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Compact perturbations of some nonlinear Hammerstein equations (**)

A GIORGIO SESTINI per il suo 70° compleanno

Introduction

Much research has been devoted in recent years to abstract nonlinear Hammerstein equations in Banach spaces and the interested reader may consult Browder's survey paper [**4**]. Those results on Hammerstein equations have been used by Browder [**4**] and by Brézis and Browder [**3**] to prove Leray-Schauder's type continuation theorems for noncompact perturbations of the identity.

The aim of this paper is first to present an unified abstract scheme for those type of continuation theorems, by the introduction in **1** of the concept of *pairs of continuously Hammerstein compatible mappings* and the proof of the corresponding continuation theorems via the usual Leray-Schauder's theory [**8**].

In **2**, we first generalize and complete, with a much simpler proof, a result of De Figueiredo-Gupta [**5**] on the existence and uniqueness of solutions of the Hammerstein equation

$$u + MNu = f \tag{0.1}$$

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in a Hilbert space where M satisfies a nonlinear version of a condition introduced by Hess [**6**] and N is not necessarily monotone. By proving for (0.1) a result on the continuous dependence of the solution of (0.1) with respect to M, N and f, we are then able to associate to the type of assumptions introduced by De Figueiredo and Gupta a class of pairs of continuously Hammerstein compatible mappings. By the results of **1** a corresponding Leray-Schauder's type continuation theorem is then available which has been shown in [**9**] to be useful in the study of the range of nonlinear perturbations of linear mappings with an infinite dimensional kernel and will be applied to other alternative problems in a subsequent paper. One also obtains in this section a result on the convergence of Galerkin's approximations to the solution.

In **3**, we give a direct and simple proof of an existence and uniqueness result due to Koscikii [**7**] and De Figueiredo and Gupta [**5**], for the solutions of (0.1) when N is strongly monotone and M is and indefinite linear mapping which splits in a certain way. Again a result of the continuous dependence of the solutions with respect to M, N, f is proved and applied to the description of an associate pair of continuously Hammerstein compatible mappings and to the convergence of the Galerkin's approximations.

1. - Pairs of continuously Hammerstein compatible mappings and continuation theorems for noncompact perturbations of the identity

Let X and Y be real Banach spaces and let $J = [0, 1]$.

Definition 1.1. A pair $(\mathscr{M}, \mathscr{N})$ of mappings $\mathscr{M}\colon Y \times J \to X$, $\mathscr{N}\colon X \times J \to Y$ is said to be *continuously Hammerstein compatible* (shortly *ch-com- patible*) if, for each $\lambda \in J$ and each $f \in X$ the Hammerstein equation

$$u + \mathscr{M}(\mathscr{N}(u, \lambda), \lambda) = f \tag{1.1}$$

has a unique solution and the corresponding mapping

$$\mathscr{S}\colon X \times J \to X\,, \qquad (f, \lambda) \to [I + \mathscr{M}(\mathscr{N}(\cdot, \lambda), \lambda)]^{-1} f$$

is continuous and bounded.

Recall that a mapping between metric spaces is called *bounded* if it takes bounded sets into bounded sets.

A simple but useful example of ch-compatible pair can be obtained by the use of the Banach fixed point theorem (see e.g. [**1**]), the details of the proof being left to the reader.

Proposition 1.1. *Let* $\mathscr{M}: Y\times J \to X$ *and* $\mathscr{N}: X\times J \to Y$ *be mappings such that, for some* $k\in[0, 1[$, *all* $\lambda\in J$, $u\in X$, $v\in X$ *one has*

$$|\mathscr{M}(\mathscr{N}(u, \lambda), \lambda) - \mathscr{M}(\mathscr{N}(v, \lambda), \lambda)| \leqslant k|u - v| ,$$

and such that, for each $u\in X$, *the mapping*

$$\lambda \to \mathscr{M}(\mathscr{N}(u, \lambda), \lambda)$$

is continuous and bounded. Then $(\mathscr{M}, \mathscr{N})$ *is ch-compatible and, for each* $(f, \lambda)\in X\times J$, *one has*

