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A generalization of the exponential functor and its connections with the SH-formulas (**)

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It is well known that the set theoretic exponential (Cartesian power), when extended to general categories, splits into three non-equivalent concepts: the Hom-functor (any category), the internal exponential (Cartesian closed categories) and the S-fold product of an object with itself (when such exists, S a set). In $[\mathbf{1}]_2$ I studied a bifunctor $G: \boldsymbol{D}^{op} \times \boldsymbol{D} \to \boldsymbol{C}$ under hypotheses which included the first two, as special cases, but failed to include the third one. In this paper I want to discuss a functor $G: \boldsymbol{E}^{op} \times \boldsymbol{D} \to \boldsymbol{C}$ which, roughly speaking, generalizes G of $[\mathbf{1}]_2$ and also includes $\Pi: \boldsymbol{S}^{op} \times \boldsymbol{D} \to \boldsymbol{D}$, $\Pi(S, A)$ being the S^{th} power of A, $\boldsymbol{D}$ a category in which $\Pi(S, A)$ exists for every $S \in \boldsymbol{S}$ and $A \in \boldsymbol{D}$ and $\boldsymbol{S}$ the category of sets. The case of Π could not be taken care of by the bifunctor of $[\mathbf{1}]_2$, since $\boldsymbol{D}$ does not coincide in general with $\boldsymbol{S}$. I still make on G strong enough assumptions as to prove the usual result: *if H is any* SH *and $\mathscr{B}$ any $\mathscr{L}$-structure (in $\boldsymbol{D}$), then H is true in $\mathscr{B}$ iff it is «uniformly true» in the $G(X, \mathscr{B})$'s (induced structures), with $X \in |\boldsymbol{E}|$.*

1. - Setting the problem

Let $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{C}$ be three categories, $\boldsymbol{D}$ with finite products (as in $[\mathbf{1}]_2$, $\boldsymbol{C}$ is not required to have *all* finite products, but it will turn out to have enough

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of them so that the interpretations of symbols of $\mathscr{L}^*$ will make sense) (1). Let $G: \boldsymbol{E}^{\mathrm{op}} \times \boldsymbol{D} \to \boldsymbol{C}$ be a bifunctor, U an object of $\boldsymbol{C}$, $K: \boldsymbol{D} \to \boldsymbol{E}$ a functor such that the following conditions hold.

(A) *The standard functor* $\boldsymbol{C}[U, -]$ *is full and faithful* (2).

(B) K *is full and faithful and has a left adjoint* $F \dashv K$.

(C) *There is a natural bijection*

$$\Phi_{-,-}: \boldsymbol{E}[-, K(-)] \to \boldsymbol{C}[U, G(-,-)] . \tag{1}$$

If $f: A \to K(B)$, we will sometimes write $\hat{f}$ for $\Phi_{A,B}(f): U \to G(A, B)$. First of all, let us check the claim above, i.e. that the present situation generalizes the one described in section **5** of $[\mathbf{1}]_2$.

Take $\boldsymbol{E} = \boldsymbol{D}$, $U = 1$ (3), $K = \mathrm{Id}_{\boldsymbol{D}}$ and since condition (C) of $[\mathbf{1}]_2$ is explicitly assumed, all the requirements of $[\mathbf{1}]_2$ are thus fulfilled.

Now we can also make Π to fit into present situation, provided we take $\boldsymbol{C} = \boldsymbol{D}$ with products and sums and assume $\boldsymbol{D}[U, -]$ to be faithful and full. For, we can set $K = \boldsymbol{D}[U, -]$ and have

$$\Sigma(-, U) \dashv \boldsymbol{D}[U, -] ,$$

where $\Sigma(S, B) = S \cdot B$ is the S-fold sum of B with itself.

This last example suggests that we consider three bifunctors

$$G: \boldsymbol{E}^{\mathrm{op}} \times \boldsymbol{D} \to \boldsymbol{C} , \quad K: \boldsymbol{C}^{\mathrm{op}} \times \boldsymbol{D} \to \boldsymbol{E} , \quad F: \boldsymbol{E}^{\mathrm{op}} \times \boldsymbol{C} \to \boldsymbol{D}$$

and natural bijections

$$\frac{F(E, C) \to D}{E \to K(C, D)} , \qquad \frac{E \to K(U, D)}{U \to G(E, D)} \qquad (U \text{ final}) ,$$

or, more symmetrically,

$$\begin{array}{c} F(E, C) \to D \\ \hline E \to K(C, D) \\ \hline C \to G(E, D) \end{array} ;$$

yet, we see no use for such complications.

