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DELFINA ROUX and ELISABETTA MALUTA (*)

Contractive Kannan maps in compact spaces (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. – Let (X, d) be a metric space and $T: X \to X$ satisfy, for every $x, y \in X$,

$$d(Tx, Ty) \leqslant b(x, y)\, d(x, Tx) + b(y, x)\, d(y, Ty) \tag{1.1}$$ (1)

with $b: X \times X \to [0, 1)$, such that

$$b(x, y) + b(y, x) \leqslant 1 \tag{1.2}$$

$$b(x, y) \to 1 \Rightarrow \text{Max}\{d(x, Tx),\, d(y, Ty)\} \to 0 \quad \text{or } \infty . \tag{1.3}$$

We call such a map *a (generalized) Kannan map*; if, moreover, in (1.1) the strict inequality holds for every x and y in X, $x \neq y$, we call T *a contractive Kannan map*.

A Kannan map can't have more than one fixed point; if (X, d) is complete and $\sup_{x,y \in X} (b(x, y) + b(y, x)) < 1$, then T has a fixed point u (and, for each $x \in X$, $T^n x \to u$). If $\sup_{x,y \in X} (b(x, y) + b(y, x)) = 1$ the result is no longer true,

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(*) Indirizzo degli AA.: Istituto Matematico, Università, Via Saldini 50, 20133 Milano, Italy.

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(1) Maps satisfying (1.1) were first considered by R. Kannan [**1**]$_1$ (in the case $b(x, y) = b(y, x) = K < \frac{1}{2}$), by S. Reich [**4**]$_1$ (in the case $K = \frac{1}{2}$); by R. M. Tiberio Bianchini [**6**] (in the case $b(x, y) + b(y, x) \leqslant K < 1$) and by S. Massa [**2**] (in a more general form).

even if T is a contractive Kannan map and, moreover, $b(x, y) = b(y, x)$. (It suffices to consider the space (X, d) of the points $x_n \in l^1$ of the form $x_n = (1 + 1/n) e_n$, where e_n is the natural basis of l^1 and the map $T\colon x_n \to x_{n+1}$).

In this paper we give a simple fixed point theorem for mappings of Kannan type in compact topological spaces. We obtain, as a consequence, that every contractive Kannan self-mapping of a closed ball in a conjugate normed space has a fixed point.

The result seems to be new and of some interest.

2. – Let $O(x)$ be the set $\bigcup_{n=0}^{\infty} \{T^n x\}$ and $\overline{O(x)}$ its closure. The following theorem holds.

Theorem 1. *Let (X, τ) be a topological Hausdorff compact space; $\varphi\colon X \times X \to R^+$ be lower semicontinuous and $T\colon X \to X$ be such that $\forall x, y \in X$*

$$(2.1) \qquad \varphi(Tx, Ty) \leqslant b(x, y)\varphi(x, Tx) + b(y, x)\varphi(y, Ty)$$

with $b\colon X \times X \to [0, 1)$ satisfying (1.2) and

$$(1.3)' \qquad b(x, y) \to 1 \Rightarrow \operatorname{Max}\{\varphi(x, Tx), \varphi(y, Ty)\} \to 0 \quad \text{or } \infty .$$

If

$$(2.2) \qquad x \neq Tx \Rightarrow \exists y \in \overline{O(x)}\colon \varphi(y, Ty) < \varphi(x, Tx)$$

then T has a fixed point in X.

Moreover, if $\lim_{n \to +\infty} \varphi(T^n x, T^{n+1} x) = \operatorname{Inf}_{x \in X} \varphi(x, Tx)$ (2)*, then the limit points of the sequence $\{T^n x\}$ (if any) are fixed points of T.*

Remarks. 1.– The theorem doesn't contain any hypothesis of continuity of T.

2.– Even if $\varphi(Tx, T^2x) < \varphi(x, Tx)$ $\forall x \neq Tx$, and moreover $b(x, y) = b(y, x)$, T can have more than one fixed point and $\lim_{n \to +\infty} \varphi(T^n x, T^{n+1} x) = \operatorname{Inf}_{x \in X} \varphi(x, Tx)$ doesn't imply that $\{T^n x\}$ converges. (Indeed, let $X = [-2, 2]$ with the usual metric d, $A = [-1, 1]$, $\varphi(x, y) = \operatorname{Min}\{d(x, A), d(y, A)\}$ and consider the map $Tx = x$ if $x \in A$, $Tx = -\frac{1}{2}(x + x/|x|)$ if $x \in X \setminus A$).

3.– If $\lim_{n \to +\infty} \varphi(T^n x, T^{n+1} x) \neq \operatorname{Inf}_{x \in X} \varphi(x, Tx)$, the limit points of $\{T^n x\}$ don't need to be fixed, even if $\varphi = d$ and T is a sequentially contractive (3) Kannan

(2) Observe that (2.1) ensures $\varphi(Tx, T^2x) \leqslant \varphi(x, Tx)$.

(3) i.e. $\forall x \neq Tx$, $d(Tx, T^2x) < d(x, Tx)$.

map and $b(x, y) = b(y, x)$. (Indeed let X, d, A as above, $\varphi = d$ and consider the map $Tx = 0$ if $x \in A$, $Tx = -\frac{1}{2}(x + x/|x|)$ if $x \in X \setminus A$).
4. – If (1.3)′ doesn't hold, the theorem fails to be true, even if in (2.1) the strict inequality holds (consider $X = [0, 1]$ with the usual metric d, $\varphi = d$, $Tx = \frac{1}{2}x$ if $x \neq 0$ and $Tx = 1$ if $x = 0$).
5. – If (2.2) does not hold, the theorem fails to be true (consider the metric space $\{0\} \cup \{1\}$ with $d(x, y) = |x - y|$, $\varphi = d$, $T(0) = 1$, and $T(1) = 0$).