$$\mathscr{S}(f, \lambda) = \lim_{n\to\infty} u_n ,$$

where $u_1\in X$ *is arbitrary and*

$$u_{n+1} = f - \mathscr{M}(\mathscr{N}(u_n, \lambda), \lambda) \qquad (n = 1, 2, \ldots) ,$$

with the estimate

$$|u_n - \mathscr{S}(f, \lambda)| \leqslant (1 - k)^{-1} k^n |f - u_1 - \mathscr{M}(\mathscr{N}(u_1, \lambda), \lambda)| .$$

We shall now show how the concept of continuously Hammerstein compatible pair is useful in formulating Leray-Schauder's type continuation theorems for some not necessarily compact perturbations of the identity. Recall that a mapping $F: A \to B$ between the metric spaces A and B is called *compact* if F is continuous on A and $F(A)$ is relatively compact in B.

Let $\Omega\subset X$ be an open bounded set, with closure $\bar{\Omega}$ and boundary $\partial\Omega$. If $F: \bar{\Omega} \to X$ is such that $F = I + G$ with I the identity on X and $G: \bar{\Omega}\to X$ is compact, and if $f\in X\setminus F(\partial\Omega)$, the *Leray-Schauder degree* [**8**] *of* F *in* Ω *at* f is defined and we shall denote it by $d[F, \Omega, f]$.

Theorem 1.1. *Let* $(\mathscr{M}, \mathscr{N})$ *be a pair of ch-compatible mappings* $\mathscr{M}$: $Y\times J \to X$, $\mathscr{N}: X\times J \to Y$ *and let* $\mathscr{C}: X\times J \to X$ *be a compact mapping such that the following conditions are satisfied.*

(1) *For each* $(u, \lambda)\in \partial\Omega\times J$, *one has*

$$u + \mathscr{M}(\mathscr{N}(u, \lambda), \lambda) \neq \mathscr{C}(u, \lambda) .$$

(2) $d[I - \mathscr{S}(\mathscr{C}(\cdot, 0), 0), \Omega, 0] \neq 0.$

Then, for each $\lambda \in J$, the equation

$$u + \mathcal{M}(\mathcal{N}(u, \lambda), \lambda) = \mathcal{C}(u, \lambda) \tag{1.2}$$

has at least one solution $u \in \Omega$.

Proof. By definition of a pair of ch-compatible mappings, equation (1.2) is equivalent to the equation

$$u = \mathcal{T}(u, \lambda)\,, \quad \text{where} \quad \mathcal{T}(u, \lambda) = \mathcal{S}(\mathcal{C}(u, \lambda), \lambda)\,, \quad (u, \lambda) \in \bar{\Omega} \times J\,,$$

so that

$$\mathcal{T} : \bar{\Omega} \times J \to X \quad \text{is continuous} \quad \text{and} \quad \mathcal{T}(\bar{\Omega} \times J) = \mathcal{S}(\mathcal{C}(\bar{\Omega} \times J) \times J]$$

is relatively compact. Thus, by assumption (1), $u \neq \mathcal{T}(u, \lambda)$ for every $(u, \lambda) \in \partial\Omega \times J$ and the homotopy invariance of Leray-Schauder's degree (see e.g. [**1**]) implies that

$$d[I - \mathcal{S}(\mathcal{C}(\cdot, \lambda), \lambda), \Omega, 0] = d[I - \mathcal{S}(\mathcal{C}(\cdot, 0), 0), \Omega, 0] \neq 0\,, \tag{1.3}$$

by assumption (2). The result then follows from (1.3) and the existence property of Leray-Schauder's degree (see e.g. [**1**]).

