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A priori estimates, continuous dependence and stability for solutions to Navier-Stokes equations on exterior domains (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

In the framework of the dynamics of incompressible fluids on exterior domains, in the last few years we have given several contributions particularly in the field of uniqueness $[\mathbf{2}]_1$, $[\mathbf{8}]_{1,2}$ continuous dependence $[\mathbf{2}]_{2,3}$ and stability $[\mathbf{1}]_1$, [7]. In the most part of these works we always tried not to assume, *a priori*, on perturbations any kind of convergence at large spatial distances. To this end, we have introduced the weight function method, by which it is possible to remove from perturbations to the weight functions, the above said convergence conditions.

In this paper, still exploiting the weight function method, we give some a priori estimates and both continuous dependence and stability theorems for classical solutions to Navier-Stokes equations on exterior domains. More precisely, after a brief subsection devolved to preliminaries (subsect. **2**), the paper is subdivided in three main sections. In the first one (**1**, subsects. **3**-**6**) we give some a priori estimates for the perturbation $\boldsymbol{u}$ in the norm of the Lebesgue spaces L^p (subsect. **3**), in the norm of function spaces involving the

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(spatial and time) derivatives of $\boldsymbol{u}$ (subsects. **4** and **5**) and, finally, in weighted L^2 norms (subsect. **6**). We should stress the relevance of the results obtained in subsect. **3**, where, among other things, it is shown that perturbations which a priori may « grow » at large spatial distances, should in fact be uniformly bounded in space and time, provided only that the data and the gradient of the pressure enjoy the same property.

In **2** (subsects. **7-9**) we give some continuous dependence theorems which, among other things, improve earlier results [**9**], [**2**]$_2$, (see subsects. **8**, **9**). It seems interesting to remark that in subsect. **7** we prove that the usually adopted definitions of continuous dependence-for Navier-Stokes equations in unbounded domains and with respect to *perturbations which may « grow » at infinity*-may be reformulated *equivalently* in terms of metrics, a fact, this last one, which *a priori* is far from beeing obvious.

Finally, in **3** (subsects. **10**, **11**) we give some stability theorems in the L^2 norm and (as far as we know) for the first time in a class of perturbations which may a priori « grow ». These theorems (subsect. **11**) are founded upon a key lemma (subsect. **10**) in which is proved an inequality of the Poincaré type for weighted L^2 spaces.

2. - Preliminaries

Let $\{\boldsymbol{v}, p\}$, $\{\boldsymbol{v}+\boldsymbol{u}, p+\pi\}$ be two classical solutions of Navier-Stokes equations in a domain $\Omega(\subseteq R^3)$ which is the exterior of a fixed, closed region Ω_0 bounded by a closed piecewise smooth surface (¹). Denoting by $\{\boldsymbol{v}_0, \boldsymbol{v}_0+\boldsymbol{u}_0\}$, $\{\boldsymbol{v}_\Sigma, \boldsymbol{v}_\Sigma+\boldsymbol{u}_\Sigma\}$, $\{\boldsymbol{f}_1, \boldsymbol{f}_1+\boldsymbol{f}\}$ the initial data, boundary data and body forces (depending on space and time) respectively, corresponding to the above solutions, it is then well known that the perturbation $\{\boldsymbol{u}, \pi\}$ satisfies the following initial boundary value problem (1)+(2)

$$(1)\qquad \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{v}+\boldsymbol{u})\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{v} = -\nabla\pi + \nu\Delta_2\boldsymbol{u} + f(P,t)\,, \qquad \nabla\cdot\boldsymbol{u} = 0 \qquad (P,t)\in\Omega\times[0,T]\,,$$

$$(2)\qquad \boldsymbol{u}(P,0) = \boldsymbol{u}_0(P) \quad P\in\Omega\,, \qquad \boldsymbol{u}(P,t) = \boldsymbol{u}_\Sigma(P,t) \quad (P,t)\in\partial\Omega\times[0,T]\,,$$

where ν is the coefficient of kinematical viscosity. Throughout this paper

(¹) Unless the contrary is esplicitely stated, Ω_0 may also be empty.

-with the exception of the last section-we shall assume, for the sake of simplicity, $\nu = 1$. Moreover, we shall be concerned with solutions of the above problem which satisfy the following generalized weighted energy equalities

$$\frac{\mathrm{d}E^{(q)}}{\mathrm{d}t} = \int_{\Omega} \Big\{\frac{u^q}{q}\Big[\frac{\partial g}{\partial t} + (\boldsymbol{u}+\boldsymbol{v})\cdot\nabla g\Big] - g u^{q-2}[\boldsymbol{u}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{u} \tag{3}$$

$$+ \nabla\boldsymbol{u}:\nabla\boldsymbol{u} + (q-2)u^{-2}(\nabla\boldsymbol{u}\cdot\boldsymbol{u})^2 + \nabla\pi\cdot\boldsymbol{u}] - u^{q-2}\nabla g\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u} + g\boldsymbol{f}\cdot\boldsymbol{u}\Big\}\,\mathrm{d}\Omega$$

$$+ \int_{\partial\Omega} g\Big[u_\Sigma \nabla\boldsymbol{u}\cdot\boldsymbol{u}_\Sigma - \frac{u^q}{q}(\boldsymbol{u}+\boldsymbol{v})\Big]\cdot\boldsymbol{n}\,\mathrm{d}\sigma\,,$$

$$\frac{\mathrm{d}\mathscr{D}}{\mathrm{d}t} = \int_{\Omega} \Big\{\frac{1}{2}\frac{\partial g}{\partial t}\nabla\boldsymbol{u}:\nabla\boldsymbol{u} - g(\Delta_2\boldsymbol{u})^2 - \nabla g\cdot\nabla\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial t} \tag{4}$$

$$- g(\boldsymbol{v}+\boldsymbol{u})\cdot\nabla\boldsymbol{u}\cdot\Delta_2\boldsymbol{u} - g\boldsymbol{u}\cdot\nabla\boldsymbol{v}\cdot\Delta_2\boldsymbol{u} + \pi\nabla g\cdot\Delta_2\boldsymbol{u} + g\boldsymbol{f}\cdot\Delta_2\boldsymbol{u}\Big\}\,\mathrm{d}\Omega$$

$$- \int_{\partial\Omega} g\Big(\nabla\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}_\Sigma}{\partial t} + \pi\Delta_2\boldsymbol{u}\Big)\cdot\boldsymbol{n}\,\mathrm{d}\sigma\,,$$

$$\frac{\mathrm{d}\mathscr{D}}{\mathrm{d}t} = \int_{\Omega} \Big\{\frac{1}{2}\frac{\partial g}{\partial t}\nabla\boldsymbol{u}:\nabla\boldsymbol{u} - g\Big(\frac{\partial\boldsymbol{u}}{\partial t}\Big)^2 - g\boldsymbol{u}\cdot\nabla\boldsymbol{v}\cdot\frac{\partial\boldsymbol{u}}{\partial t} - g\boldsymbol{v}\cdot\nabla\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial t} \tag{5}$$

$$- g\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial t} + \pi\nabla g\cdot\frac{\partial\boldsymbol{u}}{\partial t} + \nabla g\cdot\nabla\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial t} + g\boldsymbol{f}\cdot\frac{\partial\boldsymbol{u}}{\partial t}\Big\}\,\mathrm{d}\Omega$$

$$+ \int_{\partial\Omega} g\Big(\nabla\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}_\Sigma}{\partial t} - \pi\frac{\partial\boldsymbol{u}_\Sigma}{\partial t}\Big)\cdot\boldsymbol{n}\,\mathrm{d}\sigma\,,$$

where

$$E^{(q)} = \frac{1}{q}\int_{\Omega} g u^q\,\mathrm{d}\Omega \equiv \int_{\Omega} e^{(q)}\,\mathrm{d}\Omega \qquad (q>1)\,, \tag{6}$$

$$\mathscr{D} = \tfrac{1}{2}\int_{\Omega} g\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\,\mathrm{d}\Omega \tag{7}$$

and $g = g(P, t)$ is any « weight function ». The above relations are certainly fulfilled, e.g., by those solutions to (1)+(2) whose behaviour at large spatial distances is suitably related to that of g [2]. Of course, the identity (3) will be used to prove, essentially, L^p-estimates, while we shall need identities (4) or (5) each time we would estimate the derivatives of $\boldsymbol{u}$.

Remark 1. We shall later consider also solutions verifying an identity slightly different from (3). Precisely, we shall indicate by (3)′ the identity obtained from (3) with the following replacement

$$(*)\quad \begin{aligned} &-gu^{q-2}\nabla\pi\cdot\boldsymbol{u} \to u^{q-2}\pi\nabla g\cdot\boldsymbol{u} + (q-2)g\pi u^{q-4}\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u} \qquad &(\text{in } \Omega)\,,\\ &-gu^{q-2}\pi\boldsymbol{u} &(\text{in } \partial\Omega)\,. \end{aligned}$$

Of course, if $\{\boldsymbol{u}, \pi\}$ is regular and has a suitable behaviour at infinity (depending on g) the substitution $(*)$ is obtained through the simple application of the divergence theorem.

We end this section by recalling two classical inequalities which shall be frequently used in the sequel.

Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be vector functions in Ω and let f, h be scalar functions in Ω. Then

$$1)\qquad |fh| \leqslant \frac{|f|^p}{p} + \frac{|h|^q}{q}\,,\quad p^{-1} + q^{-1} = 1\,,$$

$$2)\qquad \boldsymbol{A}\cdot\nabla\boldsymbol{B}\cdot\boldsymbol{C} \leqslant \frac{1}{2\xi}A^2C^2 + \frac{\xi}{2}\nabla\boldsymbol{B}:\nabla\boldsymbol{B} \qquad (\xi > 0)\,.$$

Throughout this paper by Ω_R we shall mean the intersection of Ω with a sphere $B(R)$ of radius R and centered in $\mathring{\Omega}_0$. Moreover, we set

$$R_0 = \inf\{R\colon B(R) \supseteq \Omega_0\}\,.$$

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(2) Actually, provided that $\{\boldsymbol{u}, \pi\}$ and their first derivatives have a suitable « growth » at infinity, equalities (3), (4) and (5) can be formally obtained by first multiplying $(1)_1$ by $g\boldsymbol{u}u^{q-2}$, $g\Delta_2\boldsymbol{u}$, $g(\partial\boldsymbol{u}/\partial t)$ respectively, and then by integrating after a simple exploitation of classical identities. However, it is worth remarking that relations (3)-(5) can be possibly satisfied even by non-regular solutions. For instance, equality (3) with $q = 2$ is satisfied, generally speaking, by solutions belonging to suitable weight Sobolev spaces. This problem will be analyzed in a future work.

1. – A priori estimates

3. - L^p a priori estimates

In this section we shall give some estimates on solutions of (1)+(2) in the norm of the Lebesgue spaces L^p. These estimates will be carried out by using the identity (3). More precisely, we shall show that, though starting with *solutions which a priori may « grow » at large spatial distances*, on the assumption that the data $\boldsymbol{u}_0$, $\boldsymbol{f}$ and the gradient of pressure $\nabla\pi$ belong to $L^p(\Omega_T)$ $(p > 1, \Omega_T = \Omega\times[0, T])$, the solution in fact belongs to $L^p(\Omega)$. Moreover, if $\boldsymbol{u}_0$, $\boldsymbol{f}$ and $\nabla\pi$ are uniformly bounded in Ω_T, then the same holds for $\boldsymbol{u}(P, t)$.

The results just stated are an easy consequence of the following theorem (3).

Theorem 1. *Let $\{\boldsymbol{u}, \pi\}$ be a solution satisfying, for some $q > 1$ the identity* (3) *with $g = \exp[-\alpha r(t + t_0)^\beta]$ $(\alpha, t_0, \beta > 0)$, and such that*

$$|v_r|, \quad |u_r| \leqslant Mr, \quad |(\nabla\boldsymbol{v}) + (\nabla\boldsymbol{v})^T| \leqslant M \qquad (M > 0). \tag{8}$$

Then if

$$\boldsymbol{u}_0, \quad \boldsymbol{f}, \quad \nabla\pi \in L^q(\Omega_T), \tag{9}$$

we have (4)

$$\boldsymbol{u} \in L^\infty(0, T; L^q(\Omega)), \quad u^{q-2/2}\nabla\boldsymbol{u} \in L^2(\Omega_T), \quad \lim_{R\to\infty} R^2 \int_{\Sigma_R} u^q \, d\Sigma = 0. \tag{10}$$

Moreover if

$$\boldsymbol{u}_0, \quad \boldsymbol{f}, \quad \nabla\pi \in L^\infty(\Omega_T), \tag{11}$$

then $\boldsymbol{u}$ is essentially bounded in Ω_T, since

$$\text{ess sup } |\boldsymbol{u}| \leqslant \max\{\text{ess sup } (|\boldsymbol{u}_0|, |\boldsymbol{u}_\Sigma|, |\boldsymbol{f}|, |\nabla\pi|)\} \text{ (5)}.$$

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(3) Throughout this paper, by w_r we shall always mean the radial component of the field $\boldsymbol{w}$. Moreover, we shall use the following standard notations. Let X be a Banach space and a, b be two real numbers; the $L^s(a, b; X)$ $(s > 0)$ (resp. $L^\infty(a, b; X)$) is the class of functions on $[a, b]$ and in X whose norm in X is s-summable in $[a, b]$ (resp. in essentially bounded in $[a, b]$).

(4) By Σ_R we indicate $\partial\Omega_R \backslash \partial\Omega$.

(5) From now on it is tacitly understood that $\nabla\boldsymbol{u}$ and π are uniformly bounded on $\partial\Omega$.

Proof. From (1) and (2) of the previous subsection, the following relations follows.

(a) $\frac{\partial g}{\partial t} + (\boldsymbol{u} + \boldsymbol{v})\cdot\nabla g \leqslant 0 \quad \text{for } \beta \geqslant 2M(T + t_0)$;

(b) $g u^{q-2}\boldsymbol{u}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{u} \leqslant Mq\, e^{(q)}$,

(c) $-u^{q-2}\nabla g\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u} \leqslant \frac{\alpha}{2}(T + t_0)^{2\beta} q\, e^{(q)} + \frac{\alpha}{2} g u^{q-2}\nabla\boldsymbol{u}:\nabla\boldsymbol{u}$,

(d) $g u^{(q-2)}\nabla\pi\cdot\boldsymbol{u} \leqslant (q-1)\, e^{(q)} + g\,\frac{|\nabla\pi|^q}{q}$,

(e) $g u^{(q-2)}\boldsymbol{f}\cdot\boldsymbol{u} \leqslant (q-1)\, e^{(q)} + \frac{g}{q}\,|\boldsymbol{f}|^q$,

(f) $-[\nabla\boldsymbol{u}:\nabla\boldsymbol{u} + (q-2)u^{-2}(\nabla\boldsymbol{u}\cdot\boldsymbol{u})^2] \leqslant \begin{cases} -\nabla\boldsymbol{u}:\nabla\boldsymbol{u} & \text{if } q \geqslant 2 \\ -(q-1)\nabla\boldsymbol{u}:\nabla\boldsymbol{u} & \text{if } q \in\,]1, 2[\,. \end{cases}$

Then from (3), we get for α sufficiently small

$$\frac{\mathrm{d}E^{(q)}}{\mathrm{d}t} \leqslant \left[Mq + \frac{\alpha}{2} q(T + t_0)^{2\beta} + 2(q-1)\right]E^{(q)} - \xi \int_\Omega g u^{(q-2)}\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\,\mathrm{d}\Omega \tag{12}$$

$$+ \frac{1}{q}\int_\Omega g\big(|\nabla\pi|^q + |\boldsymbol{f}|^q\big)\,\mathrm{d}\Omega + A\int_{\partial\Omega}\{u_\Sigma^{q-1} + u_\Sigma^q\}\,\mathrm{d}\sigma \qquad (A = \text{const.} > 0),$$

where ξ is a positive constant related to the two possibilities arising from item (f). Of course, when $q = 2$, we may assume $\xi = 1/2$. Setting $N = N(q) \equiv [Mq + (\alpha/2)q(T + t_0)^{2\beta} + 2(q-1)]$ after a simple integration, the inequality (12) yields, $\forall t \in [0, T]$

$$E^{(q)} \leqslant E^{(q)}(0)\exp[Nt] - \exp(Nt)\int_0^t \exp(-Ns)g\Big(\xi u^{q-2}\nabla\boldsymbol{u}:\nabla\boldsymbol{u} \tag{13}$$

$$- \frac{|\nabla\pi|^q}{q} - \frac{|\boldsymbol{f}|^q}{q}\Big)\,\mathrm{d}\Omega\,\mathrm{d}s + A'\big(\sup_{\partial\Omega\times[0,T]}|u_\Sigma|\big)^{q-1} \qquad (A' = \text{const.} > 0).$$

Now, from (13) $\forall R \geqslant R_0$, we obtain

$$\int_{\Omega_R} u^q \, d\Omega + \xi q \int_0^t \int_{\Omega_R} u^{q-2} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \, d\Omega \leqslant \exp(\alpha R (T + t_0)^\beta) \{\exp(NT)[\int_\Omega u_0^q \, d\Omega$$
$$+ \int_0^T \exp(-Ns) \int_\Omega (|\nabla \pi|^q + |\boldsymbol{f}|^q) \, d\Omega \, ds] + A'(\sup_{\partial\Omega \times [0,T]} |\boldsymbol{u}_\Sigma|)^{q-1}\} .$$

Thus, letting *first* $\alpha \to 0$ and *then* $R \to \infty$, we get

$$\int_\Omega u^q \, d\Omega + \xi q \int_0^t \int_\Omega u^{q-2} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \, d\Omega \tag{14}$$
$$\leqslant \exp(NT) \{ \int_\Omega \boldsymbol{u}_0^q \, d\Omega + \int_0^T \int_\Omega (|\nabla \pi|^q + |\boldsymbol{f}|^q) \, d\Omega \, ds + A'(\sup_{\partial\Omega \times [0,T]} |\boldsymbol{u}_\Sigma|)^{q-1}\} .$$

From this relation we easily recover $(10)_1$ and $(10)_2$. In order to prove $(10)_3$, we notice that

$$R^2 \int_{\Sigma_R} u^q \, d\sigma = \int_{\Omega_R} \frac{1}{r^2} \frac{\partial (u^q r^2)}{\partial r} \, d\Omega + \text{const.},$$

where the const. depends on the value of $\boldsymbol{u}$ on the boundary $\partial\Omega$.

But

$$\lim_{R \to +\infty} \int_{\Omega_R} \frac{1}{r^2} \frac{\partial u^q r^2}{\partial r} \, d\Omega , \tag{15}$$

exists, since

$$|\int_{\Omega_R} \frac{1}{r^2} \frac{\partial (u^q r^2)}{\partial r} \, dr| \leqslant c\{ \int_\Omega u^q \, d\Omega + \int_\Omega u^{q-2} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \, d\Omega\} ,$$

where C is a positive constant. On the other hand, the limit (15) must be zero since $\boldsymbol{u} \in L^q(\Omega)$. Thus the first part of the theorem is completely proved. In order to prove the second part, we notice that from (12) we have,

$$\frac{dE^{(q)}}{dt} \leqslant N(q) \, E^{(q)} + \frac{1}{q} (N')^q \int_\Omega g \, d\Omega + A(N')^{q-1} \int_{\partial\Omega} (1 + u_\Sigma) \, d\sigma \tag{16}$$

$$\Rightarrow (\int_{\Omega_R} u^q \, d\Omega)^{1/q} \leqslant N' \exp(\frac{N(q)}{q} T + \frac{\alpha R (T + t_0)^\beta}{q}) \{2 \int_\Omega g \, d\Omega + \frac{A}{N'} \int_{\partial\Omega} (1 + u_\Sigma) \, d\sigma\}^{1/q},$$

where $N' = \max[\text{ess sup}\{|\boldsymbol{u}_0|, |\boldsymbol{u}_\Sigma|, |\nabla\pi|, |\boldsymbol{f}|\}$. Recalling that N is a linear function of q, the theorem follows by letting $q \to \infty$ in (16).

