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On a terminal value problem for a class of non linear ordinary differential equation (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

Consider the second-order non linear differential equation

$$y'' + a(t, y)\, y' = 0 \,, \tag{1.1}$$

where $a(t, y)$ is a real-valued function defined and continuous on $D = [\alpha, +\infty) \times \boldsymbol{R}$.

In this paper we are concerned with the existence of a solution of the following terminal value problem

$$y'' + a(t, y)\, y' = 0 \,, \qquad \lim_{t \to +\infty} y(t) = L \,, \qquad y'(t) \neq 0 \,. \tag{1.2}$$

Problems of this type have been considered by several authors (see [**1**], [**3**], [**6**], [**7**], [**8**], and the references quoted there); they arise in many fields of applied mathematics, as for instance in the theory of similarity solutions for partial differential equations (see [**2**], [**4**], [**5**]).

Our approach is rather similar to the one of [**6**]. It is based upon the con-

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struction of a sequence of approximating solution and the use of suitable comparison lemmas.

Throughout the paper we shall assume that for every (t_0, y_0, y_0'), $t_0 \geqslant \alpha$, there exists a unique solution to (1.1), continuously dependent upon the data, such that

$$y(t_0) = y_0\,, \qquad y'(t_0) = y_0'\,. \tag{1.3}$$

Note that this assumption implies that the first derivative of each non-constant solution of (1.1) cannot vanish: therefore, (1.1) admits only monotonic solutions.

2. - Preliminary results

In this section we compare the solutions of (1.1), (1.3) with the solutions of

$$w'' + h(w)\,w' = 0\,, \qquad w(t_0) = w_0\,, \qquad w'(t_0) = w_0' \tag{2.1}$$

and

$$z'' + k(t)\,z' = 0\,, \qquad z(t_0) = z_0\,, \qquad z'(t_0) = z_0'\,, \tag{2.2}$$

where $h(w)$ and $k(t)$ are real-valued functions defined and continuous on $\boldsymbol{R}$ and $[\alpha, +\infty)$, respectively.

Lemma 2.1. *If $a(t, x) \geqslant h(x)$ for every $x \in \boldsymbol{R}$, $t \geqslant \alpha$, then*

$$0 \leqslant y_0' \leqslant w_0' \quad \text{and} \quad y_0 = w_0 \quad \text{implies} \quad y(t) \leqslant w(t)\,,$$

and

$$0 \geqslant y_0' \geqslant w_0' \quad \text{and} \quad y_0 = w_0 \quad \text{implies} \quad y(t) \geqslant w(t)$$

in the common interval of existence of $y(t)$ and $w(t)$.

Proof. Let $y_0 = w_0$ and $0 \leqslant y_0' < w_0'$. Since $y'' \leqslant -h(y)\,y'$ (recall that (1.1) admits only monotonic solutions) and $w'' = -h(w)\,w'$, one obtains

$$y' - w' \leqslant y_0' - w_0' + \int_{w_0}^{w} h(s)\,\mathrm{d}s - \int_{y_0}^{y} h(s)\,\mathrm{d}s\,. \tag{2.3}$$

Let $\bar{t} = \{\inf t \colon y(t) \geqslant w(t)\}$. From (2.3), written for $t = \bar{t}$, we get $y'(\bar{t}) < w'(\bar{t})$, a contradiction. The case $y_0 = w_0$, $0 \geqslant y_0' > w_0'$ is similar while, for the case $y_0 = w_0$, $y_0' = w_0'$, it suffices to consider the sequence $y_{0,n}' = y_0' \pm 1/n$ (according to $w_0' \gtrless 0$) and then to use the continuous dependence upon the data.

