy = \overline{h}(y') < b < 1 \quad \forall y' \in (-1, 0),
\]
\[
\int_{-1}^0 k(y, y') \, dy = \overline{k}(y') < b < 1 \quad \forall y' \in (0, 1),
\]
\[2, 3, 5\]. System (1)-(3) is a one-group particle-transport problem in a homogeneous slab of thickness \( 2a \ (< \infty) \), under the assumptions of plane symmetry and isotropic scattering. The unknown function \( u(x, y; t) \) is a particles density (e.g., a photon density), i.e., \( u(x, y; t) \, dx \, dy \) is the number of particle that, at time \( t \), are between \( x \) and \( x + dx \) and are such that the cosine of the angle between their velocity \( v \) and the positive \( x \)-axis is between \( y \) and \( y + dy \). Moreover, \( q(x, y; t) \) is a source term, \( u_0(x, y) \) is the initial particle density, and \( \sigma_t, \sigma_a \) are cross sections that characterize the physical properties of the materials of the slab. Finally, the boundary conditions (2) show that particles are either captured or reflected by the boundary planes \( x = -a \) and \( x = a \), (for instance, \( h(y, y') \, dy \, dy' \) is the probability that a particle at \( x = -a \) and such that the cosine between its velocity \( v \) and the positive \( x \)-axis is between \( y' \) and \( y' + dy' \) with \( y' \in (-1, 0) \) is reflected by the plane \( x = -a \) and emerges with a cosine between \( y \) and \( y + dy \), with \( y \in (0, 1) \).

Remark 1. If
\[
\overline{h}(y') = \int_0^1 h(y, y') \, dy = 1 \quad \forall y' \in (-1, 0),
\]
then the boundary planes do not capture particles because
\[
\int_0^1 y u(-a, y; t) \, dy = \int_{-1}^0 |y'| u(-a, y'; t) \, dy',
\]
\[
\int_{-1}^0 |y| u(a, y; t) \, dy = \int_0^1 y' u(a, y'; t) \, dy'.
\]

Since
\[
N(t) = \int_{-1}^1 dy \int_{-a}^{+a} u(x, y; t) \, dx
\]
is the total number of particles in the slab at time \( t \), and \( u(x, y; t) \) is a particle density (i.e., a nonnegative function), we introduce the (real) Banach space

\[
X = L^1((-a, a) \times (-1, 1)), \quad \|f\| = \int_{-1}^{1} \int_{-a}^{a} |f(x, y)| \, dx
\]

and the (closed) positive cone of \( X \)

\[
X_0 = \{ f : f \in X; f(x, y) \geq 0 \text{ for a.e. } (x, y) \in (-a, a) \times (-1, 1) \}.
\]

To write system (1)-(3) as a problem of evolution in the space \( X \), we define the operators

\[
(6) \quad A_\sigma f = - vy \partial f \partial x, \quad D(A_\sigma) = \{ f : f \in X; y \partial f / \partial x \in X; f \text{ satisfies the boundary conditions (2)} \},
\]

\[
(7) \quad A = A_\sigma - v \sigma I, \quad D(A) = D(A_\sigma),
\]

\[
(8) \quad Jf = \frac{1}{2} \int_{-1}^{1} f(x, y') \, dy', \quad D(J) = X,
\]

where \( \partial f / \partial x \) is a distributional derivative. Then, (1)-(3) becomes

\[
(9) \quad \frac{d}{dt} u(t) = Au(t) - v \sigma J u(t) + q(t) \quad (t > 0), \quad \lim_{t \to 0^+} \|u(t) - u_0\| = 0,
\]

where \( u(t) = u(., . ; t) \) and \( q(t) = q(., . ; t) \) are now to be interpreted as functions from \([0, +\infty)\) into \( X \) (or into \( X_0 \)), \( du/dt \) is a strong derivative, and it is assumed that \( u_0 \) is a given element of \( D(A) \cap X_0 \). Moreover, \( u(t), t \in [0, +\infty) \), is said to be a (strict) solution of (9) if (i) \( u(t) \) is continuously differentiable at any \( t > 0 \), (ii) \( u(t) \in D(A) \) \( \forall t > 0 \), (iii) \( u(t) \) satisfies (9).

2. - The operators \( J \) and \( A_\sigma \)

The following lemmas are needed to prove that system (9) has a unique strict solution.