Remark 2. The estimate (10) still holds if instead of $(9)_3$, we assume the following condition to hold

$$\pi \in L^q(\Omega_T) \qquad (q > 2) \tag{17}$$

for solutions $\{\boldsymbol{u}, \pi\}$ which verify the identity (3)′ (cfr. Remark 1). In fact, in the case (17), in place of the inequality (d), we should adopt the following (cfr. (*) of Remark 1)

$$\pi u^{q-2}\nabla g\cdot\boldsymbol{u} + (q-2)\,g\pi u^{q-4}\,\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u} \leqslant k_1 e^{(q)} + k_2\pi^{(q)} + \xi u^{q-2}\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\,, \tag{17'}$$

where ξ $(< 1/2)$, k_1 and k_2 are suitable constants and where use has been made of the inequality 1).

Remark 3. A stronger estimate than those of Theorem 1 can be obtained if we replace both assumptions $(9)_3$ and $(11)_3$ with the only hypothesis $|\nabla\pi|/|\boldsymbol{u}| \leqslant M$, $\forall(P, t) \in \Omega_T$, for some suitable positive constant M. In fact, in this case, it is possible to get an estimate on $\text{ess sup}_{\Omega_T} |\boldsymbol{u}|$ which involves *only* the data. To this end, it suffices to replace the inequality (d) with the following $gu^{q-2}\nabla\pi\cdot\boldsymbol{u} \leqslant Mgu^q$, and to repeat step by step the proof.

4. - $W^{2,1}(\Omega_R)$ estimates

In this section we shall give some estimates of the solution $\boldsymbol{u}$ in the norm of $W^{2,1}(\Omega_R)$ where by $W^{2,1}(\Omega_R)$ we mean the set of function which are square summable in Ω_R together with their first derivatives.

The following theorem holds.

Theorem 2. *Let* $\{\boldsymbol{u}, \pi\}$ *be a solution satisfying the identity* (3)′ (cfr. *Remark* 1) *for* $q = 2$, *the identity* (3) *for some* $q > 1$, *the identity* (5) *and, in all cases, with* g *as in Theorem* 1.

If (11) *holds and*

$$|\boldsymbol{v}|, \quad |\nabla\boldsymbol{v}| \leqslant M, \quad |u_r| \leqslant Mr, \quad \pi \in L^{6-\varepsilon}(\Omega_T) \;\; (\varepsilon \in]0, 4[)\,, \tag{18}$$

then, $\forall R \geqslant R_0$ $(c, \sigma, \tilde{\eta} > 0)$

$$(19)\qquad \int_{\Omega_R} (u^2 + \nabla \boldsymbol{u} : \nabla \boldsymbol{u})\, d\Omega + \int_0^t \int_{\Omega_R} \left(\frac{\partial \boldsymbol{u}}{\partial s}\right)^2 d\Omega\, ds \leqslant \exp\left[\alpha(T + t_0)^\beta R\right] \cdot$$

$$c\Big\{ \int_\Omega g(u_0^2 + \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{u}_0)\, d\Omega + \alpha^{\tilde{\eta}/2} \int_0^T \int_\Omega |\pi|^{6-\varepsilon}\, d\Omega\, dt$$

$$+ \int_0^T \int_\Omega g f^2\, d\Omega\, dt + \int_0^T \int_{\partial\Omega} \left(|\boldsymbol{u}_\Sigma| + \left|\frac{\partial \boldsymbol{u}_\Sigma}{\partial t}\right|\right) d\sigma\, dt + \alpha^\sigma \Big\} .$$

Proof. Along with the inequalities (a)-(c), (e)-(f) calculated for $q = 2$, we consider the following

$$(a)'\quad \pi \nabla g \cdot \boldsymbol{u} \leqslant \frac{\alpha^2}{2} g \pi^2 + g \frac{u^2}{2} \leqslant \frac{(T + t_0)^{2\beta}(4 - \varepsilon)}{2(6 - \varepsilon)}\, g \alpha^{(2-\eta)(6-\varepsilon)/4-\varepsilon}$$

$$+ (T + t_0)^{2\beta} \frac{\alpha^{(\eta/2)(6-\varepsilon)} g}{6 - \varepsilon} |\pi|^{6-\varepsilon} + g \frac{u^2}{2} \qquad (\eta \in]0, \varepsilon/(6 - \varepsilon)) ,$$

$$(b)'\quad -g \boldsymbol{u} \cdot \nabla \boldsymbol{v} \cdot \frac{\partial \boldsymbol{u}}{\partial t} \leqslant \frac{M^2 g u^2}{2\xi} + \frac{\xi}{2} g \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)^2 \qquad (\xi > 0) ,$$

$$(c)'\quad -g \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial t} \leqslant g \frac{M^2}{2\xi} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} + \frac{\xi}{2} g \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)^2 ,$$

$$(d)'\quad \pi \nabla g \cdot \frac{\partial \boldsymbol{u}}{\partial t} \leqslant \frac{(T + t_0)^{2\beta}(4 - \varepsilon)}{\xi 2(6 - \varepsilon)}\, g \alpha^{(2-\eta)(6-\varepsilon)/4-\varepsilon}$$

$$+ \frac{(T + t_0)^{2\beta}}{\xi} \frac{\alpha^{(\eta/2)(6-\varepsilon)} g |\pi|^{6-\varepsilon}}{(6 - \varepsilon)} + \frac{\xi}{2} g \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)^2 ,$$

$$(e)'\quad \nabla g \cdot \nabla \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial t} \leqslant \frac{g}{2\xi} \alpha^2 (T + t_0)^{2\beta} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} + \frac{g\xi}{2} \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)^2 ,$$

$$(f)'\quad g f \cdot \frac{\partial \boldsymbol{u}}{\partial t} \leqslant \frac{g}{2\xi} f^2 + \frac{g\xi}{2} \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)^2 .$$

Moreover, by assumption and by the results stated in Theorem 1, we have that $\boldsymbol{u}$ is uniformly bounded in Ω_T by a costant, say N. Therefore

$$(h)\quad -g \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial t} \leqslant \frac{N^2 g}{2\xi} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} + \frac{g\xi}{2} \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)^2 .$$

Thus, summing (3)′ and (5) and exploiting the above inequalities, we easily get the following relation (with $E^{(2)} = E$)

$$\frac{\mathrm{d}}{\mathrm{d}t}(E + \mathscr{D}) \leqslant (2M + 3 + \frac{\alpha}{2} + \frac{M^2}{\xi})\,E + (-1 + \frac{M^2}{2\xi} \tag{20}$$

$$+ \frac{\alpha^2(T + t_0)^{2\beta}}{2\xi} + \frac{N^2}{2\xi})\,\mathscr{D} + (3\xi - 1)\int_\Omega g\,(\frac{\partial \boldsymbol{u}}{\partial t})^2\,\mathrm{d}\Omega + \frac{\xi + 1}{\xi}\,\frac{(T + t_0)^{2\beta}}{(6 - \varepsilon)}\,\alpha^{(\eta/2)(6-\varepsilon)}\,\cdot$$

$$\cdot \int_\Omega g\,|\pi|^{6-\varepsilon}\,\mathrm{d}\Omega + \frac{(\xi + 1)}{2\xi}\int_\Omega g f^2\,\mathrm{d}\Omega + \int_{\partial\Omega} g[\nabla\boldsymbol{u}\cdot\boldsymbol{u}_\Sigma - \frac{u_\Sigma^2}{2}\,(\boldsymbol{u}_\Sigma + \boldsymbol{v}_\Sigma)$$

$$+ \pi\,(\boldsymbol{u}_\Sigma + \frac{\partial \boldsymbol{u}_\Sigma}{\partial t}) + \nabla\boldsymbol{u}\cdot\frac{\partial \boldsymbol{u}_\Sigma}{\partial t}]\cdot\boldsymbol{n}\,\mathrm{d}\sigma + 4\pi\,(\frac{\xi + 2}{2\xi})\,\frac{(T + t_0)^{2\beta}(4 - \varepsilon)}{2(6 - \varepsilon)}\,\cdot$$

$$\cdot\,\alpha^{(2-\eta)(6-\varepsilon)/4-\varepsilon}\int_0^\infty \exp\,[-\,\alpha t_0 r]\,r^2\,\mathrm{d}r\,.$$

Since

$$\int_0^\infty \exp\,[-\,\alpha t_0^\beta r]\,r^2\,\mathrm{d}r \leqslant k\alpha^{-3} \qquad (k > 0)\,,$$

choosing $\xi < \frac{1}{3}$, the inequality (20) gives, with an obvious meaning of the positive constants c_i $(i = 1, \ldots, 6)$ and σ, $\tilde{\eta}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(E + \mathscr{D}) \leqslant c_1(E + \mathscr{D}) - c_2\int_\Omega g\,(\frac{\partial \boldsymbol{u}}{\partial t})^2\,\mathrm{d}\Omega + c_3\alpha^{\tilde{\eta}/2}\int_\Omega |\pi|^{6-\varepsilon}\,\mathrm{d}\Omega \tag{21}$$

$$+ c_4\int_\Omega g f^2\,\mathrm{d}\Omega + c_5\int_{\partial\Omega}\{|\boldsymbol{u}_\Sigma| + |\frac{\partial \boldsymbol{u}_\Sigma}{\partial t}|\}\,\mathrm{d}\sigma + c_6\alpha^\sigma\,.$$

Thus, integrating (21) from 0 to T, after a simple calculation, for a suitable value of the constant C, we get

$$E(t) + \mathscr{D}(t) + \int_0^t\int_\Omega g\,(\frac{\partial \boldsymbol{u}}{\partial s})^2\,\mathrm{d}\Omega\,\mathrm{d}s \leqslant C\{[E(0) + \mathscr{D}(0)] + \alpha^{\tilde{\eta}/2}\int_0^T\int_\Omega |\pi|^{6-\varepsilon}\,\mathrm{d}\Omega\,\mathrm{d}t$$

$$+ \int_0^T\int_\Omega g f^2\,\mathrm{d}\Omega + \int_0^T\int_{\partial\Omega}(|\boldsymbol{u}_\Sigma| + |\frac{\partial \boldsymbol{u}_\Sigma}{\partial t}|)\,\mathrm{d}\sigma\,\mathrm{d}t + \alpha^\sigma\}\,,$$

which, in turn, implies (19).