Lemma 2.2. *If* $a(t, y) \geqslant k(t)$, *then*

$$0 \leqslant y_0' \leqslant z_0' \quad \textit{and} \quad y_0 = z_0 \quad \textit{implies} \quad y(t) \leqslant z(t) \ ,$$

and

$$0 \geqslant y_0' \geqslant z_0' \quad \textit{and} \quad y_0 = z_0 \quad \textit{implies} \quad y(t) \geqslant z(t) \quad \textit{on} \quad [t_0, +\infty) \, .$$

Proof. Let $y_0 = z_0$ and $0 \leqslant y_0' < z_0'$ (the case $0 \geqslant y_0' > z_0'$ is similar). Since $y' \leqslant y_0' \exp\left(-\int_{t_0}^{t} k(s)\, ds\right)$ it follows

$$y' - z' \leqslant (y_0' - z_0') \exp\left(-\int_{t_0}^{t} k(s)\, ds\right) < 0 \, . \tag{2.4}$$

So $y(t)$ and $z(t)$ never cross each other. Moreover (2.4) implies that $y(t)$ exists on $[t_0, +\infty)$.

From now we shall assume $a(t, y) \geqslant 0$, which ensures the global existence in the future of any solution of (1.1), (1.3). Observe that, if $\lim\limits_{\substack{t \to +\infty \\ y \to L}} a(t, y) < 0$, it is obvious that (1.2) cannot have a solution, since y'' and y' would have the same sign for t greater than a suitable t_0, which excludes the possibility that $\lim\limits_{t \to +\infty} y(t) = L$.

3. - Terminal value problem: approximating solutions

We shall need the following

Lemma 3.1. *For every* $\alpha \leqslant t_0 < \tilde{t}$, y_0, L *the two-point boundary value problem*

$$y'' + a(t, y)\, y' = 0 \, , \quad y(t_0) = y_0 \, , \quad y(\tilde{t}) = L \tag{3.1}$$

possesses a solution belonging to $C^2[t_0, \tilde{t}]$.

Proof. In this case, it is easy to show, that the topological mapping $T\colon (t_0, y_0, y_0') \to (\tilde t, y(\tilde t), y'(\tilde t))$, where $y(\tilde t) = y(\tilde t; t_0, y_0, y_0')$ is the solution of (1.1), (1.3), is unbounded for fixed $t_0, y_0, \tilde t$. Indeed the assumption $y_0 \leqslant y(\tilde t) \leqslant k_2$ for given y_0, $\tilde t$ and arbitrary $y_0' \geqslant 0$ (the case $L = y_0$ in (3.1) is the only one involving $y_0' = 0$ and is trivial) would lead to the contradiction

$$0 \leqslant y_0' < \frac{A(k_2 - y_0)}{1 - \exp\left(-A(t - t_0)\right)}, \tag{3.2}$$

where $A = \max\limits_{\substack{t \in [t_0, \tilde t] \\ y \in [y_0, L]}} a(t, y)$. The argument can be repeated for $y_0' < 0$.

Let $\{t_n\}$ be an increasing sequence with $\lim\limits_{n \to \infty} t_n = +\infty$. For every n ($n = 1, 2, \ldots$) consider the two-point boundary value problem

$$y_n'' + a(t, y_n) y_n' = 0\,, \quad y_n(t_0) = y_0\,, \quad y_n(t_n) = L\,. \tag{3.3}$$

Let $\hat y_n(t)$ be a solution of (3.3) (whose existence is ensured by Lemma 3.1) and define

$$y_n(t) = \begin{matrix} \hat y_n(t)\,, & t_0 \leqslant t \leqslant t_n \\ L\,, & t > t_n\,. \end{matrix} \tag{3.4}$$

Note that $y_n(t) \in C[t_0, +\infty)$ and, moreover, $y_n(t) \in C^2[t_0, t_n)$ for any n. Clearly, the functions y_n are uniformly bounded $\left(|y_n - y_0| \leqslant |L - y_0|\right)$ and equicontinuous. This last property follows from the inequality (analogous to (3.2))

$$|y_n'(t_0)| < \frac{A_{t_1}|L - y_0|}{1 - \exp\left(-A_{t_1}(t_1 - t_0)\right)}\,, \quad A_{t_1} = \max_{\substack{t \in [t_0, t_1] \\ |y - y_0| \leqslant |L - y_0|}} a(t, y) \tag{3.5}$$

and from the assumption $a(t, y) \geqslant 0$, which implies that $|y_n'|$ is non-increasing in $[t_0, t_n)$. Therefore, an easy application of the Ascoli-Arzelà's theorem and the standard diagonalization process, ensures the existence of a subsequence (that we shall denote again by $\{y_n\}$) which converges uniformly on each interval of $[\alpha, +\infty)$ to a continuous function. Let $\hat y(t) = \lim\limits_{n \to +\infty} y_n(t)$.