Corollary 1.1. *Assume that $\mathcal{M}$, $\mathcal{N}$, $\mathcal{C}$ verify condition* (1) *of Theorem* 3.1 *and that $\mathcal{C}(u, 0) = f$ for every $u \in \bar{\Omega}$ and some $f \in X$ such that $\mathcal{S}(f, 0) \in \Omega$. Then the conclusion of Theorem* 1.1 *holds.*

Proof. By the assumptions and usual properties of the Leray-Schauder's degree (see e.g. [**1**]), one has

$$d[I - \mathcal{S}(\mathcal{C}(\cdot, 0), 0), \Omega, 0] = d[I, \Omega, \mathcal{S}(f, 0)] = 1$$

and the result follows from Theorem 1.1.

Corollary 1.2. *Assume that $\mathcal{M}$, $\mathcal{N}$, $\mathcal{C}$ verify condition* (1) *of Theorem* 3.1, *that Ω is symmetric with respect to* 0, *with $0 \in \Omega$, and that, for every $u \in \partial\Omega$, one has*

$$\mathcal{M}(\mathcal{N}(-u, 0), 0) = -\mathcal{M}(\mathcal{N}(u, 0), 0)\,, \tag{1.4}$$

$$\mathscr{C}(-u, 0) = -\mathscr{C}(u, 0). \tag{1.5}$$

Then the conclusion of Theorem 1.1 *holds.*

Proof. Clearly, (1.4) and (1.5) imply that, for each $(u, \lambda) \in \partial\Omega$, one has

$$\mathscr{S}(\mathscr{C}(-u, 0), 0) = \mathscr{S}(-\mathscr{C}(u, 0), 0) = -\mathscr{S}(\mathscr{C}(u, 0), 0),$$

and then, by the Krasnosel'skii-Borsuk's theorem (see e.g. [1]), it comes

$$d[I - \mathscr{S}(\mathscr{C}(\cdot, 0), 0), \Omega, 0] = 1 \quad (\text{mod. } 2).$$

Thus the result follows from Theorem 1.1.

2. - A class of pairs of continuously Hammerstein compatible mappings in a Hilbert space

Let now H be a real Hilbert space with inner product $(\,,\,)$ and corresponding norm $|\cdot|$. Recall that a map $F: H \to H$ is called *demi-continuous* if $u_n \to u$ implies $Fu_n \rightharpoonup Fu$.

Definition 2.1. A pair (M, N) of mappings from H into H is said to be *Hammerstein-compatible with constants a and b* (shortly *$h(a, b)$-compatible*) if the following conditions hold.

(i) $0 \leqslant b < a$; (ii) for each $u \in H$ and each $v \in H$, one has

$$(Mu - Mv, u - v) \geqslant a|Mu - Mv|^2;$$

(iii) for each $u \in H$ and each $v \in H$, one has

$$(Nu - Nv, u - v) \geqslant -b|u - v|^2;$$

(iv) N is demi-continuous on H; (v) $M(0) = 0$.

Remark 2.1. We shall see that the class of $h(a, b)$-compatible pair of mappings is related to the unique solvability of the Hammerstein equation

$$u + MNu = f, \tag{2.1}$$

for every $f \in H$. In this respect, the condition (v) is not really a restriction, because if (M, N) satisfies (i) to (iv), then equation (2.1) is equivalent to

$$u + \tilde{M}Nu = \tilde{f},$$

where $\tilde{M}(u) = M(u) - M(0)$ and $\tilde{f} = f - M(0)$ and the pair $(\tilde{M}, N)$ is now $h(a, b)$-compatible.

Remark 2.2. One shall notice that condition (ii) implies that M is monotone and Lipschitzian with Lipschitz constant a^{-1}. Consequently, M is maximal monotone. (See e.g. [**2**] for this result and the concepts of monotonicity and maximal monotonicity).

The following result generalizes and completes, with a simpler proof, a theorem due to De Figueiredo and Gupta [**5**] (see also Browder [**4**] for other special cases). It has been announced, together with Proposition 2.2., in [**9**].