Remark 4. Of course, in the above theorem we may drop assumption (11), provided only that we assume $|\boldsymbol{u}|$ uniformly bounded in Ω_T.

Remark 5. From (17), as a particular case, we have that if $\boldsymbol{f} \in L^2(\Omega_T)$, $\boldsymbol{u}_0$, $\nabla \boldsymbol{u}_0 \in L^2(\Omega)$, then $\boldsymbol{u}$, $\nabla \boldsymbol{u} \in L^2(\Omega)$ and $\partial \boldsymbol{u}/\partial t \in L^2(\Omega_T)$. In fact, in this case, we may let $\alpha \to 0$ in (19).

5. - Δ_2 estimates

In this section we shall give some estimates of the solution $\boldsymbol{u}$ which involve its second spatial derivatives.

Theorem 3. *Let $\{\boldsymbol{u}, \pi\}$ be a solution satisfying the identity* (4) *along with the assumptions made in Theorem* 2. *Then* $\forall R \geqslant R_0$

$$(22) \qquad \int_{\Omega_R} (u^2 + \nabla \boldsymbol{u} : \nabla \boldsymbol{u})\, \mathrm{d}\Omega + \int_0^t \int_{\Omega_R} [(\frac{\partial \boldsymbol{u}}{\partial s})^2 + (\Delta_2 \boldsymbol{u})^2]\, \mathrm{d}\Omega\, \mathrm{d}s$$

$$\leqslant c \exp[\alpha(T + t_0)^\beta R] \{ \int_\Omega g(u_0^2 + \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{u}_0)\, \mathrm{d}\Omega + \alpha^{\tilde{\eta}/2} \int_0^T \int_\Omega |\pi|^{6-\varepsilon}\, \mathrm{d}\Omega\, \mathrm{d}t$$

$$+ \int_0^T \int_\Omega g f^2\, \mathrm{d}\Omega\, \mathrm{d}t + \int_0^T \int_{\partial\Omega} (|\boldsymbol{u}_\Sigma| + |\frac{\partial \boldsymbol{u}_\Sigma}{\partial t}| + |\Delta_2 \boldsymbol{u}|)\, \mathrm{d}\sigma\, \mathrm{d}t \} \qquad (c, \sigma, \tilde{\eta} > 0).$$

Proof. The proof is carried out exactly as in Theorem 2, when the following estimates are beared in mind ($\xi > 0$)

$$\text{(i)} \quad -g(\boldsymbol{v} + \boldsymbol{u}) \cdot \nabla \boldsymbol{u} \cdot \Delta_2 \boldsymbol{u} \leqslant \frac{gM^2}{\xi} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} + \frac{\xi g}{2} (\Delta_2 \boldsymbol{u})^2 ,$$

$$\text{(ii)} \quad -g\boldsymbol{u} \cdot \nabla \boldsymbol{v} \cdot \Delta_2 \boldsymbol{u} \leqslant \frac{M^2}{2\xi} g u^2 + \frac{g\xi}{2} (\Delta_2 \boldsymbol{u})^2 ,$$

$$\text{(iii)} \quad \pi \nabla g \cdot \Delta_2 \boldsymbol{u} \leqslant \frac{(T + t_0)^{2\beta}}{\xi} \alpha \frac{\eta/2\,(6 - \varepsilon) g |\pi|^{6-\varepsilon}}{(6 - \varepsilon)} + \frac{g\xi}{2} (\Delta_2 \boldsymbol{u})^2 ,$$

$$\text{(iv)} \quad g\boldsymbol{f} \cdot \Delta_2 \boldsymbol{u} \leqslant \frac{g f^2}{2\xi} + \frac{g\xi}{2} (\Delta_2 \boldsymbol{u})^2 .$$

Remark 6. Remarks 4 and 5 still holds in this case. Moreover, under the assumptions of Remark 5, we have also $\Delta_2 \boldsymbol{u} \in L^2(\Omega_T)$.

6. - Weighted estimates

In this section we shall give some estimates with respect to weighted L^2-norms, with weight $r^{-\varepsilon}$ ($\varepsilon > 0$). In fact, the following Theorem holds.

Theorem 4. *Let* $\{\boldsymbol{u}, \pi\}$ *be a solution satisfying the identity* (3)′ (cfr. *Remark* 1) *for* $q = 2$. *Assume, moreover, that*

$$|u_r|\,, \quad |v_r| \leqslant M\,, \quad |(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^T| \leqslant M\,. \tag{23}$$

If $\boldsymbol{u}_0 r^{-\varepsilon} \in L^2(\Omega)$, $f r^{-\varepsilon} L^2(\Omega_T)$ $(\varepsilon > 0)$ *and*

$$\pi \in L^s(\Omega_T)\,,\ s = \frac{6}{1-\varepsilon} - \frac{2}{(1-\varepsilon)}\ (\varepsilon \in [0,1[)\,;\ \pi \in L^s(\Omega_T)\,,\ s > 1\,,\ (\varepsilon \geqslant 1)\,. \tag{24}$$

Then $\sup\limits_{t\in[0,T]} \int\limits_\Omega (u^2/r^\varepsilon)(P, t)\, \mathrm{d}\Omega < \infty$.

Proof. Taking into account the assumption (23) and that (since $\Omega_0 \supseteq B(1)$) [6]

$$|\nabla g| = \frac{\exp(-\alpha r)}{r^\varepsilon} (\varepsilon r^{-1} + \alpha) \leqslant k g \qquad (k = \varepsilon + \alpha)\,,$$

all terms which involve $\boldsymbol{u}$ and $\boldsymbol{v}$ and appearing at the right hand side of (3)′ may be increased in a standard way. As far as the term in π is concerned, it may be increased as follows

$$\pi \nabla g \cdot \boldsymbol{u} \leqslant \frac{\alpha^2 \pi^2 \exp(-\alpha r)}{2 r^\varepsilon} + \frac{\varepsilon^2 \pi^2 \exp(-\alpha r)}{2 r^{2+\varepsilon}} + g u^2\,,$$

$$\frac{\alpha^2 \pi^2 \exp(-\alpha r)}{2 r^\varepsilon} \leqslant \frac{\alpha^{2p}}{2p} g + \frac{1}{2q} g |\pi|^{2q}\,, \quad \frac{\varepsilon^2 \pi^2 \exp[-\alpha r)}{2 r^{2+\varepsilon}} \leqslant \frac{\varepsilon^2 \exp(-\alpha r)}{2 p r^{(2+\varepsilon)p}} + \frac{\varepsilon^2}{2q} |\pi|^{2q}\,.$$

From these relations, treating separately the two cases $\varepsilon \in [0, 1[$ and $\varepsilon \geqslant 1$, and choosing $q = 3/(1-\varepsilon) - 1(1-\varepsilon)$, $\varepsilon \in [0, 1[$, $q > 1$, $\varepsilon \geqslant 1$; we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leqslant k_1 E + \int\limits_\Omega |\pi|^s \mathrm{d}\Omega + \int\limits_\Omega g f^2 \mathrm{d}\Omega + k_2 \int\limits_{\partial\Omega} |\boldsymbol{u}_\Sigma| \mathrm{d}\sigma \qquad (k_1, k_2 > 0)\,. \tag{25}$$

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(6) This assumption is made here for the sake of simplicity. Actually, proceeding as in [**8**]$_1$, i.e., suitably continuing the function g, the theorem retains its validity also when $\Omega_0 = \emptyset$.

Thus, exploiting (24), integrating inequality (25) and letting $\alpha \to 0$, the theorem follows.

Remark 7. Assume that $\{\boldsymbol{u}, \pi\}$ be a solution of $(3)'$ for $q = 2$ with the replacement

$$g\boldsymbol{u}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{u} \to - g\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v} - \boldsymbol{u}\cdot\nabla g(\boldsymbol{u}\cdot\boldsymbol{v}) \quad (\text{in } \Omega), \qquad + g\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{v}) \quad (\text{in } \partial\Omega),$$

then, the above Theorem can be shown under the assumption $|u_r|$, $|\boldsymbol{v}| \leqslant M$ which is alternative to (23).