**Lemma 1.** (a) \( J \in \mathcal{B}(X), [1], [4], \) with \( \|Jf\| < \|f\| \forall f \in X \); (b) \( Jf \in X_0 \) and \( \|Jf\| = \|f\| \forall f \in X_0 \).
Proof. (a) We have from (8)

$$\|Jf\| = \int_{-1}^{1} \int_{-a}^{a} \int_{-1}^{1} f(x, y') \, dy' \, dx \leq \int_{-a}^{a} \int_{-1}^{1} |f(x, y')| \, dy' = \|f\| \quad \forall f \in X.$$ 

(b) follows directly from (8).

Lemma 2. (a) The operator \((zI - A_{\theta})^{-1}\) exists and belongs to \(\mathcal{B}(X)\) for any \(z > 0\); (b) \((zI - A_{\theta})^{-1} g \in X_{\alpha}, \forall g \in X_{\alpha}, z > 0\).

Proof. (a) If \(g \in X\) and \(z > 0\) are given, consider the equation

$$zI - A_{\theta} f = g,$$

where the unknown \(f\) must obviously be sought in \(D(A_{\theta})\). Since (10) can be written as follows

$$\frac{\partial}{\partial x} f(x, y) + \frac{z}{vy} f(x, y) = \frac{1}{vy} g(x, y) \quad \text{for a.e.} \quad (x, y) \in (-\alpha, \alpha) \times (-1, 1),$$

we have

$$f(x, y) = \frac{1}{vy} \left[ C_{1}(y) \exp \left[ \frac{z}{vy} (a - x) \right] \right]$$

$$+ \int_{-1}^{y} \exp \left[ \frac{-z}{vy} (x - x') \right] g(x', y) \, dx' \quad \text{for } y \in (-1, 0),$$

$$f(x, y) = \frac{1}{vy} \left[ C_{2}(y) \exp \left[ \frac{-z}{vy} (a + x) \right] \right]$$

$$+ \int_{-y}^{0} \exp \left[ \frac{-z}{vy} (x - x') \right] g(x', y) \, dx' \quad \text{for } y \in (0, 1),$$

where \(C_{1}(y)\) and \(C_{2}(y)\) are to be chosen so that \(f(x, y)\) satisfies the boundary conditions (2). Thus,

$$C_{1}(y) = \int_{0}^{1} k(y, y') \exp (-2a\alpha/\nu y') C_{2}(y') \, dy'$$

$$+ \int_{0}^{1} (y, y') G_{2}(y') \, dy', \quad y \in (-1, 0),$$
\( C_s(y) = \int \limits_{-1}^{0} h(y, y') \exp \left(2az/vy' \right) C_1(y') \, dy' \\
+ \int \limits_{-1}^{0} h(y, y') G_1(y') \, dy', \quad y \in (0, 1), \\
\)

where

\[ G_1(y) = \int \limits_{-a}^{a} \exp \left[ \frac{z}{vy} \left( a + x' \right) \right] g(x', y) \, dx', \quad y \in (-1, 0), \]

\[ G_2(y) = \int \limits_{-a}^{a} \exp \left[ \frac{-z}{vy} \left( a - x' \right) \right] g(x', y) \, dx', \quad y \in (0, 1). \]

To solve system (12a)-(12b), we introduce the Banach spaces

\[ Y_1 = L^1(-1, 0), \quad \| \varphi_1 \|_1 = \int \limits_{-1}^{0} |\varphi_1(y)| \, dy; \quad Y_2 = L^1(0, 1), \quad \| \varphi_2 \|_2 = \int \limits_{0}^{1} |\varphi_2(y)| \, dy; \]

\[ Y = Y_1 \times Y_2, \quad \| \varphi \|_{12} = \left\| \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right\|_{12} = \| \varphi_1 \|_1 + \| \varphi_2 \|_2, \]

and the operators

\[ (14a) \quad B_{21} \varphi_1 = \int \limits_{-1}^{0} k(y, y') \exp \left(2az/vy' \right) \varphi_1(y') \, dy', \quad D(B_{21}) = Y_1, \quad R(B_{21}) \subset Y_1, \]

\[ (14b) \quad B_{12} \varphi_2 = \int \limits_{0}^{1} k(y, y') \exp \left(-2az/vy' \right) \varphi_2(y') \, dy', \quad D(B_{12}) = Y_2, \quad R(B_{12}) \subset Y_1, \]

\[ (15) \quad B = \begin{pmatrix} 0 & B_{12} \\ B_{21} & 0 \end{pmatrix}, \quad D(B) = Y, \quad R(B) \subset Y. \]