Proposition 2.1. *Let (M, N) be a pair of $h(a, b)$-compatible mappings from H into itself. Then, for each $f \in H$, equation* (2.1) *has a unique solution u which satisfies the estimate*

$$|u - f| \leqslant (a - b)^{-1} |Nf| . \tag{2.2}$$

Proof. If we define $T: H \to 2^H$ by

$$Tu = - M^{-1}(f - u) + Nu ,$$

equation (2.1) is clearly equivalent to equation

$$0 \in Tu , \tag{2.3}$$

and the existence and uniqueness result will follow if we show that T is maximal monotone and strongly monotone (see e.g. [**2**]). Using (ii) and (iii) in Definition 2.1 we obtain easily that, for every (u_1, v_1) and (u_2, v_2) in the graph of T, one has

$$(v_1 - v_2, u_1 - u_2) \geqslant (a - b)|u_1 - u_2|^2$$

so that, by (i) in Definition 2.1, T is strongly monotone. Define now $A: H \to 2^H$ and $B: H \to 2^H$ respectively by

$$Au = - M^{-1}(f - u) - bu , \qquad Bu = Nu + bu ,$$

so that $T = A + B$. By conditions (iii) and (iv), B is monotone, demi-continuous on H and hence maximal monotone (see e.g. [**2**]). By conditions (i) and (ii), A is monotone and $A + bI = -M^{-1}(f - \cdot)$ is maximal monotone because M is maximal monotone. Consequently, by Minty's characterization of maximal monotone operators (see e.g. [**2**]), one has using (i),

$$\operatorname{Im}(A + aI) = \operatorname{Im}\big((A + bI) + (a - b)I\big) = H ,$$

and hence, by the same characterization, A is maximal monotone. A result of Rockafellar (see e.g. [**2**]) then implies that $T = A + B$ is maximal monotone. Now, to prove (2.2), let us notice that, using (ii), (iii), (v) and (2.1), one has

$$\begin{aligned} -b|f-u|^2 &\leqslant (Nf - Nu, f - u) = (Nf - Nu, MNu) \\ &\leqslant (Nf, MNu) - a|MNu|^2 \leqslant |u - f||Nf| - a|f - u|^2, \end{aligned}$$

which gives (2.2) and completes the proof.

The following proposition, which is modelled after a result of Brézis and Browder [**3**] for another situation, shows how the solution of (2.1) depends on M, N, f.

Proposition 2.2. *Let (M, N) be a pair of $h(a', b')$-compatible mappings from H to H, (M_n, N_n) be a sequence of pairs of $h(a, b)$-compatible mappings from H to H and let (f_n) be a sequence in H which converges to $f \in H$. Assume that the following conditions hold.*

(1) *For each bounded subset $S \subset H$, $\bigcup_{n=1}^{\infty} N_n(S)$ is bounded.*

(2) *For each $u \in H$, $M_n N u \to MNu$ if $n \to \infty$.*

(3) *For each $u \in H$, $N_n u \to Nu$ if $n \to \infty$.*

Then, if we write

$$u = (I + MN)^{-1} f , \qquad u_n = (I + M_n N_n)^{-1} f_n \qquad (n = 1, 2, \ldots) ,$$

we have $u_n \to u$ for $n \to \infty$.

Proof. Assumption (1) and the estimate (2.2) imply that the sequence (u_n) is bounded, and so the same is true for the sequence $(N_n u_n)$. On the other hand, by the definition of u_n, one has

$$(2.3)\qquad (M_nN_nu_n - M_nNu, N_nu_n - Nu) = (f_n - u_n - M_nNu, N_nu_n - Nu)$$
$$= (u - u_n + g_n, N_nu_n - N_nu + h_n)\,,$$

where

$$g_n = f_n - u - M_nNu \quad \text{and} \quad h_n = N_nu - Nu \qquad (n = 1, 2, \ldots)\,.$$

By assumptions 2 and 3, one has

$$g_n \to 0 \quad \text{and} \quad h_n \to 0 \quad \text{if } n \to \infty\,.$$

Now, using conditions (ii) and (iii) of Definition 2.1 and (2.3) we obtain

$$(2.4)\qquad a|M_nN_nu_n - M_nNu|^2 - b|u - u_n|^2$$
$$\leqslant (M_nN_nu_n - M_nNu, N_nu_n - Nu) + (N_nu - N_nu_n, u - u_n)$$
$$= (g_n, h_n) + (u - u_n, h_n) + (g_n, N_nu_n - N_nu) = \delta_n \qquad (n = 1, 2, \ldots)\,.$$

Clearly, $\delta_n \to 0$ if $n \to \infty$.