2. - Continuous dependence

7. - General facts about continuous dependence

The problem of continuous dependence (and more particularly, of stability) in the frame of abstract dynamical systems has been widely treated in several recent papers (see, e.g., [**3**], [**5**]). More precisely, let $\mathscr{B} = \mathscr{B}(\mathscr{T}, X; X^*) = \{\varphi_\xi : \mathscr{T} \to X; \xi \in X^*\}$ be a dynamical system, where $\mathscr{T} = [0, \tilde{T}]$ $(\tilde{T} > 0)$, X is the set of the « states » and X^* is a suitable space from which the « solutions » φ_ξ are labelled. The meaning of such spaces is most natural. In particular, the space X^* represents the space of « parameters » (different from « initial data ») with which a « solution » may vary [7]. Moreover, a « motion » $\varphi_\xi(\tau + t) \equiv \varphi_{\tau,\xi}(t)$, $t \in \mathscr{T}$, corresponding to the initial data $\varphi(\tau) \in X(\tau) \subseteq X$ $(\tau \in \mathscr{T}^* \subseteq \mathscr{T})$ and to $\xi \in X^*$, can be characterized by the following map (which a priori need not be single-valued)

$$T \equiv T_{\tau,\xi} : \big(\varphi(\tau); \xi\big) \in X(\tau) \times X^* \to \varphi_{\tau,\xi} \in \mathscr{B}\,.$$

Denote, now, by r a metric in X and put $\bar{r}_{t_1}(\varphi_\xi, \psi_\eta) = \sup\limits_{t \in [0,t_1]} r\big(\varphi_\xi(t), \psi_\eta(t)\big)$ for $t_1 \in \mathscr{T}$ and ξ, $\eta \in X^*$. Obviously, $\bar{r}_{t_1}$ is a metric on $\mathscr{B}_{t_1} = \mathscr{B}\big([0, t_1], X; X^*\big)$. Furthermore, let ϱ be a metric on $Y = X(\tau) \times X^*$. Following [**3**], [**5**] one says that a « solution » φ_ξ *continuously depends upon the data with respect to* ϱ *and* $\bar{r}_{t_1}$ iff the « restriction » map T_{t_1} of T to $\mathscr{B}_{t_1}$ is continuous in φ_ξ, $\forall t_1 \in \mathscr{T}$ and $\forall \tau \in \mathscr{T}^*$, i.e. iff

$$\forall \varepsilon > 0, \quad \exists \delta(\varepsilon, \tau, t_1) > 0 : \quad \varrho(\zeta_1, \zeta_2) < \delta \Rightarrow \bar{r}_{t_1}(\varphi_{\xi_1}, \psi_{\xi_2}) < \varepsilon\,.$$

[7] For example, if we consider a dynamical system associated to the motion of a continuous system, the parameters ξ may represent, e.g., boundary data, forces, coefficients characterizing the material, etc.

Moreover, if $\varepsilon = k\delta^p (p \in [0, 1[)$ with $k=k(t_1)$ we shall say that φ_ξ *depends Hölder continuously*. Finally, we shall say that is *stable* iff T_{t_1} is continuous in φ_ξ uniformly in t_1, i.e., δ does not depend on t_1.

We should now remark that this abstract scheme, though it well applies to the most part of physical situations, at a first sight seems to fail when one is dealing with exterior problems—both in linear elasticity and in hydrodynamics—and when *the perturbations are allowed to « grow » at large spatial distances* [**2**]$_{2,3}$, [**9**], [**6**]. For instance, in the case of Navier-Stokes equations, a typical example is of the following kind. Retaining the same notations of **1**, in [**9**] the null solution of problem (1)+(2) is said to « depend continuously upon the data » iff (8) $(k, p, s > 0)$

$$(26) \qquad \sup\{|\boldsymbol{u}_0|, |\boldsymbol{f}|, |\boldsymbol{u}_\Sigma|\} < \delta \Rightarrow \int_{\Omega_R} u^2(P, t)\, d\Omega < k\delta^p, \quad \forall R \leqslant \delta^{-s}, \ \forall t \in [0, T].$$

It is easy to see that (26) cannot be considered at once as a particular case of the previous definition of continuous dependence, since the perturbation $\boldsymbol{u}(P, t)$ is not « measured » through a metric. In view of this fact, in order that (26) be meaningful from the physical point of view (i.e. in order that (26) merit the attribute of « continuous dependence ») it is fundamental to investigate the following two aspects: (i) whether the space X might be topologized in such a way that the continuity of the map T be *equivalent* to a request of the kind (26); (ii) whether in X is possible to introduce a metric such that the continuity of T with respect to the above topology, implies the continuity of T with respect to the topology induced by such a metric. We shall show below that, in fact, both items (i) and (ii) are met.

To this end let $\mathscr{R} = \{r_\alpha\}_{\alpha \geqslant \alpha_0 > 0}$ be a family of quasi-metrics on X, such that:

(1) $r_{\alpha_1}(x, y) \leqslant r_{\alpha_2}(x, y), \ \forall \alpha_1 \leqslant \alpha_2$,

(2) $\mathscr{R}$ separates points, i.e., if $x_1 \neq x_2$ there exists $r_\alpha \in \mathscr{R}: r_\alpha(x_1, x_2) \neq 0$.

Moreover, let h from $]0, 1]$ onto $[\alpha_0, +\infty[$ be a non-increasing function of the argument. Then, for $x \in X$, $\varepsilon \in [0, 1]$, we set $U_\varepsilon^{(x)} = \{y \in X: r_{h(\varepsilon)}(x, y) < \varepsilon\}$. The family $B = \{U_\varepsilon^{(x)}\}_{x \in X; \varepsilon \in]0,1]}$ is readily seen to enjoy the following properties (9)

(1) $\forall x \in X, \ \exists U \in B: \ x \in U$,

(2) $\forall U_1, U_2 \in B$ with $U_1 \cap U_2 \neq \emptyset$ then $\forall x \in U_1 \cap U_2, \ \exists U \in B: x \in U \subseteq U_1 \cap U_2$.

(8) In this case $X = \{\boldsymbol{u}: u_i \in C^2(\Omega), \ |\boldsymbol{u}| \leqslant Mr^{1-\varepsilon}, \ |\nabla \boldsymbol{u}| \leqslant M \ (M, \varepsilon > 0)\}$.

(9) Property (1) is obvious. To show (2) it suffices to prove that $\forall U_\varepsilon^{(x)} \in B$ and $\forall y \in U_\varepsilon^{(x)} \ \exists U_\sigma^{(y)} \in B: \ U_\sigma^{(y)} \subseteq U_\varepsilon^{(x)}$. To show this, let us pick $\sigma \in]0, \varepsilon - r_{h(\varepsilon)}(x, y)[$. We want to show that $r_{h(\sigma)}(z, y) < \sigma \Rightarrow r_{h(\varepsilon)}(z, x) < \varepsilon \ \ \forall z \in U_\sigma^{(y)}$. In fact, since h is non-increasing, we get

$$r_{h(\varepsilon)}(z, x) \leqslant r_{h(\varepsilon)}(z, y) + r_{h(\varepsilon)}(x, y) \leqslant r_{h(\sigma)}(z, y) + r_{h(\varepsilon)}(x, y) < \sigma + r_{h(\varepsilon)}(x, y) < \varepsilon.$$

As is well known, then, the family B is a basis for a (uniquely determined) topology on X (cfr. e.g. [4]) and the open sets A are of the form $A = \bigcup_{U\in B} U$. Moreover, this topology is Hausdorff ([10]). By standard arguments, it then follows that $f: \tilde{Y} \to X$ ($\tilde{Y}$ metric space with metric d) is continuous in $y_0 \in \tilde{Y}$ with respect to the above topology in X, iff

$$(26)' \qquad \forall\varepsilon > 0, \quad \exists\delta(\varepsilon) > 0: \quad d(y, y_0) < \delta \Rightarrow r_{h(\varepsilon)}(f(y), f(y_0)) < \varepsilon ,$$

which, in its abstract form, just coincides with (26).

Of course, when X is topologized in such a way, we can also introduce a natural topology on $\mathscr{B}_{t_1}$ ($\forall t_1 \in \mathscr{T}$), by setting $\bar{r}_{\alpha,t_1}(\varphi_\xi, \psi_\eta) = \sup_{t\in[0,t_1]} r_\alpha(\varphi_\xi(t), \psi_\eta(t))$ or, if $r_\alpha(\varphi_\xi(t), \psi_\eta(t))$ is a continuous function of $t \in [0, t_1]$, $\forall\alpha \geqslant \alpha_0$, we may set

$$\bar{r}_{\alpha,t_1}(\varphi_\xi, \psi_\eta) = \int_0^{t_1} r_\alpha(\varphi_\xi(t), \psi_\eta(t))\, dt .$$

Now, starting with the family $\mathscr{R}$, let us introduce in $X \times X$ the following function

$$\lambda: (x_1, x_2) \in X \times X \to \int_0^1 \frac{r_{h(t)}(x_1, x_2)}{1 + r_{h(t)}(x_1, x_2)}\, dt \equiv \int_0^1 \tilde{r}(t; x_1, x_2)\, dt .$$

Taking into account that $\mathscr{R}$ is a family of quasi-metric which separates points and that the function $z \to z/(1+z)$ is increasing in z, it can be readily shown that λ enjoys all the properties of a metric ([11]).

We end this section, by noting that the following theorem (relating the continuity in the sense of (26)′ to that associated with the topology induced by λ) holds.

Theorem 5. *Let $f: \tilde{Y} \to X$ be continuous in some $y_0 \in \tilde{Y}$ in the sense of (26)′. Then f is continuous in y_0 also with respect to the topology induced in X by λ.*

([10]) Let $x \neq y$. Since $\mathscr{R}$ separates points, we have for some $\gamma > 0$,

$$(*) \qquad m = r_{h(\gamma)}(x, y) \neq 0 .$$

Choose $\varepsilon < \min(\gamma, \frac{1}{2}m)$ and let $U = U_\varepsilon^{(x)} \cap U_\varepsilon^{(y)}$. If $U \neq \emptyset$, given $z \in U$, by the monotonicity properties of $\mathscr{R}$ and h, we would have

$$r_{h(\gamma)}(x, y) \leqslant r_{h(\varepsilon)}(x, z) + r_{h(\varepsilon)}(y, z) < 2\varepsilon$$

which contradicts $(*)$.