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(1) For symbols and terminology we will stick to $[\mathbf{1}]_{2,3}$.

(2) We don't require U to be final; notice that this hypothesis, although assumed, was not used in $[\mathbf{1}]_2$.

(3) The final object; see footnote # 2.

2. - The functor $\hat{G}$

As in [**1**]$_2$, we want $\hat{G}: \boldsymbol{D} \to \boldsymbol{C}^{\boldsymbol{E}^{\mathrm{op}}}$ to be the functor associated with G in the adjunction of the exponentiation, i.e. $\hat{G}(x)(y) = G(y, x)$, where $x \in \boldsymbol{D}$, $y \in \boldsymbol{E}$. With this notation we can prove the following theorem (see theorems 4 and 5 of [**1**]$_2$).

Theorem 1. *G is full and faithful.*

Proof. Let $\hat{G}(D) \to \hat{G}(D')$; from bijection

$$\frac{K(D) \to K(D)}{U \to G(K(D), D)}$$

one gets $\varrho_D = \widehat{1_{K(D)}}: U \to G(K(D), D)$, hence $\varphi_{K(D)}\varrho_D: U \to G(K(D), D')$. Thus $\Phi^{-1}_{K(D),D}(\varphi_{K(D)}\varrho_D): K(D) \to K(D')$. Using the fullness of K, let $f: D \to D'$ be such that

$$K(f)\Phi^{-1}_{K(D),D}(\varphi_{K(D)}\varrho_D) .$$

Proving $\varphi = \hat{G}(f)$ and $\hat{G}$ faithful requires now no more ingenuity than the proof of theorem 4 in [**1**]$_2$. (Same thing for faithfulness of $\hat{G}$, with respect to theorem 5 in [**1**]$_2$).

Theorem 2. *For every $X \in |\boldsymbol{E}|$, the functor $G(X, -)$ preserves limits.*

Proof. Since K has a left adjoint, it preserves I-limits, for every diagram scheme I. Now use an argument similar to theorem 6 in [**1**]$_2$.

Corollary 1. *For every $X \in |\boldsymbol{E}|$, the functor $G(X, -)$ preserves finite products and monomorphisms.*

Corollary 2. *$\hat{G}$ preserves finite products and monomorphisms.*

Notation. Given any finite limit preserving functor M, we will write $\alpha_n: M(A^n) \to M(A)^n$ and $\beta_n: M(A)^n \to M(A^n)$ for the (inverse to each other) canonical isomorphisms (4).

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(4) It is assumed that $A \times B$ is an arbitrary product, chosen once and for all.

3. - The lemmas leading to the conclusion

The lemma 6 in [**1**]$_2$ and its proof can be recorded as they stand:

Lemma 1. *Let* $\mathscr{C}^* = (C, \Psi^*)$ *be any* $\mathscr{L}$*-interpretation (in* $\mathbf{C}$*), let* H_0 *be an atomic* SH*-formula of rank* n *and let* $\xi\colon Y \to C^n$. *Then* H_0 *is satisfied in* $\mathscr{C}^*$ *by* ξ *if and only if it is satisfied by* ξx, *for every* $x\colon U \to Y$.

Proof. As in [**1**]$_2$, because of condition (A).

The discussion which follows lemma 6 in [**1**]$_2$ can now be accepted as it is, since all the conditions used thereby still hold. Lemma 7 in [**1**]$_2$, on the contrary, splits into the following two lemmas, due to the presence of (non trivial) K (5).

Lemma 2. *Let* $\mathscr{D}^* = (D, \Psi^*)$ *be any* $\mathscr{L}$*-interpretation (in* $\mathbf{D}$*) and let* H_0 *be an atomic* SH *of rank* n. *For every* $g\colon Y \to D^n$, *put* $\hat{g} = \alpha_n \Phi_{K(Y),D^n}(K(g))\colon U \to G(K(Y), D)^n$. *Then* H_0 *is satisfied in* $\mathscr{D}^*$ *by* g *if and only if it is satisfied in* $\mathscr{D}^*_{K(Y)}$ *by* $\hat{g}$.