By (2.4) and the definition of u, u_n and g_n, we obtain

$$a|g_n + u - u_n|^2 - b|u - u_n|^2 \leqslant \delta_n \qquad (n = 1, 2, \ldots)\,,$$

i.e.

$$(a - b)|u - u_n|^2 \leqslant \delta_n - a|g_n|^2 - 2a(u - u_n, g_n)$$

and the result follows from the fact that $a > b$ and the boundedness of the sequence (u_n).

An immediate consequence of Proposition 2.2 is the following

Corollary 2.1. *Let (M, N) be a pair of $h(a, b)$-compatible mappings from H to H. Then, if N is bounded, $(I + MN)^{-1}\colon H \to H$ is continuous.*

Another important consequence of Proposition 2.2 is the following

Theorem 2.1. *Let $\mathscr{M}\colon H \times J \to H$ and $\mathscr{N}\colon H \times J \to H$ be mappings such that the following conditions are satisfied.*

(a) *There exist $0 \leqslant b < a$ such that, for each $\lambda \in J$, $(\mathscr{M}(\cdot, \lambda), \mathscr{N}(\cdot, \lambda))$ is $h(a, b)$-compatible.*

(b) $\mathcal{N}: H \times J \to H$ *is bounded.*

(c) *For each* $u \in H$, *the mapping* $\mathcal{N}(u, \cdot)$ *is continuous on* J.

(d) *For each* $u \in H$ *and each* $v \in J$, *the mapping* $\mathcal{M}(\mathcal{N}(u, v), \cdot)$ *is continuous on* J.

Then the pair $(\mathcal{M}, \mathcal{N})$ *is ch-compatible.*

Proof. It follows from assumption (a) and Proposition 2.1 that equation (1.1) has a unique solution for each $f \in H$.

Now let $(f, \lambda) \in H \times J$ and (f_n) and (λ_n) be sequences such that $f_n \to f$, $\lambda_n \to \lambda$ if $n \to \infty$, and let

$$M_n = \mathcal{M}(\cdot, \lambda_n), \quad N_n = \mathcal{N}(\cdot, \lambda_n), \quad M = \mathcal{M}(\cdot, \lambda), \quad N = \mathcal{N}(\cdot, \lambda).$$

By assumptions (a) to (d), the conditions (1) to (3) of Proposition 2.2 are satisfies and hence

$$\mathcal{S}(f_n, \lambda_n) \to \mathcal{S}(f, \lambda) \qquad \text{if } n \to \infty.$$

On the other hand, by the estimate (2.2), one has for each $(f, \lambda) \in H \times J$,

$$|\mathcal{S}(f, \lambda) - f| \leqslant (a-b)^{-1} |\mathcal{N}(f, \lambda)|$$

and the boundedness of $\mathcal{S}$ follows from that of $\mathcal{N}$.

Under the assumptions of Theorem 2.1 it is no more possible in general to approximate $\mathcal{S}(f, \lambda)$ by an iteration process, but the following consequence of Proposition 2.2 will show that the solution $S(f)$ of (2.1), and hence the solution $\mathcal{S}(f, \lambda)$ of (1.1) for each fixed $\lambda \in J$, can often be obtained as the limit of Galerkin approximations.