([11]) $\lambda(x, y) = 0 \Rightarrow x = y$, since $\mathscr{R}$ separates points; moreover $\lambda(x, y) \leqslant \lambda(x, z) + \lambda(y, z)$ since $r_\alpha(x, y)/(1 + r_\alpha(x, y)) \leqslant r_\alpha(x, z)/(1 + r_\alpha(x, z)) + r_\alpha(y, z)/(1 + r_\alpha(y, z))$ (the function $h/(1+h)$ beeing increasing in h). The other properties are obvious.

Proof. Since $\tilde{r}(t; x_1, x_2) < 1$, $\forall t \in]0, 1]$, $\forall x_1, x_2 \in X$, we have that, given $\varepsilon \in]0, 1]$

$$\lambda(f(y), f(y_0)) = \int_0^1 \tilde{r}(t; f(y), f(y_0))\, dt < \varepsilon + \int_\varepsilon^1 \tilde{r}(t; f(y), f(y_0))\, dt.$$

On the other hand, corresponding to the given ε, there exists $\delta > 0$, such that (cfr. (26)′) $d(y, y_0) < \delta \Rightarrow r_{h(\varepsilon)}(f(y), f(y_0))$. Thus, by the monotonicity properties of $\mathscr{R}$ and h, we have

$\lambda(f(y), f(y_0)) < \varepsilon + r_{h(\varepsilon)}(f(y), f(y_0))(1-\varepsilon) < 2\varepsilon$, and the Theorem is proved.

8. - Continuous dependence theorems with respect to a metric of the type λ.

In the light of the results obtained in **1** and of the considerations made in the previous section, we shall now give some continuous dependence theorems with respect to suitable families of quasi-metrics and hence (cfr. previous section) with respect to the associated metric λ. We should remark that the families we shall consider will involve all the first derivatives of the perturbation $\boldsymbol{u}$ and, in some cases, the laplacian of $\boldsymbol{u}$. Thus, the results we shall obtain concern a stronger type of continuous dependence than that considered in earlier works [**9**], [**2**]$_2$.

We shall later use some function spaces which we are going to define. By $C^k(A)$ $(k = 0, 1, \ldots)$ [resp. $C^k(\bar{A})$] we mean the set of vectorial functions in A which are continuous in A [resp. up to the boundary of A] up to the k-th derivative inclusive. Now, retaining the notations of the previous section, by X we shall mean the space of vectorial functions $\boldsymbol{w}$, such that

$$\boldsymbol{w} \in C^0(\bar{\Omega}) \cap C^2(\Omega) \qquad |w_r| \leqslant Mr.$$

Furthermore we choose $\tau = 0$ (12) and set $X(0) = \{z : z_i \in C^1(\Omega) \cap L^\infty(\Omega)\}$, $Y_1 = \{\boldsymbol{h} : h_i \in C^0(\Omega_T) \cap L^\infty(\Omega_T)\}$, $Y_2 = \{\boldsymbol{l} : l_i \in C^1(\partial\Omega \times [0, T])\}$, $X^* = Y_1 \times Y_2$. Now, by $\mathscr{B}(\mathscr{T}, X; X^*)$ we mean the set of functions $\boldsymbol{u}$ from $[0, T]$ into X, parametrized from X^* and such that: (1) $\boldsymbol{u}$ satisfies problem (1) + (2) with $\boldsymbol{u}_0 \in X(0)$, $\boldsymbol{f} \in Y_1$, $\boldsymbol{u} \in Y_2$, for a suitable choice of the pressure $\pi \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap L^{6-\varepsilon}(\Omega_T)$ $(\varepsilon \in [0, 4[)$; (2) $\boldsymbol{u}$ satisfies the identity

(12) This choice is due to the fact that, as will be later clear, the results we shall obtain will be independent of the choice of the initial time.

(3)' for $q = 2$, the identity (3) for some $q > 1$ (with $\nabla\pi \in \cap L^\infty(\Omega_T)$) and the identity (5). In all the above cases g is assumed to be $\exp[-\alpha(t + t_0)^\beta r]$.

Finally, *in the case* $\Omega \equiv R^3$, by $\mathscr{B}^*(\mathscr{T}, X; X^*)$ we shall mean the subset of $\mathscr{B}(\mathscr{T}, X; X^*)$ of functions $\boldsymbol{u}$ which satisfy also the identity (4) ([13]).

Let us now introduce in X the following families of quasi-metrics (parametrized in R), with $\boldsymbol{w}_2 = \boldsymbol{w}_1 + \boldsymbol{w}$

$$(27)\qquad \begin{aligned} r_R^2(\boldsymbol{w}_1, \boldsymbol{w}_2) &= \int_{\Omega_R} \left(\boldsymbol{w}^2 + \nabla\boldsymbol{w}:\nabla\boldsymbol{w} + \left(\frac{\partial \boldsymbol{w}}{\partial s}\right)^2\right) \mathrm{d}\Omega\,, \\ r_R^{*2}(\boldsymbol{w}_1, \boldsymbol{w}_2) &= r_R(\boldsymbol{w}_1, \boldsymbol{w}_2) + \int_{\Omega_R} (\Delta_2 \boldsymbol{w})^2 \,\mathrm{d}\Omega\,. \end{aligned}$$

It can be readily seen that $\{r_R\}_{R>0}$ and $\{r_R^*\}_{R>0}$ are non-decreasing families of quasi-metrics which separate points. Starting with $(27)_1$ resp. $(27)_2$ we now introduce in $\mathscr{B}(\mathscr{T}, X; X^*)$ resp. in $\mathscr{B}^*(\mathscr{T}, X; X^*)$ the following family (with $\boldsymbol{u}_2 = \boldsymbol{u}_1 + \boldsymbol{u}$)

$$(28)_1\qquad \bar{r}_R^2(\boldsymbol{u}_1, \boldsymbol{u}_2) = \sup_{t\in[0,T]} \int_{\Omega_R} (\boldsymbol{u}^2 + \nabla\boldsymbol{u}:\nabla\boldsymbol{u})\,\mathrm{d}\Omega + \int_0^T\int_{\Omega_R} \left(\frac{\partial \boldsymbol{u}}{\partial s}\right)^2 \mathrm{d}\Omega\,\mathrm{d}s\,,$$

$$(28)_2\qquad [\text{resp. } \bar{r}_R^{*2}(\boldsymbol{u}_1, \boldsymbol{u}_2) = \bar{r}_R^2(\boldsymbol{u}_1, \boldsymbol{u}_2) + \int_0^T\int_{\Omega_R} (\Delta_2 \boldsymbol{u})^2 \,\mathrm{d}\Omega\,\mathrm{d}s]\,.$$

As far as the function $h(\varepsilon)$ is concerned, we shall *always* assume $h(\varepsilon) = k\varepsilon^{-n}$ $(k, n > 0)$. Moreover, as regards the metric in $Y = X(0) \times X^*$, we shall choose

$$\varrho(\zeta_1, \zeta_2) = \sup_{\Omega} |\boldsymbol{z}_1 - \boldsymbol{z}_2| + \sup_{\Omega} |\nabla\boldsymbol{z}_1 - \nabla\boldsymbol{z}_2| + \sup_{\Omega_T} |\boldsymbol{h}_1 - \boldsymbol{h}_2| + \sup_{\partial\Omega\times[0,T]} |\boldsymbol{l}_1 - \boldsymbol{l}_2| + \sup_{\partial\Omega\times[0,T]} \left|\frac{\partial \boldsymbol{l}_1}{\partial t} - \frac{\partial \boldsymbol{l}_2}{\partial t}\right|\,.$$

The following theorems hold.

Theorem 6. *Let $|\boldsymbol{v}|$, $|\nabla\boldsymbol{v}| \leqslant M$. Then the null solution of problem* (1)+(2) *dependes Hölder continuously upon the data, when $\mathscr{B}$ is topologized with* $(28)_1$.

Proof. Starting with (19) and choosing $\alpha = 1/R = \delta^p$ $(p \in]0, \frac{2}{3}[)$ the theorem follows when the following inequality is taken into account $(\tilde{k} > 0)$

$$\int_0^T\int_{\Omega} A^2(P, t) g \,\mathrm{d}\Omega\,, \quad \int_{\Omega} A^2(P, t) g\,\mathrm{d}\Omega \leqslant k\alpha^{-3}\delta^2 \quad \text{whenever} \quad \sup|\boldsymbol{A}(P, t)| < \delta\,.$$

([13]) Of course, in this case, $Y_2 = \emptyset$.

Theorem 7. *Assume* $\Omega \equiv R^3$. *Moreover, let* $|\boldsymbol{v}|$, $|\nabla \boldsymbol{v}| \leqslant M$. *Then the null solution of problem* (1) + (2) *depends Hölder continuously upon the data, when* $\mathscr{B}^*$ *is topologized with* $(28)_2$.

Proof. Starting with (22) and noticing that for $\partial\Omega = \emptyset$ (22) is of the same king of (23), the theorem follows.

Remark 8. We note that, standing Remarks 4 and 6, the results stated in the above theorems, may be also obtained if we replace assumption (11) with the hypothesis that $\boldsymbol{u} \in L^\infty(\Omega_T)$.