Proof. If $u\colon R \rightarrowtail D^m$ interprets an m-ary predicate in $\mathscr{D}^*$, put

$$u_{K(Y)} = \alpha_m G(K(Y), u)\colon G(K(Y), R) \rightarrowtail G(K(Y), D)^m \tag{2}$$

for the corresponding interpretation in $\mathscr{D}^*_{K(Y)}$. Since $G(K(Y), -)$ preserves finite products, if t is an n-ary term interpreted in $\mathscr{D}^*$ and $t_{K(Y)}$ is the same term interpreted in $\mathscr{D}^*_{K(Y)}$, then the following holds

$$t_{K(Y)} = G(K(Y), t)\beta_n\,. \tag{3}$$

Let $\{t\}$, $\{t_{K(Y)}\}$ stand for the generic bracket of m-ple of terms interpreted in $\mathscr{D}^*$ and $\mathscr{D}^*_{K(Y)}$ respectively; for every $y\colon Y \to R$, put $\hat{y} = \Phi_{K(Y),R}(K(y))\colon U \to G(K(Y), R)$. By the same method as in [**1**]$_2$, prove first that

$$uy = \{t\}g \quad \textit{iff} \quad u_{K(Y)}\hat{y} = t_{K(Y)}\hat{g}\ : \tag{4}$$

from

$$uy = \{t\}g\,, \tag{5}$$

get $\Phi_{K(Y),D^m}(K(u)K(y)) = \Phi_{K(Y),D^m}(K(\{t\})K(g))$, hence

$$G(K(U), u)\Phi_{K(Y),R}(K(y)) = G(K(Y), \{t\})\,\Phi_{K(Y),D^n} K(g))\,.$$

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(5) Remember (see section **1**) that, in [**1**]$_2$, K is the identity functor.

Now, using (2) and (3), get

$$\beta_m u_{K(Y)} \Phi(K(y)) = \beta_m \{t_{K(Y)}\} \alpha_n \Phi(K(g)) ,$$

hence

$$u_{K(Y)} \hat{y} = \{t_{K(Y)}\} \hat{g} . \tag{6}$$

Vice versa, get (5) from (6) using Φ injective and K faithful in last two passages.

Then, using K full, notice that every $z\colon U \to G(K(Y), R)$ is of the form $\hat{y}$. The desired conclusion follows: there is a y such that $uy = \{t\} g$ if and only if there is a z such that $u_{K(Y)} z = \{t_{K(Y)}\} \hat{g}$.

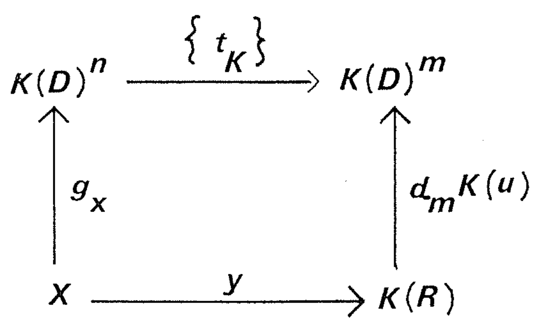
Lemma 3. *Let $\mathscr{D}^*$ and H_0 be as in lemma 2 and let $X \in |\mathbf{E}|$. For every $\xi\colon Z \to G(X, D)^n$ and every $x\colon U \to Z$,*

$$\models_{K(\mathscr{D}^*)} H_0[g_x] \quad (in\ \mathbf{E}) \qquad iff \qquad \models_{\mathscr{D}^*_X} H_0[\xi x] \quad (in\ \mathbf{C}) ,$$

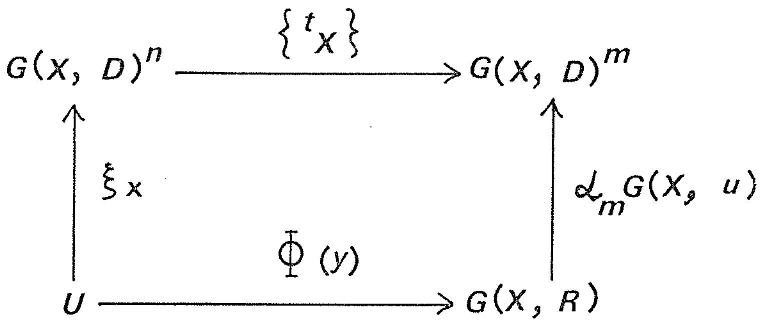
where $g_x\colon X \to K(D^n) \xrightarrow[\approx]{\alpha_n} K(D)^n$ is obtained via

$$\frac{U \xrightarrow{x} Z \xrightarrow{\xi} G(X, D)^n \xrightarrow{\approx} G(X, D^n)}{X \to K(D^n)} .$$

Proof. Very similar to that of Lemma 2. For every $y\colon X \to K(R)$, the diagram



commutes if and only if the diagram



does. Since Φ is a bijection, the claimed statement follows.

