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A new solution of Marathe gravitational equation (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

The mathematical model of the modern theories of gravitation is essentially a 3-plet $(\boldsymbol{M}, \boldsymbol{g}, \boldsymbol{T})$, where M is the « space time », i.e. a differentiable manifold with dimension 4, $\boldsymbol{g}$ is a Lorentz metric on M, $\boldsymbol{T}$ is a symmetric tensor field on M with rank 2 (energy-momentum tensor) which represents the matter content of the universe, and which is related to $\boldsymbol{g}$ by a « field equation » on M.

The first field equation that has been proposed, and still the most studied one, is Einstein equation. The main reasons of its interest are simplicity, conservation of energy and good experimental tests (if we exclude the atomic and cosmological scale). However, nobody tells us that energy is conserved on a cosmological scale, and the physical laws we use to describe phaenomena at present time may well fail for times in which the conditions of the universe were quite different from now. It is therefore natural that other field equations have been proposed; one of these is Marathe equation $[\mathbf{1}]_{1,2}$

$$\boldsymbol{r} - (1/4) r_0 \boldsymbol{g} = \boldsymbol{T} - (1/4) T_0 \boldsymbol{g} \ (^1), \tag{1}$$

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(1) We use a unit system in which $\chi = c = 1$, where χ is the coupling constant of Einstein equation and c is the speed of light, and the unit of length is the light year (ly).

(where $\boldsymbol{r}$ is the Ricci tensor on M and r_0 and T_0 are the traces respectively of $\boldsymbol{r}$ and $\boldsymbol{T}$).

This equation was considered by Einstein himself (who however preferred an equation which, as we shall see, is more restricted, in order to have conservation of energy), and it was applied by Penney [**5**] to the construction of a non-quantistic electron (he supposes the conservation of energy to be an average macroscopic property of matter).

Recently, Marathe $[\mathbf{1}]_{1,2}$ has reproposed this equation showing its interesting mathematical properties; a further study has been made by Marathe himself and Modugno [**2**], [**4**], who among other things have shown that Marathe equation gives the same results as Einstein's in the cases of spherically symmetric and cilindrically symmetric vacuum fields, and in the study of shock waves in gravitational fields.

To see that (1) is more general than Einstein's, observe that it may be also written

$$\boldsymbol{r} - (1/2)r_0\boldsymbol{g} + \Phi\boldsymbol{g} = \boldsymbol{T}, \tag{1a}$$

where $\Phi \equiv (1/4)(r_0 + T_0)$ is the « cosmological function ». Applying the divergence (or left-codifferentiation, see [**4**]) operator δ to the two members of eq. (1a), we obtain its differential form

$$\delta\boldsymbol{T} = \mathrm{d}\Phi\,; \tag{2}$$

then, we see immediately that, if Marathe equation holds on M, the following conditions are equivalent

(E1) Einstein equation holds on M.

(E2) $\Phi =$ constant (cosmological constant).

(E3) $\delta\boldsymbol{T} = 0$ (conservation law).

The problem of finding new solutions of Marathe equation is then closely connected to the problem of replacing the conservation law with another physically interesting condition; moreover, we should bear in mind that the usual state equations have a form that takes into account the conservation of energy; looking for new solutions of Marathe equation we shall then have to consider new kinds of state equations, which (as also suggested by the form of eq. (2)) may contain further geometrical parameter (for example curvature), considering a deeper interaction between the gravitational field and matter.

The non-constancy of the cosmological function distinguishes between

Einstein and Marathe equations; the distinction might be important in the past of the universe, for small values of the scale function R: so, new cosmological models are possible; we are particularly interested to see if, among them, there are some which show no singularity; however, we think that there are other reasons of interest in a deeper physical study of Marathe equation. For example:

— creation of mass (e.g. it is easily seen that Marathe equation admits, differently from Einstein, a stationary non-static solution, with non-zero density and pressure);

— the presence of the cosmological function might account for the diversity between the observed and calculated (through the standard model) density of mass of the universe.