Then, system (12a)-(12b) becomes

\[ (16) \quad C = BC + P, \quad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad P = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \]

with

\[ F_1(y) = \int \limits_{0}^{1} k(y, y') G_2(y') \, dy', \quad y \in (-1, 0), \]

\[ F_2(y) = \int \limits_{-1}^{0} k(y, y') G_1(y') \, dy', \quad y \in (0, 1). \]
Now, for each $z > 0$, we have from (14) and from (15)

$$
\|B_2 \varphi_1\|_2 < b \exp(-2az/v) \|\varphi_1\|_1, \quad \|B_1 \varphi_2\|_1 < b \exp(-2az/v) \|\varphi_2\|_2,
$$

$$
\|B \varphi\|_2 = \|B_1 \varphi_2\|_1 + \|B_2 \varphi_1\|_2 < b \exp(-2az/v) \|\varphi\|_2,
$$

because of (4). Moreover, we obtain from the two (17)

$$
\|F_1\|_1 < \int_{-1}^{0} dy \int_{0}^{1} k(y, y') |G_2(y')| \, dy' < b \int_{0}^{1} dy' \int_{-a}^{+a} |g(x', y')| \, dx',
$$

$$
\|F_2\|_2 < b \int_{-1}^{0} dy' \int_{-a}^{+a} |g(x', y')| \, dx'
$$

because of (4) and (13), and so $F \in \mathcal{Y}$ with $\|F\|_2 < b \|g\| < \infty$, $\forall g \in X$. Since $\{b \exp(-2az/v)\} < 1$ $\forall z > 0$, $(I-B)^{-1} \in \mathcal{B}(\mathcal{Y})$,

with $\|(I-B)^{-1} g\|_2 < \{1 - b \exp(-2az/v)\}^{-1} \|g\|_2$ and (16) gives

$$
C = (I-B)^{-1} F = \sum_{j=0}^{\infty} B^j F,
$$

with $\|C\|_x < \{1 - b \exp(-2az/v)\}^{-1} \|F\|_2 < b \{1 - b \exp(-2az/v)\}^{-1} \|g\|$. Relation (18) shows that $C_1$ and $C_2$ are uniquely determined by $g$ and depend linearly on $g$. In other words

$$
C_1 = \chi_1 g, \quad C_2 = \chi_2 g,
$$

with $\chi_j \in \mathcal{B}(X, Y_j)$, $j = 1, 2$, because

$$
\|C_j\|_j = \|\chi_j g\|_j < \|C\|_2 < b \{1 - b \exp(-2az/v)\}^{-1} \|g\|.
$$

Going back to (11b) we have for a.e. $y \in (0, 1)$

$$
\int_{-a}^{+a} |f(x, y)| \, dx < z^{-1} \{1 - \exp(-2az/vy)\} |C_2(y)|
$$

$$
+ \left( \frac{1}{vy} \right) \int_{-a}^{+a} dx' \int_{-a}^{+a} \exp(-z(x-x')/vy) |g(x', y)| \, dx < z^{-1} |C_2(y)|
$$

$$
+ z^{-1} \int_{-a}^{+a} \{1 - \exp(-z(a-x')/vy)\} |g(x', y)| \, dx'
$$

$$
< z^{-1} \{ |C_2(y)| + \int_{-a}^{+a} |g(x', y)| \, dx' \},
$$
\[
\int_0^1 dy \int_{-a}^{+a} |f(x, y)| \, dx < z^{-1} \{ \| C_2 \|_1 + \int_0^1 dy \int_{-a}^{+a} |g(x', y)| \, dx' \}.
\]

Since in an analogous way we obtain from (11a)
\[
\int_{-1}^{+1} dy \int_{-a}^{+a} |f(x, y)| \, dx < z^{-1} \{ \| C_1 \|_1 + \int_{-1}^{+1} dy \int_{-a}^{+a} |g(x', y)| \, dx' \},
\]
we conclude that
\[
\|f\| = \|(zI - A_o)^{-1} g\| < z^{-1} \{ \| C \|_1 + \| g \| \}
\]
\[
< z^{-1} \{ b [1 - b \exp (-2az/v)]^{-1} + 1 \} \| g \| \quad \forall z > 0, \quad g \in X,\]
and so \((zI - A_o)^{-1} \in \mathcal{B}(X), \forall z > 0.\)

