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The Burgers' equation in radiative magnetogasdynamics (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Field equations

We consider the unsteady flow of a viscous, heat-conducting and electrically-conducting gas at a temperature sufficiently high ($T > 10^5$ °K) for thermal radiation to be consistent.

Within the differential approximation the basic equations are [**2**]: *the continuity equation*

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \boldsymbol{v}) = 0 , \tag{1}$$

where ϱ and $\boldsymbol{v}$ are the material density and the gas velocity, respectively; *the momentum equation*

$$\varrho \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right) \boldsymbol{v} = - \nabla (p + p_R) + \mu (\nabla \wedge \boldsymbol{H}) \wedge \boldsymbol{H} + \nabla \cdot \tau , \tag{2}$$

where p and p_R are the gas dynamical pressure and the radiation pressure, respectively, μ is the magnetic permeability, $\boldsymbol{H}$ is the magnetic field and τ is the viscous stress tensor: $\tau^{rs} = \eta(\partial v^r/\partial x^s + \partial v^s/\partial x^r) + \theta(\partial v^k/\partial x^k)\delta^{rs}$, η and θ being the viscosity coefficients; *the Maxwell equations*

$$\frac{\partial H}{\partial t} = \nabla \wedge (\boldsymbol{v} \wedge H) - \nabla \wedge (\nu \nabla \wedge H) , \tag{3}$$

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ν being the magnetic diffusivity. In equations (3) the magnetogasdynamics approximation is used, in which the displacement current is negligibly small compared with the curl of the magnetic field. In addition there is also: *the energy equation*

$$(4)\qquad \varrho\,(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla)\;(C_V T+\frac{E_R}{\varrho})=(p+p_R)(\nabla\cdot\boldsymbol{v})+(\boldsymbol{\tau}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}+\frac{J^2}{\sigma}$$

$$+\nabla(\chi\nabla T)+\boldsymbol{\nabla}(D_R\nabla E_R)\,,$$

where C_V is the specific heat at constant volume, T is the absolute temperature, $\boldsymbol{J}=\nabla\wedge\boldsymbol{H}$, σ is the electric conductivity, E_R is the radiation energy density, χ is the heat conductivity and D_R the diffusion coefficient of the radiation. Equation (4) is obtained by using the Milne-Eddington relation $p_R=\frac{1}{3}E_R$ and assuming that $E_R=a_R T^4$, a_R being the Stefan-Boltzmann constant. This last assumption confines our research to the socalled thick gas approximation. More general situations embodying the case of a grey gas of arbitrary opacity are under current investigation and will be published elsewhere. Finally we have *the equation of state of the plasma*

$$(5)\qquad p=R\varrho T\,.$$

2. - One-dimensional propagation

We consider now one-dimensional propagation in which the following assumptions are made: (A) the flow is parallel to the x-axis: $\boldsymbol{v}=(u,0,0)$; (B) all unknown functions depend on x and t only; (C) the magnetic field is planar and perpendicular to the flow direction so that $\boldsymbol{H}=(0,H_2,H_3)$.

Under the above assumptions the system (1)-(4), taking into account equation (5), can be written in the form

$$(6)\qquad U_t+A\,U_x+K_1(K_2\,U_x)_x=0\,,$$

with

$$U=\begin{bmatrix}\varrho\\ u\\ H_2\\ H_3\\ T\end{bmatrix},\qquad A=\begin{bmatrix}u & \varrho & 0 & 0 & 0\\ \dfrac{RT}{\varrho} & u & \dfrac{\mu}{\varrho}H_2 & \dfrac{\mu}{\varrho}H_3 & \phi\\ 0 & H_2 & u & 0 & 0\\ 0 & H_3 & 0 & u & 0\\ 0 & NM & 0 & 0 & u\end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\zeta/\varrho & 0 & 0 & 0 \\ 0 & 0 & -\nu & 0 & 0 \\ 0 & 0 & 0 & -\nu & 0 \\ -N & N\zeta u & \frac{N}{\sigma}H_2 & \frac{N}{\sigma}H_3 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & u & \frac{1}{\sigma}H_2 & \frac{1}{\sigma}H_3 & \tilde{\chi} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\zeta = 2\eta + \theta$ is the longitudinal viscosity and

$$\phi = R + \frac{4}{3} a_R \frac{T^3}{\varrho}, \qquad M = p + \frac{4}{3} E_R,$$

$$N = (\varrho C_V + 4 a_R T^3)^{-1}, \qquad \tilde{\chi} = \chi + 4 D_R a_R T^3 .$$