2. - Gravitational pressure

Let us consider the following case

$$T = pg + (\mu + p)\boldsymbol{v}\otimes \boldsymbol{v}\,, \tag{3}$$

(energy-momentum tensor of a perfect fluid with pressure p, density μ, velocity $\boldsymbol{v}$). Eq. (2), when decomposed to its space and time components with respect to $\boldsymbol{v}$, gives then

(2a) $(\mu + p)\nabla_v \boldsymbol{v} = \mathrm{d}^{\perp}(\Phi - p)\,,$ (2b) $\boldsymbol{v}.\,\mu + (\mu + p)\,\delta\boldsymbol{v} = -\,\boldsymbol{v}.\,\Phi\,,$

(where ∇ is the covariant differentiation operator, $\mathrm{d}^{\perp}$ is the space component of the exterior differential and $\boldsymbol{v}.$ is the Lie derivative operator in the direction of $\boldsymbol{v}$.

Eq. (2a) is the new « law of motion »; the spatial gradient of the cosmological function appears in the expression of the force, besides the pressure's gradient. It can be seen that if the cosmological function is not constant, the gravitational and inertial mass may not coincide.

Eq. (2b) is the new « conservation law », the word « conservation » having a more general sense than usual: the last term being non-zero we may have, for example, creation or destruction of mass.

Let us now consider the limiting case $\mu = 0$; eq. (2b) becomes then

$$p\delta\boldsymbol{v} = -\,\boldsymbol{v}.\,\Phi \tag{4}$$

if the last term is non-zero we have that in absence of matter there is a residual pressure, that we regard as a « gravitational pressure »: in other words we may consider a more general case of vacuum field, in which the pressure is non-zero. Putting off to further works the not easy task of studying in a general way the problem of the state equations and of the conditions which may replace the conservation law, we are going now to make a simple hypothesis, which will lead us to an example of the interest and of the possibilities of the new theory of gravitation.

Let us suppose $\delta \boldsymbol{v} \neq 0$ (that is we consider the case, which is physically interesting, of a « non-static universe »), and, in the general case $\mu \neq 0$, the pressure to be the sum of two terms, $p = p' + p''$. Let it be

$$p' = \frac{-\boldsymbol{v}.\Phi}{\delta \boldsymbol{v}}, \tag{5}$$

that is, p' has same expression as the gravitational pressure in the case $\mu = 0$. From eq. (2b) it is immediately seen that

$$\boldsymbol{v}.\mu + (\mu + p'')\,\delta \boldsymbol{v} = 0, \tag{6}$$

that is, μ and p'' satisfy a conservation law, analogous to the law which holds for μ and p in the case of the Einstein equation. We shall then regard p' and p'' respectively as the « gravitational term » and the « material term » of pressure, the second being related to the presence of matter, and the first having a cosmological character (in our model we shall request that its present value is very small).

3. - Spatially homogeneous and isotropic universe

We are looking now for a spatially homogeneous and isotropic solution. Let the metric tensor have the Robertson-Walker expression

$$g = -\,\mathrm{d}t \otimes \mathrm{d}t + R^2\,[\frac{\mathrm{d}r \otimes \mathrm{d}r}{1 - kr^2} + r^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta\,\mathrm{d}\varphi \otimes \mathrm{d}\varphi)]. \tag{7}$$