(b) If \( z > 0 \) and \( g \in X_o \), then \( F_1 \) and \( F_2 \) are both nonnegative functions (see (13) and (17)), and (14a), (14b) show that \( B_2 F_1 \) and \( B_1 F_2 \) are nonnegative. It follows that \( C_1 \) and \( C_2 \) are nonnegative because of (15) and (18). Thus, \( f(x, y) \) is nonnegative (see the two (11)), i.e., \( f = (zI - A_o)^{-1} g \in X_o.\)

**Lemma 3.** \( \|(zI - A_o)^{-1} g\| < \|g\|/z, \forall z > 0, g \in X.\)

**Proof.** If \( z > 0 \) and \( f \in D(A_o) \cap X_o \), we have
\[
\|(zI - A_o)f\| \geq \left| \int_{-1}^{+1} dy \int_{-a}^{+a} [(zI - A_o)f] \, dx \right|
\]
\[
= \|f\| + v \int_{-1}^{+1} dy \left[ \int_{-a}^{+a} (\partial f/\partial x) \, dx \right] = \|f\| + v \int_{-1}^{+1} dy \left[ \int_{y(-a, y)}^{y(a, y)} dy \right] \|f\|
\]
\[
= \|f\| + v \left\{ \int_{-1}^{+1} dy \left[ \int_{y(-a, y)}^{y(a, y)} dy \right] \right\} + v \left\{ \int_{-1}^{+1} dy \left[ \int_{y(-a, y)}^{y(a, y)} dy \right] \right\}.
\]
But
\[
\int_{-1}^{+1} dy \left[ \int_{y(-a, y)}^{y(a, y)} dy \right] = \int_{-1}^{+1} dy \left[ \int_{y(-a, y)}^{y(a, y)} \right] = \int_{-1}^{+1} dy [1 - k(y)] \|f(a, y)\| \, dy > 0,
\]
\[
\int_{-1}^{+1} dy \left[ \int_{y(-a, y)}^{y(a, y)} dy \right] = \int_{-1}^{+1} dy [1 - k(y)] \|f(-a, y)\| \, dy > 0,
\]
because of (2) and (4) and because \( f(x, y) > 0 \) for a.e. \( (x, y) \in (-a, a) \times (-1, 1).\)
Hence,

\[(22) \quad \| (zI - A_0) f \| > \| f \| \quad \forall z > 0, \quad f \in D(A_0) \cap X_0.\]

Now, if \( g \in X_0 \) and \( f = (zI - A_0)^{-1} g \), then \( f \in X_0 \) because of (b) of Lemma 2, \((zI - A_0) f = g\), and (22) gives

\[(23) \quad \| (zI - A_0)^{-1} g \| < \| g \|/z \quad \forall z > 0, \quad g \in X_0.\]

Finally, if \( g \in X \), let \( g^-(x, y) = -g(x, y) \) if \( g(x, y) < 0 \), \( g^+(x, y) = 0 \) if \( g(x, y) > 0 \), \( g^+ g^- = 0 \) if \( g(x, y) < 0 \) and \( g^+ g^- = 0 \) if \( g(x, y) > 0 \). Then, \( g = g^+ - g^- \), \( g^+ \in X_0 \), \( g^- \in X_0 \), \( \| g \| = \| g^+ \| + \| g^- \| \) and we obtain from (23) (see also Appendix A)

\[\| (zI - A_0)^{-1} g \| < \| (zI - A_0)^{-1} g^+ \| + \| (zI - A_0)^{-1} g^- \| < (\| g^+ \| + \| g^- \|)/z = \| g \|/z.\]

**Remark 2.** If the conditions \((4)'\) are satisfied, then (22) becomes \(\| (zI - A_0) f \| = \| f \|, \forall z > 0, f \in D(A_0) \cap X_0\), and so

\[(23)' \quad \| (zI - A_0)^{-1} g \| = \| g \|/z \quad \forall z > 0, \quad g \in X_0.\]

3. - The semigroups generated by \( A_0, A, A + \nu \sigma, J \)

The operator \( A_0 \) is densely defined because \( D(A_0) \supset C_0^\omega ((-a, a) \times (-1, 1)) \). Furthermore, \( A_0 \) is closed because \(-(zI - A_0)^{-1}\in B(X) \subset C(X)\) for each \( z > 0 \), and so \((A_0 - zI) \in C(X)\) and \( A_0 = (A_0 - zI) + zI \in C(X)\) because \( zI \in B(X) \).