It is easy to show that for any given $[t_0, \bar t]$, $\{y_n\}$ contains a subsequence whose first derivatives are uniformly bounded and equicontinuous on $[t_0, \bar t]$.

Indeed, using (3.5), for any n such that $t_n > \bar{t}$ we have

$$(3.6) \qquad |y_n''(t)| \leq \frac{A_{t_1}|L - y_0|}{1 - \exp(-A_{t_1}(t_1 - t_0))} \cdot A_{\bar{t}}, \quad A_{\bar{t}} = \max_{\substack{t \in [t_0, \bar{t}] \\ |y - y_0| \leq |L - y_0|}} a(t, y) .$$

Call $\{\bar{y}_n\}$ a subsequence of $\{y_n\}$ whose derivative (again by Ascoli-Arzelà's theorem) converge uniformly (to $\hat{y}'$) on $[t_0, \bar{t}]$. Moreover, for any $t', t'' \in [t_0, \bar{t}]$, it is

$$(3.7) \qquad |\bar{y}_n''(t') - \bar{y}_n''(t'')| \leq A_{\bar{t}} |\bar{y}_n'(t') - \bar{y}_n'(t'')| + \frac{A_{t_1}|L - y_0|}{1 - \exp(-A_{t_1}(t_1 - t_0))} \cdot |a(t'', \bar{y}_n(t'')) - a(t', \bar{y}_n(t'))| .$$

Since the functions $\bar{y}_n'$ are equicontinuous on $[t_0, \bar{t}]$ and $a(t, y)$ is uniformly continuous for $t \in [t_0, \bar{t}]$ and $|y - y_0| \leq |L - y_0|$, from (3.7) we obtain that $\{\bar{y}_n''\}$ is equicontinuous. So we have proved

Lemma 3.2. *Let $a(t, y)$ be nonnegative. Then, there exists a sequence $\{\bar{y}_n\}$ converging uniformly on any subinterval of $[\alpha, +\infty)$ to function $\hat{y}(t)$ which is a solution of* (1.1).

Remark. At this point it is not clear whether $\hat{y}(t)$ satisfies (1.2). Notice: this is not the case if we take, for instance, $a(t, y) = 0$.

4. - Solution of the terminal value problem

We wish to establish some sufficient conditions for the existence of a solution to the terminal value problem (1.2). We begin with a rather simple case.

Theorem 4.1. *If* $\liminf\limits_{\substack{t \to +\infty \\ y \to L}} a(t, y) > 0$, *then there exists a solution to the terminal value problem* (1.2).

Proof. From the assumption on $a(t, y)$ we can choose t_0 and y_0 (set e.g. $y_0 < L$) such that

$$(4.1) \qquad a(t, y) \geq a_0 > 0 , \quad t \geq t_0 , \quad y \geq y_0 .$$

Consequently, for any $n = 1, 2, \ldots$, we have

$$\bar{y}'_{0,n} \geqslant \frac{a_0(L - y_0)}{1 - \exp\left(-a_0(t_n - t_0)\right)} \geqslant a_0(L - y_0) . \tag{4.2}$$