3. - Dispersion relation

The system (6) is a general dissipative system [**3**], the study of which can be reduced to the study of the Burgers' equation. To achieve this we first deduce the linear dispersion relation. Looking for U in the form $U = U_0 + U'$, $|U'| \ll |U_0|$, with U_0 being a constant solution of (6) given by

$$U_0 = \begin{bmatrix} \varrho_0 \\ 0 \\ H_{2_0} \\ H_{3_0} \\ T_0 \end{bmatrix},$$

and linearizing the system (6) with respect to U', and assuming that U' is proportional to $\exp[i(kx - \omega t)]$, we obtain the linear dispersion relation

$$\det(-VI + A_0 + ikK_0) = 0, \tag{7}$$

where $V = \omega/k$ is the phase velocity, I is the unit matrix, $A_0 = A(U_0)$ and $K_0 = K(U_0)$, with $K = K_1 K_2$. Excluding the case $V + ik\nu = 0$, equation (7) gives

$$V^4 + ikV^3\left(\frac{\zeta}{\varrho_0} + \nu + N_0\tilde{\chi}_0\right) - V^2\{RT_0 + M_0 N_0 \phi_0 +$$

$$\frac{H_0^2}{\varrho_0} + k^2\left(\frac{\zeta\nu}{\varrho_0} + \frac{\zeta N_0 \tilde{\chi}_0}{\varrho_0} + \nu N_0 \tilde{\chi}_0\right)\} - ikV\{\nu M_0 N_0 \phi_0 + \frac{\mu N_0 H_0^2 \tilde{\chi}_0}{\varrho_0} +$$

$$+ RT_0(\nu + N_0\tilde{\chi}_0) + k^2 \frac{\zeta\nu N_0 \tilde{\chi}_0}{\varrho_0}\} + k^2 RT_0 \nu N_0 \tilde{\chi}_0 = 0 .$$

For small values of k, expanding V around a non-degenerate finite and non-zero eigenvalue of A_0, $V_0 = \pm\,(RT_0 + M_0 N_0\,\phi_0 + \mu(H_0^2/\varrho_0))^{\frac{1}{2}}$, we obtain

$$V = V_0\,\{1 + a_1 k + O(k^2)\}, \quad a_1 = i\,\frac{\alpha}{V_0}, \quad \alpha = -\,\frac{l_0 k_0 d_0}{l_0 d_0},$$

where l_0 and d_0 are the right and the left normalized eigenvectors of A_0 corresponding to the eigenvalue V_0. We have

$$d_0 = D\begin{bmatrix}\varrho_0\\ V_0\\ H_{20}\\ H_{30}\\ M_0 N_0\end{bmatrix}, \qquad D = (\varrho_0^2 + V_0^2 + H_0^2 + M_0^2 N_0^2)^{-\frac{1}{2}},$$

$$l_0 = L(RT_0,\,\varrho_0 V_0,\,\mu H_{20},\,\mu H_{30},\,\varrho_0\,\phi_0)\,,$$

$$L = \{(RT_0)^2 + (\varrho_0 V_0)^2 + (\mu H_0)^2 + (\varrho_0\,\phi_0)^2\}^{-\frac{1}{2}},$$

so that

$$\alpha = \frac{1}{2V_0^2}\,\Big(\frac{\zeta V_0^2}{\varrho_0} + \frac{H_0^2}{\sigma\varrho_0} + \tilde{\chi}_0 N_0^2 M_0\,\phi_0\Big),$$

$$V = V_0 - \frac{i}{2V_0^2}\,\Big(\frac{\zeta V_0^2}{\varrho_0} + \frac{H_0^2}{\sigma\varrho_0} + \tilde{\chi}_0 N_0^2 M_0\,\phi_0\Big)\,k + O(k^2)\,.$$

4. - The Burgers' equation

Since the phasor of the wave becomes

$$kx - \omega t = k(x - V_0 t) - V_0 a_1 k t\,,$$

following the perturbation method given in [**4**], we introduce the coordinate-stretching of Gardner and Morikawa

$$\xi = \varepsilon(x - V_0 t)\,, \qquad \tau = \varepsilon^2 t\,,$$

where ε is a parameter so that $\varepsilon = O(k)$.