Since \( A_0 \) is densely defined and closed, it follows from Lemma 3 that \( A_0 \in \mathcal{D}(1, 0; X), [1], [4] \). Hence, \( A_0 \) generates the strongly continuous semigroup \([Z_0(t), t > 0]\), with \( \| Z_0(t) g \| < \| g \|, \forall t > 0 \), and with

\[\lim_{n \to \infty} \| Z_0(t) g - (I - (t/n) A_0)^{-n} g \| = 0 \quad \forall t > 0, \quad g \in X.\]

Now, if \( t > 0 \) and \( g \in X_0 \), then

\[(I - {t \over n} A_0)^{-n} g = [i {n \over t} I - A_0)^{-1}]^n g \in X_0\]

because of (b) of Lemma 2 and so \( Z_0(t) g \in X_0 \). If \( t = 0 \) and \( g \in X_0 \), then \( Z_0(t) g = g \) obviously belongs to \( X_0 \). Thus, we have
Theorem 1. (a) \( A_0 \in \mathcal{D}(1, 0; X) \); (b) the semigroup \( \{Z_0(t), t \geq 0\} \) generated by \( A_0 \) maps \( X_0 \) into itself for any \( t > 0 \).

Remark 3. If the conditions (4)' are satisfied, then (23') gives

\[
\| (I - \frac{t}{n} A_0)^{-1} g \| = \frac{n}{t} \| (\frac{t}{n} I - A_0)^{-1} g \| = \| g \| \quad \forall g \in X_0,
\]

\( n = 1, 2, \ldots, t > 0 \). Hence, \( \| [(I - (t/n) A_0)^{-1}]^n g \| = \| g \| \), and so

\[
(24) \quad \| Z_0(t) g \| = \| g \| \quad \forall t > 0, \quad g \in X_0.
\]

The physical meaning of relation (24) will be discussed later on, (see (28), (29) and the discussion that follows).

Theorem 2. (a) The operator \( A \) generates the semigroup \( \{Z(t), t \geq 0\} \), which is such that \( Z(t) = \exp(-
\nu \sigma t) Z_0(t), \forall t \geq 0 \); (b) \( A + \nu \sigma J \in \mathcal{D}(1, \nu (\sigma - \sigma); X) \); (c) if \( \{S(t), t \geq 0\} \) is the semigroup generated by \( A + \nu \sigma J \), then \( S(t) g \in X_0, \forall g \in X_0, t \geq 0 \).

Proof. (a) is obvious because \(-\nu \sigma I \) commutes with \( A_0 \). (b) \( A \in \mathcal{D}(1, -\nu \sigma; X) \) because \( \| Z(t) f \| < \exp(-\nu \sigma t) \| f \| \), and \( \nu \sigma J \in \mathcal{D}(X) \) with \( \| \nu \sigma J f \| < \nu \sigma \| f \| \) because of (a) of Lemma 1. Hence, \( A + \nu \sigma J \in \mathcal{D}(1, -\nu \sigma + \nu \sigma; X) \). (c) We have

\[
\lim_{n \to \infty} \| S(t) g - \sum_{j=0}^{n} Z_j(t) g \| = 0, \quad t \geq 0, \quad g \in X,
\]

with

\[
Z_0(t) g = Z(t) g, \quad Z_{j+1}(t) g = \nu \sigma \int_{0}^{t} Z(t-s) J Z_j(s) g \, ds, \quad j = 0, 1, \ldots.
\]

Thus, if \( g \in X_0 \), \( Z_0(t) g = \exp(-\nu \sigma t) Z_0(t) g \), \( Z_1(t) g \), \( Z_2(t) g \), \( Z_3(t) g \), \( Z_4(t) g \), \ldots, all belong to \( X_0 \) because of (b) of Theorem 1. If follows that \( S(t) g \) also belongs to the closed cone \( X_0 \).