Let (P_n) be a sequence of orthogonal projectors from H into itself such that the following condition

$$\text{for each } f \in H, \quad P_n f \to f \quad \text{if } n \to \infty \tag{2.5}$$

is satisfied. By a *Galerkin approximation of order* n for the solution of (2.1) we mean any solution u_n of the equation

$$u_n + P_n M P_n N u_n = f_n,$$

where $f_n = P_n f$ $(n = 1, 2, \ldots)$. Obviously, $u_n \in \operatorname{Im} P_n$.

Proposition 2.3. *Let (M, N) be a pair of $h(a, b)$-compatible mappings from H into itself such that $N: H \to H$ is bounded. Assume that there exists a sequence (P_n) of orthogonal projectors in H such that (2.5) holds. Then, for each $n = 1, 2, \ldots$, there exists a unique Galerkin approximation u_n of order n for the solution u of (2.1) and $\lim_{n\to\infty} u_n = u$.*

Proof. Define, for each $n = 1, 2, \ldots$, $M_n: H \to H$ and $N_n: H \to H$ by $M_n = P_n M P_n$, $N_n = N$. For each $u, v \in H$ and each $n = 1, 2, \ldots$, one has

$$(M_n u - M_n v, u - v) = (MP_n u - MP_n v, P_n u - P_n v)$$
$$\geqslant a\,|MP_n u - MP_n v|^2 \geqslant a\,|M_n u - M_n v|^2,$$

and hence, as obviously $M_n(0) = 0$, the pair (M_n, N_n) is $h(a, b)$-compatible for each $n = 1, 2, \ldots$, which insures the existence and uniqueness of u_n for each $n = 1, 2, \ldots$. To complete the proof by the use of Proposition 2.2, it remains to show that, for each $u \in H$,

$$M_n N u \to MNu \qquad \text{if } n \to \infty .$$

But, for each $v \in H$,

$$|M_n v - Mv| \leqslant |P_n(MP_n v - Mv)| + |P_n Mv - Mv|$$
$$\leqslant |MP_n v - Mv| + |P_n Mv - Mv| \leqslant a^{-1}|P_n v - v| + |P_n Mv - Mv|,$$

using Remark 2.2, so that the result follows from (2.5).

3. - Another class of pairs of continuously Hammerstein compatible mappings in a Hilbert space

Let still H be a real Hilbert space with inner product (,) and corresponding norm $|\cdot|$, and let H_1 and H_2 be two closed orthogonal vector subspaces of H such that $H = H_1 \oplus H_2$. We shall denote by P the orthogonal projector onto H_1 and write $Q = I - P$, so that Q is the orthogonal projector onto H_2.

The following result completes a former result of Kosicky [7] and De Figueiredo-Gupta [5] with a substantially simpler proof.

Proposition 3.1. *Let $K: \operatorname{dom} K \subset H_1 \to H_1$ and $L: \operatorname{dom} L \subset H_2 \to H_2$ be linear mappings and let $N: H \to H$ be such that the following conditions hold.*

(1) *K is maximal monotone.*

(2) *L is closed, one-to-one and onto and* $|L^{-1}| \leqslant a$.

(3) *N is hemi-continuous and strongly monotone with constant b.*

Then, if $b > a$, the Hammerstein equation

$$u + (KP + LQ)Nu = f \tag{3.1}$$

has for each $f \in H$ a unique solution u. Moreover, the following estimate holds:

$$|u - f| \leqslant (b - a)^{-1} |Nf| \,. \tag{3.2}$$

Proof. Let $v = u - f$, $N_f = N(f + \cdot)$, $M = KP + LQ$ with $\operatorname{dom} M = \{x\colon Px \in \operatorname{dom} K \text{ and } Qx \in \operatorname{dom} L\}$.