Let it be $\boldsymbol{v} \equiv -\,\mathrm{d}t$; let the energy-momentum tensor have the expression (3), with μ and p constant on the spatial surfaces perpendicular to $\boldsymbol{v}$. From eq. (1) we have then

$$\frac{-R\ddot{R} + \dot{R}^2 + k}{R^2} = \frac{1}{2}(\mu + p), \tag{8}$$

where the dot means derivative with respect to the cosmic time t. (This equation follows from both time and space components of eq. (1), while from Einstein equation we obtain, in the analogous case, two independent conditions). Taking into account the definition of Φ it becomes

$$3\left(\frac{\dot{R}^2 + k}{R^2}\right) = \mu + \Phi . \tag{8a}$$

Eq. (5) becomes

$$p' = -\frac{1}{3}\frac{R}{\dot{R}}\dot{\Phi} . \tag{5a}$$

We suppose moreover the state equation $p'' = 0$ to hold, that is we consider a « pure matter » universe, in which the pressure is only gravitational. Eq. (6) gives then

$$\mu R^3 = m = \text{constant}, \tag{6a}$$

which is the mass conservation law. Besides eqs. (8) and (5a), that we rewrite in this case

$$p = -\frac{1}{3}\frac{R}{\dot{R}}\dot{\Phi}, \tag{9}$$

we need, to determine a solution, another independent condition in the unknown functions R, μ and p.

4. - Choice of a particular expression for Φ

Let us now observe that if the universe is represented by a cosmological model which satisfies Marathe equation, the present value of the cosmological function must be small, so that at present Einstein equation holds approximately; however this may be not true for some time, and in particular for the period (if it has effectively occurred) in which the value of the scale function was very small.

As an example of such a possibility we consider interesting the case, although it does not descend from a general study of the state equations, in which $\Phi \propto R^{-\alpha}$, where α is a positive constant: Φ tends to $\pm\infty$ when $R \to 0$, and tends to 0 when $R \to +\infty$. Moreover, as $\mu \propto R^{-3}$, for $\alpha > 3$ the effects of the cosmological function will be predominant over those of density for very small R, and the converse will happen for large R.

Let it be

$$\Phi = \Phi_1 R^{-\alpha}\,, \qquad \text{with } \Phi_1 \in \mathbf{R}\,. \tag{10}$$

From eq. s (8), (9) and (10) we obtain

$$p = nR^{-\alpha}, \qquad \text{with} \qquad n = \frac{\alpha}{3}\,\Phi_1 \ (\text{that is } p = \frac{\alpha}{3}\,\Phi)\,, \tag{11}$$

$$3H^2 = \frac{\Phi_1 + mR^{\alpha-3} - 3kR^{\alpha-2}}{R^\alpha}\,, \tag{12}$$

$$q = \frac{1}{2}\{1 + \frac{1}{H^2}(\frac{k}{R^2} + \frac{(\alpha/3 - \Phi_1)}{R^\alpha})\}\,, \tag{13}$$

where $H \equiv \dot{R}/R$ and $q \equiv -R\ddot{R}/\dot{R}^2$ are respectively the « Hubble parameter » and the « deceleration parameter ».

In eq. (12) we can easily separate the variables R and t, but we obtain an integral that in general cannot be calculated in closed form. Sometimes however it is possible to get the qualitative behaviour of the function R by means of the study of its extremal points: at the maximum and minimum points the parameter H will vanish and the parameter q will take respectively positive and negative values; at the inflection points the parameter q will vanish.

Let now R_b, H_b and q_b be the values of R, H and q at the time t_b. From eq.s (11), (12) and (13) we obtain easily that

(a) we have $H_b = 0$ if and only if $f(R_b) \equiv \Phi_1 + mR_b^{\alpha-3} - 3kR_b^{\alpha-2} = 0$,

(b) we have $q_b > 0$ ($q_b < 0$) if and only if $R_b^{\alpha-3} > -(\alpha-2)\dfrac{\Phi_1}{m}$