Remark 4. Under the assumptions (4'), (24); and (a) of Theorem 2 give:

\[
\| Z_0(t) g \| = \| Z(t) g \| = \exp(-\nu \sigma t) \| g \|, \quad g \in X_0, \quad t \geq 0, \quad \text{and so}
\]

\[
\| Z_1(t) g \| = \nu \sigma \int_{0}^{t} \| Z(t-s) J Z_0(s) g \| \, ds
\]

\[
= \nu \sigma \int_{0}^{t} \| Z(t-s) J Z_0(s) g \| \, ds = \nu \sigma \int_{0}^{t} \exp[-\nu \sigma (t-s)] \| J Z_0(s) g \| \, ds
\]

\[
= \nu \sigma \int_{0}^{t} \exp[-\nu \sigma (t-s)] \| Z_0(s) g \| \, ds = \nu \sigma t \exp(-\nu \sigma t) \| g \|
\]
because \(Z(t-s)jZ_0(s)g \in X_\alpha \ \forall s \in [0, t] \). By a similar procedure, we have
\[
\|Z_j(t)g\| = \sum_{j=0}^{\infty} (v|\sigma_j|/|j|! \exp(-v\sigma|t||j|! \exp(-v\sigma t)\|g\|), \ \forall g \in X_\alpha, \ t > 0,
\]
and so
\[
\|S(t)g\| = \sum_{j=0}^{\infty} (v|\sigma_j|/|j|! \exp(-v\sigma|t||j|! \exp(-v\sigma t)\|g\| = \exp [v(\sigma - \sigma)t] \|g\|, \ \forall g \in X_\alpha, \ t > 0.
\]

4. - The abstract problem (9)

If \(u_0 \in D(A) \cap X_\alpha \) and \(q = q(t)\) is continuously differentiable and belongs to \(X_\alpha\) at any \(t > 0\), then the unique strict solution of (9) can be written as follows

\[
u(t) = S(t)u_0 + \int_0^t S(t-s)q(s)ds, \quad t > 0,
\]

and \(u(t) \in D(A) \cap X_\alpha \ \forall t > 0\) because of (c) of Theorem 2, [1], [4]. We have from (26)

\[
\|u(t)\| = \|S(t)u_0\| + \int_0^t \|S(t-s)q(s)\|ds \\
< \exp [v(\sigma - \sigma)t] \|u_0\| + \int_0^t \exp [v(\sigma - \sigma)(t-s)] \|q(s)\|ds,
\]

where \(\|u(t)\| = N(t)\), \(\|u_0\| = N(0)\) are the total numbers of particles in the slab at time \(t\) and at time \(t = 0\), see (5).

If in particular assumptions (4)' are satisfied, then (25) and (26) give

\[
\|u(t)\| = \exp [v(\sigma - \sigma)t] \|u_0\| + \int_0^t \exp [v(\sigma - \sigma)(t-s)] \|q(s)\|ds,
\]

and so

\[
\frac{d}{dt} N(t) = v(\sigma - \sigma) N(t) + \|q(t)\|, \quad (29b) \quad N(0) = \|u_0\|.
\]

(29a) is an equation of balance which takes into account that particles are not captured by the boundary planes. Note that (29a) can be derived in a heuristic way by integrating both sides of (1) with respect to \(x\) and \(y\) and taking into account (2a), (2b), and (4'). However, as it was proved above, (29a) can be obtained by a rigorous procedure from (26) and by using (24), which shows
that the evolution operator $Z_0(t)$ generated by the free-streaming operator $A_0$
does not change the total number of particles in the slab (provided that $(4')$ are satisfied).

The above results can be summarized as follows.

**Theorem 3.** Assume that $u_0 \in D(A) \cap X_0$ and that $q(t)$ is continuously
differentiable and belongs to $X_0$ at any $t \geq 0$. Then if the assumptions $(4)$ are satisfied
the evolution problem (9) has a unique strict solution $u = u(t)$ that is
defined by (26), belongs to $D(A) \cap X_0$, and satisfies the inequality (27).
Moreover, under the assumptions $(4')$, $N(t) = \|u(t)\|$ is the solution of system (29a)-(29b).