In terms of the variables ξ and τ the system (6) can be re-written as

$$\varepsilon\,\frac{\partial U}{\partial \tau} + (A - V_0 I)\,\frac{\partial U}{\partial \xi} + \varepsilon^2 K\,\frac{\partial^2 U}{\partial \xi^2} = 0\,. \tag{8}$$

We now expand U around the uniform state U_0 as a power series in the scale parameter ε

$$U = \sum_{q\to 0}^{\infty} \varepsilon^q U_q , \tag{9}$$

and impose the boundary conditions

$$\lim_{X\to -\infty} U = U_0 \Rightarrow \lim_{X\to -\infty} U_q = 0 \qquad (q \geq 1) . \tag{10}$$

By introducing expression (9) into equation (8), equating to zero corresponding powers of ε and taking into account the conditions (10), we find to the first order $U_1 = d_0 u^{(1)}$, where $u^{(1)}$ is one of the non-zero components of U_1.

Finally, from the second order terms, we see that $u^{(1)}$ satisfies the following Burgers' equation

$$\frac{\partial u^{(1)}}{\partial \tau} + \beta u^{(1)} \frac{\partial u^{(1)}}{\partial \xi} = \alpha \frac{\partial^2 u^{(1)}}{\partial \xi^2} \qquad \left(\beta = \frac{l_0\, [d_0(\nabla_U A)_0]\, d_0}{l_0 d_0}\right) , \tag{11}$$

where α, l_0, d_0 are given in **3** and ∇_U denotes the gradient in the U space. A straightforward calculation gives

$$\beta = \frac{D}{V_0} \{RT_0 + \frac{1}{2}(P_0\varrho_0 + V_0^2 + M_0 N_0 Q_0) + \mu \frac{H_0^2}{\varrho_0} + \frac{1}{2}\phi_0(W_0 + M_0 N_0)\} , \tag{12}$$

with

$$P = \frac{R}{\varrho}(MN - T) , \quad Q = \frac{4a_R T^2}{\varrho}(MV - \frac{1}{3}T), \qquad W = \varrho (MN)'_\varrho + MN(MN)'_T .$$

It is well known that the non-linear equation (11) is exactly solvable by means of the Hopf transformation $u^{(1)} = -(\alpha/\beta)(\log z)_\xi$, which reduces it to the linear heat equation

$$\frac{\partial z}{\partial \tau} = \alpha \frac{\partial^2 z}{\partial \zeta^2} .$$

In this manner it is possible to construct [1] the steady solution of the Burgers' equation

$$u_0^{(1} = -u_\infty^{(1)} \tan h \left(\frac{u_\infty^{(1)} \xi}{2\alpha}\right) \qquad (u_\infty^{(1)} > 0) ,$$

known as the Burgers' shock wave.

5. - A particular thermodynamical case

Because of its remarkable simplicity we only consider a perfect polytropic gas $R = C_p - C_V$ in the particular case $\gamma = C_p/C_V = 4/3$. We have $MN = \frac{1}{3}T$ and

$$V_0^2 = \gamma \frac{p_0 + p_R}{\varrho_0} + \mu \frac{H_0^2}{\varrho_0},$$

which is the same as in non-radiative magnetogasdynamics except for the replacement of the gas dynamical pressure with the total pressure (gas dynamical + radiation pressure). On the other hand we find that

$$P = -\frac{2}{3}\frac{RT}{\varrho}, \qquad Q = 0, \qquad W = \frac{1}{9}T, \qquad N\phi = \frac{1}{3\varrho},$$

while the coefficients α and β in the equation (11) reduce to

$$\alpha = \frac{1}{2\varrho_0 V_0^2}\left(\zeta V_0^2 + \frac{H_0^2}{\sigma} + \frac{1}{9}\tilde{\chi}_0 T_0\right),$$

$$\beta = \frac{2D}{V_0\varrho_0}\left[\frac{7}{9}(p_0 + p_R) + \frac{3}{4}\mu H_0^2\right].$$

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Abstract

The system of equations governing the radiative magnetogasdynamics of a thick gas layer is written in the form of a general dissipative system in the case of one-dimensional propagation. This system is reduced via the perturbation method of Taniuti and Wei [**4**], *to the Burgers' equation, which is exactly solvable. A particular thermodynamic case is considered.*

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