$$(R_b^{\alpha-3} < -(\alpha-2)\frac{\Phi_1}{m}).$$

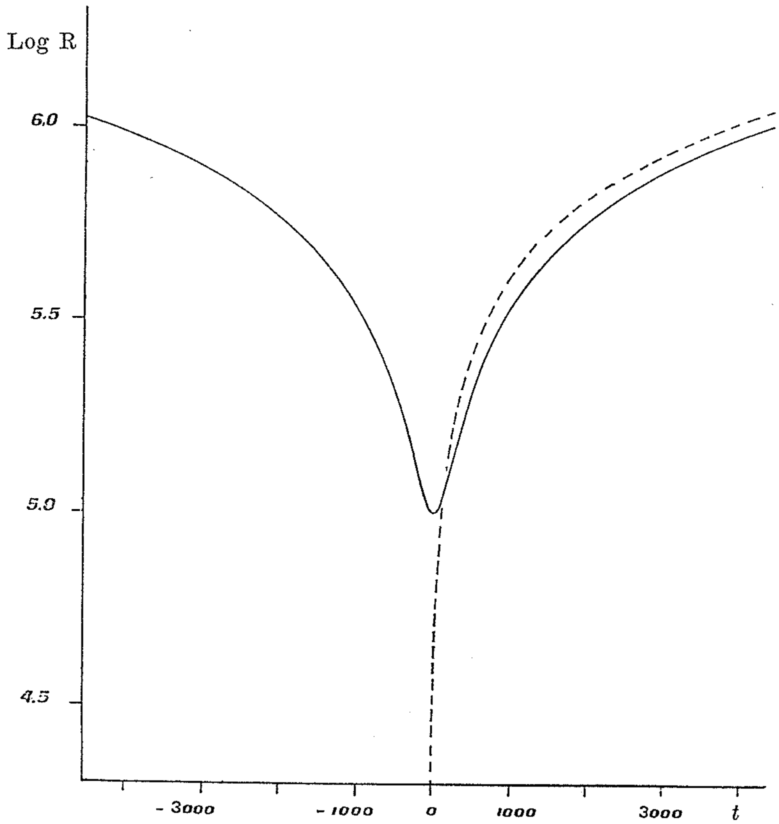
5. - Model with α integer > 3 and $\Phi_1 < 0$

We are particularly interested in the case $\alpha > 3$ in which, as we have already observed, we have a model that approaches the standard model for large R, but that may be different from it for small R. In particular, we see from condition (b) that if $\Phi_1 < 0$ (that is, if the cosmological function and the gravitational pressure are negative), we may have some minimum points, unlike the case of Einstein equation. If moreover α is an integer, $f(R) \equiv \Phi_1 + mR^{\alpha-3} - 3kR^{\alpha-2}$ is a polynomial and its real roots can be studied by means of the Sturm succession associated to it: the positive roots correspond to extremal points of R.

Let it be: α *integer* > 3; $\Phi_1 < 0$. We now distinguish three cases according to the value of k.

1st *case*: $k = +1$. The equation $f(R) = 0$ has two real positive and distinct solutions R_1, R_2 if and only if the following condition is verified

$$|\Phi_1| < \frac{m^{\alpha-2}(\alpha-3)^{\alpha-3}}{3^{\alpha-3}(\alpha-2)^{\alpha-2}}, \tag{14}$$



The minimum of $R(t)$ for $\alpha = 4$, $m = 10^{11}\, ly$, $|\Phi_1| = 10^{16}\, (ly)^2$, $k = +1$ (units are those of note (***)). For large R the behaviour is substantially that of the standard model (dotted line). The cases $k = 0$ and $k = -1$ are analogous.

we have $f(R) > 0$ for values of R belonging to the interval $]R_1, R_2[$ and then eq. (12) has real solution with range $\mathscr{R} = [R_1, R_2]$. Moreover we have $R_1^{\alpha-3} < (\alpha - 2)(|\Phi_1|/m) < R_2^{\alpha-3}$, that is R_1 and R_2 correspond respectively to minima and maxima of the function R. $R_f \equiv \sqrt[\alpha-3]{(\alpha - 2)|\Phi_1|/m}$ corresponds to oblique inflection points. Moreover, the smaller $|\Phi_1|$, the smaller will be R_f and R_1, and the nearer will be R_2 to $m/3$, which is the maximum of the standard model. The solution then approaches the standard model but has no singularity, since when R reaches the minimum it starts growing again, repeating a cycle which is identical to the preceding one (see figure).