**Appendixes**

**A.** - The properties $(zI - A_0)^{-1} \in B(X)$ and $\| (zI - A_0)^{-1} \| < \| g \| / z$ \forall $z > 0$,
g $\in X$ can be derived directly from (11a)-(11b) as follows.

If $y \in (-1, 0)$, then we have from (11a)

$$\int_{-a}^{+a} |f(x, y)| dx \leq z^{-1} [1 - \exp (2z \alpha / \nu y)] |C_1(y)|$$

$$+ z^{-1} \int_{-a}^{+a} \{1 - \exp [z(a + x')/\nu y]\} |g(x', y)| dx',$$

whereas (11b) gives for $y \in (0, 1)$

$$\int_{-a}^{+a} |f(x, y)| dx \leq z^{-1} [1 - \exp (-2z \alpha / \nu y)] |C_2(y)|$$

$$+ z^{-1} \int_{-a}^{+a} \{1 - \exp [-z(a - x')/\nu y]\} |g(x', y)| dx'.$$

On the other hand, we obtain from (12a) and from (12b)

$$\int_{-1}^{0} |C_1(y)| dy \leq \int_{0}^{1} \tilde{h}(y') \exp (-2z \alpha / \nu y') |C_2(y')| dy'$$

$$+ \int_{0}^{1} [\tilde{h}(y') \int_{-a}^{+a} \exp (-z(a - x')/\nu y') |g(x', y')| dx'] dy',$$

$$\int_{0}^{1} |C_2(y)| dy \leq \int_{-1}^{0} \tilde{h}(y') \exp (2z \alpha / \nu y') |C_1(y')| dy'$$

$$+ \int_{-1}^{0} [\tilde{h}(y') \int_{-a}^{+a} \exp [z(a + x')/\nu y'] |g(x', y')| dx'] dy'.$$
and so

\[
\begin{align*}
\|z\| & \leq \int_{-1}^{0} |C_1(y)|\,dy - \int_{-1}^{0} \exp(2za/y'') |C_2(y')|\,dy' \\
& \quad + \int_{0}^{1} |C_2(y')|\,dy' - \int_{0}^{1} \exp(-2za/y'') |C_2(y')|\,dy' \\
& \quad + \int_{-1}^{0} dy'' \int_{-a}^{+a} \{1 - \exp[z(a + x')/y'']\} |g(x', y')|\,dx' \\
& \quad + \int_{0}^{1} dy'' \int_{-a}^{+a} \{1 - \exp[-z(a - x')/y'']\} |g(x', y')|\,dx'.
\end{align*}
\]

\[
\leq \int_{0}^{1} [\tilde{h}(y'') - 1] \exp(-2za/y'') |C_2(y')|\,dy' \\
\quad + \int_{-1}^{0} [\tilde{h}(y'') - 1] \exp(2za/y'') |C_1(y')|\,dy' \\
\quad + \int_{0}^{1} dy'' [\tilde{h}(y'') - 1] \int_{-a}^{+a} \exp[-z(a - x')/y''] |g(x', y')|\,dx' \\
\quad + \int_{-1}^{0} dy'' [\tilde{h}(y'') - 1] \int_{-a}^{+a} \exp[z(a + x')/y''] |g(x', y')|\,dx'.
\]

Hence, if \( z > 0 \) and \( g \in X \), we have

\[
(30) \quad \|z\| \leq \|g\| + (b - 1) \exp(-2za/v)\|C_{12}\| + (b - 1)\|g\| \\
\leq b\|g\| + \frac{b(b - 1) \exp(-2za/v)}{1 - b \exp(-2za/v)} \|g\| = b \frac{1 - \exp(-2za/v)}{1 - b \exp(-2za/v)} \|g\|
\]

and so

\[
(31) \quad \|z\| \leq \|g\|
\]

provided that \( b < 1 \).

Remark 5. If \( \tilde{h}(y'') = 1 \ \forall y'' \in (0, 1), \tilde{h}(y'') = 1 \ \forall y'' \in (-1, 0), \) (see (4')), and if \( g \in X', \) then the above procedures give \( \|z\| = \|g\| \).
Remark 6. Inequality (30) holds even if \( b > 1 \). However, in this case, \( z \) must be taken larger than \( z_0 \), with \([b \exp(-2z_\kappa v)] = 1\). In fact, if \( z > z_0 \), then \([b \exp(-2z_\kappa v)] < 1\) and \((I-B)^{-1} \in \mathcal{B}(X)\), i.e., \( C \) is uniquely determined by \( g \), (see the proof of (a) of Lemma 2).