Corollary 3. *If H is an* SH, *true in $\mathscr{D}^*_X$ for every $X \in |\mathbf{E}|$, then H is true in $\mathscr{D}^*$.*

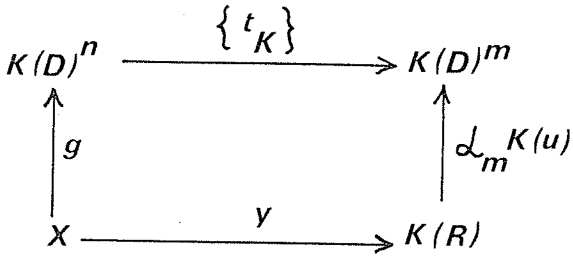
Proof. Let H be $H_1 \to H_0$, with H_0 atomic, H_1 conjunction of atomic SH's, let $g\colon Y \to D^n$ be such that $\models_{\mathscr{D}^*} H_1[g]$. Then, by Lemma 2, $\models_{\mathscr{D}^*_{K(Y)}} H_1[\hat{g}]$. But H is true in $\mathscr{D}^*_{K(Y)}$, therefore $\models_{\mathscr{D}^*_{K(Y)}} H_0[\hat{g}]$ and hence $\models_{\mathscr{D}^*} H_0[g]$.

Lemma 4. *Let $\mathscr{D}^*$, $H_0 X$ be as before and let $g\colon X \to K(D)^n$ (in $\mathbf{E}$). Let $\bar{g}\colon F(X) \to D^n$ be obtained through isomorphism $K(D)^n \approx K(D^n)$ and adjunction $F \dashv K$. Then*

$$\models_{\mathscr{D}^*} H_0[\bar{g}] \qquad \textit{iff} \qquad \models_{K(\mathscr{D}^*)} H_0[g]\,.$$

Proof. Let $y\colon X \to K(R)$ and $x\colon F(X) \to R$ be adjoint morphisms in the adjunction $F \dashv K$, and consider the following two diagrams.

(7)



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(8) \qquad \begin{array}{ccc}
D^n & \xrightarrow{\{t\}} & D^m \\
\uparrow{\scriptstyle \bar g} & & \uparrow{\scriptstyle u} \\
F(X) & \xrightarrow{x} & R
\end{array}
$$

Being $\{t_K\} = \alpha_m K(\{t\})\beta_n$, it is straightforward to check that diagram (7) commutes if and only if diagram (8) does. Conclusion easily follows.

Corollary 4. *Let H be an* SH *true in $\mathscr{D}^*$ and let $X \in |\boldsymbol{E}|$. Then H is true in $\mathscr{D}^*_X$.*

Proof. Using Lemmas 1, 3 and 4, one has that for each atomic H_0,

$$\underset{\mathscr{D}^*_X}{\models} H_0[\xi] \quad \text{iff} \quad \text{for each } x\colon U \to Z, \quad \underset{\mathscr{D}^*}{\models} H_0[\overline{g_x}]\,.$$

Conclusion follows easily.

4. - Behaviour of G with respect to SH's

Theorem 3. *Let H be an* SH *formula and let $\mathscr{D}$ be an $\mathscr{L}$-structure (in $\boldsymbol{D}$). Then H is true in $\mathscr{D}$ iff it is uniformly true in the $\mathscr{D}_X$'s ($X \in |\boldsymbol{E}|$):*

$$\underset{\mathscr{D}}{\models} H \qquad \textit{iff} \qquad \overset{X\in E}{\underset{\mathscr{D}_X}{\models}} H\,.$$

Proof. It follows immediately from Corollaries 3 and 4 (see also $[\mathbf{1}]_{2,1}$).

References

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Sunto

Si introduce un bifuntore $G\colon \mathbf{E}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{C}$ *che generalizza quello introdotto in* $[\mathbf{1}]_2$ *e tale che se* H *è una* SH *e* $\mathscr{D}$ *una struttura in* $\mathbf{D}$, *allora* $\models_{\mathscr{D}} H$ *sse* $\overset{X \in \mathbf{E}}{\models_{\mathscr{D}_X}} H$.

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