If $|\Phi_1| \geqslant (m^{\alpha-2}(\alpha - 3)^{\alpha-3})/(3^{\alpha-3}(\alpha - 2)^{\alpha-2})$, there is no real positive root of $f(R)$, or at the most one double real positive root. Since in this case $f(R) \leqslant 0$ for $R > 0$, eq. (12) has no real non-static solution.

2^{nd} *case*: $k = 0$. The equation $f(R) = 0$ has always one and only one real positive root, $R_1 = \sqrt[\alpha-3]{|\Phi_1|/m}$, which is immediately seen to correspond to a minimum of R. There is an inflection point for $R = R_f$. For large values of R we have again substantially the standard model, but there is no singularity.

3^{rd} *case*: $k = -1$. The equation $f(R) = 0$ has one and only one real positive root, which corresponds to a minimum of R. For $R = R_f$ there is an inflection point; for large values of R we have substantially the standard model, but there is no singularity.

6. - A numerical example

If $\alpha = 4$ it is easy to calculate explicitely $R(t)$ by integrating eq. (12), and for this value of α we have drawn the graph; moreover, the extremal values of R are simply found since $f(R)$ is of the second (if $k \neq 0$) or of the first degree. For example, if $k = +1$, we have $R_1 = (1/6)(m - \sqrt{m^2 - 12|\Phi_1|})$, $R_2 = (1/6)(m + \sqrt{m^2 - 12|\Phi_1|})$. Let Φ_0, R_0, H_0 etc. be the present values of the corresponding functions. Let us suppose $|\Phi_1|$ to be small enough so that we have substantially $q = (1/2)(1 + k/\dot{R}^2)$ throughout the period which can be observed by our instruments, that is since the time in which R was about $(1/4)R_0$: in this case the method used to estimate the value of q_0 (see [5]XA2) is substantially correct; taking $q_0 \simeq 1$; $H_0 \simeq 7.3 \cdot 10^{-11}(1y)^{-1}$ we obtain $k = +1$; $\dot{R}_0^2 \simeq 1$; $R_0 \simeq 1.4 \cdot 10^{10} ly$, and then the requested condition is surely verified if $|\Phi_0| \simeq 10^{-24}(ly)^{-2}$ (which is smaller by five orders of magnitude than the value which is estimated to be the largest possible one for the cosmological constant in the Einstein equation).

We have then $|\Phi_1| \simeq 3.7 \cdot 10^{16}(ly)^2$ and, from eq. (8a), $\mu_0 \simeq 3.2 \cdot 10^{-20}(ly)^{-2}$

(corresponding to $\mu_0 \simeq 2 \cdot 10^{-30} g\ cm^{-3}$); $m \simeq 8.6 \cdot 10^{10} ly$. We have then $m^2 \gg 12 |\Phi_1|$, and the values of R at the minimum and at the inflection points result $R_1 \simeq 4.3 \cdot 10^5 ly$; $R_f \simeq 2R_1$. The value of $|\Phi_0|$ could be experimentally estimated by precise measures on very low density systems; however even if it is to small to be detected in this way, the cosmological function may well be important in the past; from this point of view it may be interesting a study of the problem of the aboundance of the elements.

Notice that the smaller the value of $|\Phi_0|$, the lower will be the reached minimum, and then more pronounced will be the « big bang » effect.

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Riassunto

Proseguiamo, da un punto di vista fisico, uno studio dell'equazione, più generale di quella di Einstein, proposta da Marathe $[\mathbf{1}]_{1,2}$ *e le cui interessanti proprietà sono state studiate da Marathe* $[\mathbf{1}]_{1,2}$, **[2]** *e da Modugno* **[2]**, **[4]**.

Discutiamo l'interesse e la possibilità di nuove soluzioni che non siano soluzioni dell'equazione di Einstein, e proponiamo un modello cosmologico che si avvicina a quello standard ma non ha la singolarità iniziale.

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