Now, consider the solution $u(t) = u_0 + (L - u_0)(1 - \exp(-a_0 t))$ of the problem

$$u'' + a_0 u' = 0 , \quad u(t_0) = u_0 < y_0 , \quad u'(t_0) = a_0(L - u_0) \tag{4.3}$$

and observe that $u(t) < L$ and $\lim_{t \to +\infty} u(t) = L$. It is easily seen that, for any n, $\bar{y}_n(t)$ and $u(t)$ cannot cross in the time interval $[t_0, t_n]$. Indeed, in such case, we can find a value T of t $(T \in [t_0, t_n])$ such that $u(T) = \bar{y}_n(T)$, $u'(T) \geqslant \bar{y}'_n(T)$ and so, by virtue of (4.1) and the results of section **2**, $\bar{y}_n(t) \leqslant u(t)$ for $t \in [T, t_n)$. But this is impossible since $\bar{y}_n(t_n) = L > u(t_n)$. Therefore, for any $t \in [t_0, +\infty)$ (the case $t \geqslant t_n$ is trivial) and for any $n = 1, 2, \ldots$, we have

$$u(t) \leqslant \bar{y}_n(t) \leqslant L , \tag{4.4}$$

which implies that $\hat{y}(t)$ is a solution of (1.2).

Theorem 4.2. *If $a(t, y)$ is non negative and, for suitable t_0, y_0 satisfies*

$$\int_{t_0}^{+\infty} \exp\left(-\int_{t_0}^{s} \min_{|y - y_0| \leqslant |L - y_0|} a(t, y)\, dt\right) ds = \lambda < +\infty , \tag{4.5}$$

then there exists a solution to the terminal value problem (1.2).

Proof. Let us define $\gamma(t) = \min_{|y - y_0| \leqslant |L - y_0|} a(t, y)$ and consider the solution of the problem (suppose $y_0 < L$)

$$z'' + \gamma(t) z' = 0 , \quad z(t_0) = z_0 < y_0 , \quad z'(t_0) = (L - z_0)/\lambda . \tag{4.6}$$

For any n we have $\bar{y}'_{0,n} > (L - y_0)/\lambda$.

Then, by an argument similar to that of Theorem 4.1 and using the results of section **2** one easily obtains the result.

Finally we have the following

Theorem 4.3. *Suppose that there exists a Lipschitz continuous non* *negative function* $h(w)$ *satisfying, for suitable* t_0, y_0, *the inequalities*

$$h(w) \leqslant a(t, w) \qquad \text{for } |w - y_0| < |L - y_0|, \qquad (t \geqslant t_0), \tag{4.7}$$

$$\left|\int_w^L h(s)\,ds\right| > 0 \qquad \text{for } |w - y_0| < |L - y_0|. \tag{4.8}$$

Then the terminal value problem (1.2) *has a solution.*

Proof. Assume $y_0 < L$ and consider the problem

$$w'' + h(w)\,w' = 0\,, \qquad w(t_0) = w_0 < y_0\,, \qquad w'(t_0) = \int_{w_0}^L h(s)\,ds\,. \tag{4.9}$$

It is $w'(t) = \int_{w_0}^L h(s)\,ds - \int_{w_0}^{w(t)} h(s)\,ds$, hence $w(t) < L$, since $w'(t)$ cannot change its sign: therefore $\lim_{t\to+\infty} w(t) = L$. Moreover, note that for any n $\bar{y}'_{0,n} \geqslant \int_{y_0}^L h(s)\,ds$. The results of section **2** enable us to affirm that $w(t) \leqslant \bar{y}_n(t) \leqslant L$ and to conclude the proof of the theorem.

Remark. The analysis performed above can be applied equally well, with only minor modifications, for the study of the problems

$$y'' + a(t, y)\,y' = 0\,, \qquad y(t_0) = y_0\,, \qquad \lim_{t\to+\infty} y(t) = L\,, \tag{4.10}$$

$$y'' + a(t, y)\,y' = 0\,, \qquad \lim_{t\to-\infty} y(t) = L_1\,, \qquad \lim_{t\to+\infty} y(t) = L_2\,. \tag{4.11}$$

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Sommario

Si studia il seguente problema ai limiti su intervallo non limitato

$$y'' + a(t, y)y' = 0\,, \qquad \lim_{t\to+\infty} y(t) = L\,, \qquad y'(t) \neq 0\,.$$

Si costruiscono soluzioni di appropriate successioni di problemi ai limiti su intervalli limitati e si prova che esse convergono alla soluzione del problema dato. Si suppone che la funzione $a(t, y)$ soddisfi appropriate condizioni quando y tende ad L e t tende a $+\infty$.

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