Then (3.1) reduces to

$$-v = MN_f v \,. \tag{3.3}$$

Now, $Mx = y$, $y \in H$, if and only if $KPx = Py$, $LQx = Qy$, i.e.

$$Px \in K^{-1}Py \,, \qquad Qx = L^{-1}Qy \qquad \text{i.e.} \qquad x \in (K^{-1}P + L^{-1}Q)\,y = M^{-1}y$$

with $K^{-1}\colon H_1 \to 2^{H_1}$ maximal monotone. Consequently, (3.3) is equivalent to

$$0 \in N_f(v) + M^{-1}(v) = T_f(v) \,, \tag{3.4}$$

with $T_f\colon H \to 2^H$. Now, $K^{-1}P\colon H \to 2^H$ is maximal monotone; in fact, for all (u, v), (u', v') in the graph of $K^{-1}P$, one has

$$(v - v', u - u') = (v - v', Pu - Pu') \geqslant 0 \,,$$

because $K^{-1}\colon H_1 \to 2^{H_1}$ is monotone, so that $K^{-1}P$ is monotone; moreover, for each $y \in H$, the equation

$$y \in (K^{-1}P + I)\,x \tag{3.5}$$

is equivalent to (with I_1 the identity in H_1)

$$Py \in (K^{-1} + I_1)\,Px \,, \qquad Qy = Qx \tag{3.6}$$

and, $K^{-1}: H_1 \to 2^{H_1}$ being maximal monotone, the first equation in (3.6) has by Minty's characterization (see e.g. [**2**]), a unique solution

$$Px = (K^{-1} + I_1)^{-1} Py$$

so that (3.5) has the unique solution

$$x = (K^{-1} + I_1)^{-1} Py + Qy .$$

By Minty's characterization again, $K^{-1}P$ is maximal monotone. Now, $N_f + L^{-1}Q$ is defined on H and hemicontinuous; moreover, for $u, v \in H$

$$(N_f u + L^{-1}Qu - N_f v - L^{-1}Qv,\ u - v) \geqslant b\,|u - v|^2 - (L^{-1}Q(u - v),\ u - v)$$
$$\geqslant (b - a)\,|u - v|^2,$$

so that $N_f + L^{-1}Q$ is maximal monotone and strongly monotone. By a result of Rockafellar (see e.g. [**2**]), T_f is maximal monotone and strongly monotone, so that (3.3) and hence (3.1) has a unique solution. Now let us notice that, for each $v \in \operatorname{dom} M$,

$$\text{(3.7)} \qquad (Mv, v) = (KPv + LQv, v) = (KPv, Pv) + (LQv, Qv)$$
$$\geqslant (LQv, Qv) = (LQv, L^{-1}LQv) \geqslant -a\,|LQv|^2 \geqslant -a\,|Mv|^2.$$

Therefore, if u is the solution of (3.1), one has $Nu \in \operatorname{dom} M$ and

$$b\,|f - u|^2 \leqslant (Nf - Nu, f - u) = (Nf, f - u) - (Nu, MNu)$$
$$\leqslant (Nf, f - u) + a\,|MNu|^2 \leqslant |Nf|\,|f - u| + a\,|f - u|^2$$

which gives (3.2).

The following results shows how the solutions of (3.1) depend on K, L, N and f.

Proposition 3.2. *Let* K, L, N, f *be like in Proposition* 3.1 *and let* (K_n), (L_n), (N_n), (f_n) *be sequences such that* $f_n \to f$ *if* $n \to \infty$ *and the following conditions hold.*

(1) *For each* $n = 1, 2, \ldots$, $K_n : \operatorname{dom} K \subset H_1 \to H_1$ *is linear maximal monotone,* $L_n : \operatorname{dom} L \subset H_2 \to H_2$ *is linear, closed, one-to-one onto, and, for some* $a \geqslant 0$, $|L_n^{-1}| \leqslant a$.

(2) *For each* $n = 1, 2, \ldots$ *and some* $b > a$, $N_n \colon H \to H$ *is demicontinuous and strongly monotone with constant* b.

(3) *For each bounded subset* $S \subset H$, $\bigcup_{n=1}^{\infty} N_n(S)$ *is bounded.*

(4) *For each* $u \in H$, *such that* $PNu \in \operatorname{dom} K$ *and* $QNu \in \operatorname{dom} L$, $M_n N u \to MNu$ *if* $n \to \infty$, *with* $M_n = K_n P + L_n Q$, $n = 1, 2, \ldots$; $M = KP + LQ$.