We end this section, by remarking that a continuous dependence theorem may be also obtained from Theorem 1. Precisely, in the light of Remark 3 concerning the assumption $|\nabla\pi|/|\boldsymbol{u}| \leqslant M$, we get continuous dependence of the Hölder type with respect to the following family of quasi-metrics

$$r_R(\boldsymbol{u}_1, \boldsymbol{u}_2) = \sup_{\Omega_R} |\boldsymbol{u}_1(P, t) - \boldsymbol{u}_2(P, t)| \, .$$

9. - Continuous dependence theorems with respect to metrics of the L^2 type

We shall now give some continuous dependence theorems (in the Hölder sense) with respect to metrics of the L^2 type. More precisely, we shall prove that if only the boundary data are perturbed, then we have continuous dependence with respect to the following metric

$$\sup_{t\in[0,T]} \Big(\int_\Omega \big\{|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2 + |\nabla\boldsymbol{u}_1 - \nabla\boldsymbol{u}_2|^2\big\}\, d\Omega + \int_0^T\!\!\int_\Omega \Big|\frac{\partial \boldsymbol{u}_1}{\partial t} - \frac{\partial \boldsymbol{u}_2}{\partial t}\Big|^2 d\Omega\, dt\Big)^{1/2} . \tag{29}$$

Moreover, if *all the data* are perturbed, we shall prove continuous dependence theorems with respect to the following metric

$$\sup_{t\in[0,T]} \Big(\int_\Omega \frac{|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2}{r^\gamma}\, d\Omega\Big)^{1/2} \qquad (\gamma > 0) . \tag{30}$$

To this end, let X_i $(i = 1, 2, 3)$ be the subset of functions of X, which are, respectively, square summable in Ω together with their first (spatial and time) derivatives, which have bounded radial component $\big(|w_r| \leqslant M\big)$ in Ω_T and which square summable in Ω with weight $r^{-\varepsilon}$ $(\varepsilon > 0)$. Moreover, let

$$Y_1^* = \{\boldsymbol{z}\colon \boldsymbol{z} r^{-\varepsilon/2} \in L^2(\Omega) \text{ and } \boldsymbol{z} \in L^\infty(\Omega)\} ,$$
$$Y_2^* = \{\boldsymbol{h}\colon \boldsymbol{h} \in C_0(\Omega_T) \cap L^\infty(\Omega_T) \text{ and } \boldsymbol{h} r^{-\varepsilon/2} \in L^2(\Omega_T)\} ,$$
$$X_2^* = Y_2 \times Y_2^* .$$

Furthermore, by $\mathscr{B}_2(\mathscr{T}, X_2; X_2^*)$ we mean the set of functions from $[0, T] \to X_2$ parametrized from X_2^* which satisfy problem (1)+(2) with $\boldsymbol{u}_0 \in Y_1(= X_2(0))$, $\boldsymbol{u}_\Sigma \in Y_2$, $\boldsymbol{f} \in Y_2^*$ for some $\pi \in L^{6-\eta}(\Omega_T) \cap C^0(\bar{\Omega}) \cap C^1(\Omega)$ $(\eta \in]0, 4[)$ and the identity (3)′ with $q = 2$ and $g = \exp[-\alpha r]$.

The following theorems hold.

Theorem 8. *Assume* $\boldsymbol{u}_0 = \boldsymbol{f} = 0$ *and* $|\boldsymbol{v}|$, $|\nabla \boldsymbol{v}| \leqslant M$. *Then* $\mathscr{B}(\mathscr{T}, X; X^*)$ *is embedded in* $\mathscr{B}(\mathscr{T}, X_1; X^*)$. *Moreover the null solution of problem* (1)+(2) *depends Hölder continuously when* $\mathscr{B}(\mathscr{T}, X_1; X^*)$ *is topologized with the metric* (29).

Proof. From (19) we get (for $\alpha > 0$ and $t \in [0, T]$)

$$\int_{\Omega_R} (u^2 + \nabla \boldsymbol{u} : \nabla \boldsymbol{u})\, d\Omega \int_0^T \int_{\Omega_R} \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)^2 d\Omega\, dt \leqslant c \exp[\alpha(T + t_0)^\beta R] \cdot \Big\{\alpha^{\tilde{\eta}/2} \int_0^T \int_\Omega |\pi|^{6-\varepsilon} d\Omega\, dt$$

$$+ \int_0^T \int_{\partial\Omega} \left(|\boldsymbol{u}_\Sigma| + \left|\frac{\partial \boldsymbol{u}_\Sigma}{\partial t}\right|\right) d\sigma\, dt + \alpha^\sigma \Big\}\,.$$

Thus, the theorem follows by letting first $\alpha \to 0$ and then $R \to \infty$.

Theorem 9. *Assume* $|v_r|$, $|(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^T| \leqslant M$. *Then* $\mathscr{B}_2(\mathscr{T}, X_2; X_2^*)$ *is embedded in* $\mathscr{B}_2(\mathscr{T}, X_3; X_2^*)$. *Moreover, the null solution of the problem* (1)+(2) *depends Hölder continuously upon the data, when* $\mathscr{B}_2$ *is topologized with the metric* (30) *with* $\gamma = \varepsilon + \beta$, $(\beta > 0)$.

Proof. The first part of the theorem follows from the results of section **6**. So far as the last statement of the theorem is concerned, in [**2**]$_2$ we have shown, on the given assumptions, that

$$\text{(31)} \qquad \sup\{|\boldsymbol{u}_0|, |\boldsymbol{f}|, |\boldsymbol{u}_\Sigma|\} < \delta \Rightarrow \sup_{t \in [0,T]} \int_{\Omega_{\bar{R}}} u^2(P, t)\, d\Omega < k\delta^p \qquad (k, p > 0)\,,$$

where $\bar{R} = \delta^{-s}$ $(s > 0)$. Now, since $B(1) \subseteq \Omega_0$ (cfr. footnote (6)), we have

$$\text{(32)} \qquad \int_\Omega \frac{u^2}{r^{\varepsilon+\eta}}\, d\Omega \leqslant \int_{\Omega_{\bar{R}}} u^2\, d\Omega + \int_{\Omega - \Omega_{\bar{R}}} \frac{u^2}{r^{\varepsilon+\eta}}\, d\Omega\,.$$

On the other hand

$$\text{(33)} \qquad \frac{1}{r} \leqslant \delta^s \qquad \forall r \geqslant \bar{R}\,,$$

and by Theorem 4, we get

$$\sup_{t\in[0,T]} \int_{\Omega} \frac{u^2(P,t)}{r^{\varepsilon}} \, d\Omega = N < \infty . \tag{34}$$

From (31)-(34) we thus obtain $\int_{\Omega} (u^2/r^{\varepsilon+\eta})\, d\Omega \leqslant k\delta^p + N\delta^s\eta$, which proves the theorem.

3. - Stability

10. - A key lemma to stability

In several previous papers [**2**]$_4$, [**1**]$_2$, [7] we have given some sufficient conditions to the stability of fluid motions in exterior domains and for weak solutions, which we have shown to exist and to belong to suitable Sobolev spaces. In order to formulate the above conditions, a foundamental role has been played by the inequality (35) below, which should be considered as the inequality corresponding to that of Poincaré, in unbounded domains

$$\int_{\Omega} \frac{u^2}{r^2} \, d\Omega \leqslant 4 \int_{\Omega} \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \, d\Omega . \tag{35}$$

Of course such an inequality holds for functions $\boldsymbol{u}$ such that the right hand side makes sense. However, in the cases we are treating along this paper, the perturbations $\boldsymbol{u}$ need not satisfy *a priori* such a requirement since they are only allowed to « grow » at large spatial distances. Thus, in order to give stability theorems for this class of perturbations, it seems quite natural to seek for an inequality of the type (35) but in *weighted spaces*. As usual, the weight we shall use will be of the type $g = \exp(-\alpha r)$. Therefore, we shall look for an inequality of the kind

$$\int_{\Omega} g \frac{u^2}{r^2} \, d\Omega \leqslant k \int_{\Omega} g \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \, d\Omega + \alpha \int_{\Omega} g u^2 \, d\Omega + \text{boundary terms} . \tag{36}$$

Now, since we want to obtain again stability in the L^2 norm (as we did in [**2**]$_4$, [**1**]$_2$, [7]) it will be necessary to let $\alpha \to 0$. As a consequence, it is fundamental that the constant k which appears in (36) be a bounded function of α in the neighborhood of zero (of course, so much the better if k is independent of α). More precisely we shall prove the following lemma.

Lemma. *Let* $\boldsymbol{u} \in C_0(\bar{\Omega})^3 \cap C^1(\Omega)$, $|\boldsymbol{u}| \leqslant Mr^k$ $(M, k > 0)$ *and let* $\boldsymbol{u}$, $\nabla\boldsymbol{u}$ *be square summable with weight* $g = \exp(-\alpha r)$. *Then the following inequality holds*

$$\int_\Omega g\frac{u^2}{r^2}\,\mathrm{d}\Omega \leqslant 4\int_\Omega g\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\,\mathrm{d}\Omega + \frac{\alpha}{2}\int_\Omega gu^2 + k\sup_{\partial\Omega}|\boldsymbol{u}|^2 \tag{37}$$

where k *does not depend on* $\boldsymbol{u}$.