B. – Assume that (i) \( \sigma = \sigma_1 \); (ii) \( h(y, y') = 2y \) for a.e. \( (y, y') \in (0, 1) \times (-1, 0) \), \( k(y, y') = -2y \) for a.e. \( (y, y') \in (-1, 0) \times (0, 1) \). Note that (i) and (ii) imply that the materials of the slab under consideration and the boundary planes do not capture particles. Assumption (ii) also shows that an outgoing isotropic density is reflected isotropically (i.e., if for instance \( f(a, y') = c = a \) constant for a.e. \( y' \in (-1, 0) \), then \( f(a, y) = c \) for a.e. \( y \in (0, 1) \)).

Under the assumptions (i), (ii), it is not difficult to show that the equation

\[
(A_0 - \nu \sigma I + \nu \sigma J)f = 0
\]

has the solution \( f(x, y) = c \). In other words, (i) and (ii) imply that the transport operator \((A_0 - \nu \sigma I + \nu \sigma J)\) has the eigenvalue \( z = 0 \) and that the corresponding eigenfunction \( f \) does not depend on \( x \) and \( y \). Thus, if \( u_\theta(x, y) = c \) and \( q(t) = 0 \quad \forall t > 0 \), then the unique solution of (9) is \( u(t) = u(x, y; t) = c, \forall t > 0 \).

To show that \( f(x, y) = c \) satisfies (32), we re-write (32) into the equivalent form

\[
f = \nu \sigma (\nu \sigma I - A_0)^{-1} J f;
\]

where \( \nu \sigma (\nu \sigma I - A_0)^{-1} J f \) is given by (11a)-(11b) with \( z = \nu \sigma \) and with \( g = \nu \sigma J f \). Now, if \( f(x, y) = c \), then \( g = \nu \sigma c \) and (11a)-(11b) with \( z = \nu \sigma \) give

\[
c = (-1/y \nu) C_1(y) \exp[\sigma(a - x)/y]
\]

\[
+ c \{1 - \exp[-\sigma(x - a)/y]\}, \quad y \in (-1, 0),
\]

\[
c = (1/y \nu) C_2(y) \exp[-\sigma(a + x)/y]
\]

\[
+ c \{1 - \exp[-\sigma(x + a)/y]\}, \quad y \in (0, 1).
\]

On the other hand, we have from (12a)-(12b) with \( z = \nu \sigma \)

\[
C_1(y)/(-y) = 2 \int_0^1 \exp(-2a \sigma /y') \, G_2(y') \, dy' + 2 \int_0^1 \, G_2(y') \, dy', \quad y \in (-1, 0),
\]

\[
C_2(y)/y = 2 \int_{-1}^0 \exp(2a \sigma /y') \, C_1(y') \, dy' + 2 \int_{-1}^0 \, G_1(y') \, dy', \quad y \in (0, 1),
\]
and so $C_1(y) / (-y) = c_1$, $C_2(y) / y = c_2$. Hence

$$c_1 = 2c_2 \int_0^1 y' \exp(-2a\sigma/y') \, dy' + 2ve \int_0^1 y' [1 - \exp(-2a\sigma/y')] \, dy',$$

$$c_2 = -2c_1 \int_{-1}^0 y' \exp(2a\sigma/y') \, dy' - 2ve \int_{-1}^0 y' [\exp(2a\sigma/y') - 1] \, dy',$$

i.e.,

$$c_1 = 2(c_2 - ve) \int_0^1 y' \exp(-2a\sigma/y') \, dy' + ve,$$

$$c_2 = 2(c_1 - ve) \int_0^1 y' \exp(-2a\sigma/y') \, dy' + ve.$$

It follows that $c_1 = c_2 = ve$, $C_1(y) = -yve$, $C_2(y) = yve$, and that consequently the two (34) are identically satisfied. Thus, $f(x, y) = e$ satisfies (33) and (32).

References


Sommario

Si prova esistenza ed unicità di una soluzione sommabile per un problema integrodifferenziale della teoria del trasporto di particelle in un muro omogeneo limitato da piani capaci di catturare e di riflettere particelle. Si studiano quindi alcune proprietà di tale soluzione.

* * *