Proof of Theorem 1. For each $r \in R^+$ let us set

$$A_r = \{x \in X : \varphi(x, Tx) \leqslant r\}$$

and, if $A_r \neq \emptyset$, $B_r = \operatorname{cl} T(A_r)$. Let $r_0 = \operatorname{Inf}\{r : A_r \neq \emptyset\}$.

Lemma. *$A_{r_0} \neq \emptyset$ and $B_{r_0} \subset A_{r_0}$* [4].

Indeed let $A_r \neq \emptyset$ and $x \in B_r$. For each $\varepsilon > 0$, there exists $y \in A_r$ such that

$$\varphi(x, Tx) - \varepsilon \leqslant \varphi(Ty, Tx) \leqslant b(y, x)\,\varphi(y, Ty) + b(x, y)\,\varphi(x, Tx)\,.$$

From (1.2) and (1.3)′ we get $x \in A_r$; hence $B_r \subset A_r$ and $B_r \subset B_{r'}$ if $r < r'$. Then $C = \bigcap_{r > r_0} B_r \neq \emptyset$. But $x \in C \Rightarrow x \in A_{r_0}$ and the lemma follows.

Now observe that, if $x \in A_{r_0}$, the lemma gives $\overline{O(x)} = \{x\} \cup \overline{O(Tx)} \subset A_{r_0}$, absurd if $x \neq Tx$ when (2.2) holds. So $x \in A_{r_0} \Rightarrow x = Tx$.

Finally, if $\varphi(T^n x, T^{n+1} x) \to r_0$ and z is a limit point of $\{T^n x\}$, the lower semicontinuity of φ implies $z \in A_{r_0}$ and so $z = Tz$ [5] and Theorem 1 is proved.

3. – Now let us list some consequences of Theorem 1.

Theorem 2. *If (X, d) is a compact metric space, and $T : X \to X$ is a Kannan map such that $x \neq Tx \Rightarrow \exists y \in \overline{O(x)} : d(y, Ty) < d(x, Tx)$, then T has a fixed point u and, if T is asymptotically regular at x* [6], *$T^n x \to u$.*

If $b(x, y) = b(y, x) = \frac{1}{2}$, we obtain Proposition 1 of **[4]**$_2$.

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(4) The Lemma doesn't require use of (2.2).

(5) More generally, if $\varphi(y_n, Ty_n) \to r_0$, the limit points of the sequence $\{y_n\}$ are fixed.

(6) i.e. $d(T^n x, T^{n+1} x) \to 0$. This condition cannot be omitted in this and in the following Theorems 3 and 4: see the counter-example in Remark 3.

Corollary 1. *Every sequentially slowly contractive* (7) *Kannan map on a compact metric space* (X, d) *has a fixed point* (8).

If $b(x, y) = b(y, x) = \frac{1}{2}$, we obtain Theorem 2 of **[1]**$_2$ without any hypothesis of continuity.

Corollary 2. *Every contractive Kannan map on a compact metric space* (X, d) *has a fixed point.*

Theorem 3. *Let* X *be a weakly compact subset of a normed space* S. *If* $T: X \to X$ *is a sequentially slowly contractive Kannan map, then* T *has a fixed point* u *in* X *and, if* T *is asymptotically regular at* x, $T^n x \to u$.

Proof. From Theorem 1, assuming $\varphi(x, y) = \|x - y\|$, we obtain that the fixed point u exists. Moreover, if T is asymptotically regular at x,

$$\|u - T^n x\| = \|Tu - T^n x\| \leqslant b(T^{n-1}x, u)\, \|T^{n-1}x - T^n x\|$$

and the theorem follows.

Corollary 3. *If* X *is a closed convex bounded subset of a reflexive Banach space, every sequentially contractive Kannan map* $T: X \to X$ *has a fixed point in* X.

If $b(x, y) = b(y, x) = \frac{1}{2}$, we obtain a result of **[5]**, p. 111.

Theorem 4. *Let* X *be a weakly*compact subset of a conjugate normed space* S. *If* $T: X \to X$ *is a sequentially slowly contractive Kannan map, then* T *has a fixed point* u *in* X *and, if* T *is asymptotically regular at* x, $T^n x \to u$.

The proof is similar to that one of Theorem 3.

Corollary 4. *If* X *is a closed ball in a conjugate normed space, every contractive Kannan map* $T: X \to X$ *has a fixed point in* X.

The problem whether this result holds in any Banach space is still open.

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(7) i.e. $\forall x \neq Tx$, $\exists\, n = n(x)$: $d(T^n x, T^{n-1}x) < d(x, Tx)$.

(8) Observe that this and the following results cannot be derived from Theorem 1 of **[3]** (indeed $d(x, Tx)$ is not, in general, lower semicontinuous).

References

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Summary

We give a fixed point theorem for maps of Kannan type in compact topological spacesg We obtain as a consequence, among other things, that every contractive Kannan selfmappin. of a closed ball in a conjugate normed space has a fixed point.

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