Proof. Let $r = f(\theta, \varphi)$ be the (local) equation of the closed surface $\partial\Omega$. We then have ($\int \mathrm{d}\gamma = \int_0^\pi\int_0^{2\pi} \sin\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi$) [14]

$$\int_{\Omega_R} g\frac{u^2}{r^2}\,\mathrm{d}\Omega = \int \mathrm{d}\gamma\int_{r=f}^R \exp(-\alpha r)u^2\,\mathrm{d}r = \int \mathrm{d}\gamma\int_{r=f}^R \frac{\partial}{\partial r}[\exp(-\alpha r)u^2 r]\,\mathrm{d}r$$

$$+\alpha\int \mathrm{d}\gamma\int_f^R \exp(-\alpha r)u^2 r\,\mathrm{d}r - 2\int \mathrm{d}\gamma\int_f^R \boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial r}r\,\mathrm{d}r$$

$$\leqslant \exp(-\alpha R)R\int u^2(R,\gamma)\,\mathrm{d}\gamma + k\sup_{\partial\Omega}|\boldsymbol{u}_\Sigma|^2 + \alpha\int_{\Omega_R} gu^2\,\mathrm{d}\Omega$$

$$+\frac{2}{\xi}\int_{\Omega_R} g\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\,\mathrm{d}\Omega + \frac{\xi}{2}\int_{\Omega_R} g\frac{u^2}{r^2}\,\mathrm{d}\Omega\,.$$

Thus, we obtain (on choosing $\xi < 2$)

$$\int_\Omega g\frac{u^2}{r^2}\,\mathrm{d}\Omega \leqslant \frac{4}{\xi(2-\xi)}\int_\Omega g\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\,\mathrm{d}\Omega + \frac{2-\xi}{2}\alpha\int_\Omega gu^2\,\mathrm{d}\Omega + K\frac{(2-\xi)}{2}\sup_{\partial\Omega}|\boldsymbol{u}_\Sigma|^2\,.$$

Now, inequality (37) follows, by minimizing the function $4/\xi(2-\xi)$ over $\xi \in\,]0, 2[$.

11. - A stability theorem in the L^2 norm for perturbations which may « grow » at large spatial distances

Let us first introduce the following function spaces.
$Z_1 = L^2(\Omega)$, $Z_2 = \{l\colon l \in C^0(\partial\Omega\times R^+)^3 \cap L^\infty(\partial\Omega\times R^+) \cap L^\eta(R^+; L^\infty(\partial\Omega)), \eta < 1\}$, X^4 the subset of X_2 such that $|w_r| \leqslant M$ and $|w| \leqslant Mr^k$ $(M, k > 0)$. Further-

(14) Of course, one should think of integrating along the charts defined by the equation defining $\partial\Omega$ in spherical coordinates and then to add the various contributions.

more by $\widetilde{\mathscr{B}}(R^+, X_4; Z_2)$ we mean the set of solutions of the problem (1)+(2) from R^+ into X_4, for some $\pi \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap L^\gamma(\Omega_T)$ $(\forall T > 0,\ \gamma < 6/(1+2\varepsilon),\ \varepsilon \in]0, 1[)$, with $\boldsymbol{u}_0 \in Z_1(= X_4(0))$, $\boldsymbol{f} = 0$, $\boldsymbol{u}_\Sigma \in Z_2$ and satisfying the identity (3)′ with $q = 2$ and $g = \exp(-\alpha r)$. Moreover, we equip Z_1 with its natural metric and Z_2 with the supremum metric.

Indicating by $D = \sup\limits_{\Omega_T} |(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^T|$ and by d a comparison length, we shall now prove the following stability theorem.

Theorem 10. *Let* $\Omega_0 \neq \emptyset$, $|v_r| \leqslant M$, $|(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^T| \leqslant Dd^2/r^2$. *Then, if the Reynolds number* $\mathrm{Re} = Dd^2/\nu$ *associated to the unperturbed motion* $\{\boldsymbol{v}, p\}$ *is such that* $\mathrm{Re} < \frac{1}{4}$ *it turns out that* $\widetilde{\mathscr{B}}(R^+, X_4; Z_2)$ *is embedded in* $\widetilde{\mathscr{B}}(R^+, L^2(\Omega); Z_2)$. *Moreover, the null solution is stable when* $\widetilde{\mathscr{B}}(R^+, L^2(\Omega); Z_2)$ *is topologized with* $\sup\limits_{t \in R^+} \int\limits_\Omega u^2 \, \mathrm{d}\Omega$.

Proof. From (3)′ with $q = 2$, it easily follows that

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leqslant \int\limits_\Omega \Big\{\frac{u^2}{2}(\boldsymbol{u} + \boldsymbol{v}) \cdot \nabla g - g[\boldsymbol{u} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{u} + \nabla \boldsymbol{u} : \nabla \boldsymbol{u}] - \nabla g \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u}\Big\} \, \mathrm{d}\Omega + \int\limits_\Omega \pi \nabla g \cdot \boldsymbol{u} \, \mathrm{d}\Omega + N \int\limits_{\partial\Omega} |\boldsymbol{u}_\Sigma| \, \mathrm{d}\sigma \quad (N > 0)\,. \tag{38}$$

Thus from (38) and from the previous Lemma we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leqslant \Big(2M\alpha + \frac{\alpha^2}{\xi} + \alpha\Big) E + \Big(4D\,\mathrm{d}^2 - \nu + \frac{\xi}{2}\Big) \int\limits_\Omega g \nabla \boldsymbol{u} : \nabla \boldsymbol{u} \, \mathrm{d}\Omega + \alpha \int\limits_\Omega \pi g |\boldsymbol{u}| \, \mathrm{d}\Omega + N' \sup\limits_{\partial\Omega} |\boldsymbol{u}_\Sigma(P, t)| \qquad (N' > 0)\,. \tag{39}$$

Now $\alpha \int\limits_\Omega \pi g |\boldsymbol{u}| \, \mathrm{d}\Omega \leqslant \alpha^{2\varepsilon} E + c_1 \alpha^{\sigma_1} + c_2 \alpha^{\sigma_2} \int\limits_\Omega |\pi|^\gamma \, \mathrm{d}\Omega$ $(c_i, \sigma_i > 0;\ i = 1, 2)$, and then from (39) and by assumption, for ξ sufficiently small, putting $N(\alpha) = (2M\alpha + \alpha^2/\xi + \alpha^{2\varepsilon} + \alpha)$, we obtain (with $C = \max(c_1, c_2, N')$)

$$E(t) \leqslant E(0) \exp\big(N(\alpha) t\big) + c \exp\big(N(\alpha) t\big) \int\limits_0^t \Big\{\alpha^{\sigma_1} + \alpha^{\sigma_2} \int\limits_\Omega |\pi^\gamma| \, \mathrm{d}\Omega + \sup |\boldsymbol{u}_\Sigma(P, t)|\Big\} \, \mathrm{d}t\,. \tag{40}$$

As a consequence, since $s \equiv \sup\limits_{\partial\Omega} |\boldsymbol{u}_\Sigma(P, t)| = s^\eta s^{1-\eta} \leqslant s^\eta \delta^{(1-\eta)}$, the inequality (40) in the limit $\alpha \to 0$ gives the desired result.

Remark 9. Assume that $\boldsymbol{u}$ is a solution satisfying (3)′ with $q = 2$ with the replacement (cfr. Remark 7)

$$(41) \qquad g\boldsymbol{u}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{u} \to - g\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v} - \boldsymbol{u}\cdot\nabla g(\boldsymbol{u}\cdot\boldsymbol{v}) \ (\text{in } \Omega)\,, \ + g\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{v}) \ (\text{in } \partial\Omega)\,.$$

Then the assumptions $|(\nabla\boldsymbol{v}) + (\nabla\boldsymbol{v})^T| \leqslant Dd^2/r^2$ and $\mathrm{Re} < \frac{1}{4}$, may be replaced with $|\boldsymbol{v}| \leqslant Vd/r$ and $Vd/\nu < \frac{1}{2}$, where $V = \sup_{\Omega_T}|\boldsymbol{v}|$. In fact, all goes as in the previous proof, provided that the « new » terms arising from $(41)_1$ are increased in the following way

$$- g\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v} \leqslant g\,\frac{V^2\,\mathrm{d}^2}{2}\,\frac{u^2}{r^2} + \frac{1}{2}\,g\nabla\boldsymbol{u}\,{:}\,\nabla\boldsymbol{u}\,, \qquad - \boldsymbol{u}\cdot\nabla g(\boldsymbol{u}\cdot\boldsymbol{v}) \leqslant V\alpha g u^2$$

and then increased again by the inequality proved in the Lemma.

Remark 10. In the light of Theorem 1, the assumption $|u_r| \leqslant M$ may be replaced by $\boldsymbol{u}_0 \in L^\infty(\Omega)$, $\nabla\pi \in L^\infty(\Omega_T)$ for any $T > 0$.

Remark 11. From the estimate (40) an interesting continuous dependence theorem with respect to a family of quasi-metrics can be also obtained, in which the measure T of the interval $[0, T]$ where continuous dependence holds, can be polynomially related to the measure δ of the « smallness » of the initial and boundary data. Actually, assuming $\sup_{R^+}\int_{\Omega}|\pi(P, t)|^\gamma\,\mathrm{d}\Omega < \infty$ and $\sup_{\Omega}|\boldsymbol{u}_0| < \delta$, from (40) we get $\forall R \geqslant R_0$

$$(42) \qquad \int_{\Omega_R} u^2\,\mathrm{d}\Omega \leqslant k\,\Big\{\Big(\frac{\delta^2}{\alpha^3} + \alpha^\sigma T\Big)\exp\big(\alpha(T + R)\big) + \int_0^\infty s(t)\,\mathrm{d}t\Big\}\,,$$

where K and σ are suitable positive constants *independent of* T and R. Thus, choosing $\alpha = \delta^s$ $(s < \frac{2}{3})$, from (42) we have for some $p > 0$ that

$$\int_{\Omega_R} u^2(P, t)\,\mathrm{d}\Omega < \delta^p \qquad \forall R \leqslant \delta^{-n}\,, \quad \forall t \leqslant \delta^{-m} \qquad\qquad (n < s,\ m < s\sigma)\,,$$

which is the claimed result.

Remark 12. We would finally notice that the condition $\mathrm{Re} < \frac{1}{4}$ is also sufficient for stability with respect to perturbations which *a priori* belong to $L^2(\Omega)$ together with their first (spatial) derivatives (cfr. $[\mathbf{2}]_4$, $[\mathbf{1}]_2$, $[\mathbf{7}]$). It is then worth remarking that such assumption on perturbations is not made here.

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Summary

See the Introduction.

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