(5) *For each* $u \in H$, $N_n u \to Nu$ *if* $n \to \infty$.

Then, if we write

$$u = (I + MN)^{-1} f\,, \qquad u_n = (I + M_n N_n)^{-1} f_n \qquad (n = 1, 2, \ldots)\,,$$

we have $u_n \to u$ *for* $n \to \infty$.

Proof. Assumption (3) and the estimate (3.2) imply that the sequences (u_n) and $(N_n u_n)$ are bounded. On the other hand, we have, by assumptions (1), (2) and relation (3.7),

$$\begin{aligned} b\,|u - u_n|^2 &\leqslant (N_n u - N_n u_n,\, u - u_n) - a\,|M_n N_n u_n - M_n N u|^2 \\ &\leqslant (M_n N_n u_n - M_n N u,\, N_n u_n - Nu) \\ &= (u - u_n + g_n,\, N_n u_n - N_n u + h_n) \end{aligned}$$

if we write

$$g_n = f_n - u - M_n N u\,, \qquad h_n = N_n u - Nu \qquad (n = 1, 2, \ldots)\,,$$

so that $g_n \to 0$, $h_n \to 0$ if $n \to \infty$. Consequently,

$$b\,|u - u_n|^2 - a\,|u - u_n + g_n|^2 \leqslant (u - u_n,\, h_n) + (g_n,\, N_n u_n - N_n u) + (g_n, h_n) = \delta_n \qquad (n = 1, 2, \ldots)\,,$$

with $\delta_n \to 0$ if $n \to \infty$, so that

$$(b - a)\,|u - u_n|^2 \leqslant \delta_n + 2a(u - u_n,\, g_n) + a\,|g_n|^2 \quad (n = 1, 2, \ldots)\,,$$

and the result follows.

An immediate consequence of Proposition 3.2 is the following.

Corollary 3.1. *Let K, L, N be like in Proposition 3.1 with moreover N bounded. Then*

$$[I + (KP + LQ)N]^{-1} \colon H \to H$$

is continuous.

Another consequence of Proposition 3.2, whose proof can be modelled on the one of Theorem 2.1 and then can be ommitted is the following,

Theorem 3.1. *For each $\lambda \in J$, let $\mathscr{H}(\cdot, \lambda) \colon D_1 \subset H_1 \to H_1$ and $\mathscr{L}(\cdot, \lambda)\colon D_2 \subset H_2 \to H_2$ be linear mappings and let $\mathscr{N} \colon H \times J \to H$ be such that the following conditions hold.*

(1) *For each $\lambda \in J$, $\mathscr{H}(\cdot, \lambda)$ is maximal monotone and $\mathscr{L}(\cdot, \lambda)$ is linear, closed, one-ot-one, onto and such that*

$$|[\mathscr{L}(\cdot, \lambda)]^{-1}| \leqslant a$$

with $a \geqslant 0$ independant of λ.

(2) *$\mathscr{N} \colon H \times J \to H$ is bounded, there exists $b > a$ such that, for each $\lambda \in J$, $\mathscr{N}(\cdot, \lambda) \colon H \to H$ is strongly monotone with constant b, and, for each $u \in H$, the mapping $\mathscr{N}(u, \cdot)$ is continuous on J.*

(3) *For each $u \in H$, and for each $\nu \in J$ such that $P\mathscr{N}(u, \nu) \in D_1$ and $QN(u, \nu) \in D_2$, the mapping*

$$\mathscr{M}(\mathscr{N}(u, \nu), \cdot) = \mathscr{H}(P\mathscr{N}(u, \nu), \cdot) + \mathscr{L}(Q\mathscr{N}(u, \nu), \cdot)$$

is continuous on J.

Then the pair $(\mathscr{M}, \mathscr{N}) = (\mathscr{H}(P, \cdot) + \mathscr{L}(Q, \cdot), \mathscr{N})$ is ch-compatible.

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