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Nonhomogeneous boundary conditions in evolution problems (**)

A Giorgio Sestini per il suo 70° compleanno

1. - Introduction

The theory of the semigroups of linear operators is an useful tool in the study of evolution problems in Banach spaces. The operator that is the generator of a semigroup is linear and hence it has a domain that is a linear manifold in a complete normed space. Since the boundary conditions of the problem characterize the domain, it is clear that an operator is linear if not only it is formally linear, but if the boundary conditions are such that they are satisfied by linear combinations of elements that satisfy them. It is the case that certain boundary conditions can make nonlinear an operator that is formally linear. This remark makes clear how it is useful to transform eventual boundary conditions in source terms in order to be able to deal with evolution problems in the frame of the linear semigroup theory.

In this work we show how the above ideas may be adapted to a neutron transport problem. Indeed, the most of the papers published on this topic deal with problems with homogeneous boundary conditions, that is, either it is assumed no neutron density entering the system, or it is assumed a perfectly reflecting boundary condition. These are linear boundary conditions.

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Case and Zweifel $[\mathbf{3}]_{1,2}$, considered nonhomogeneous boundary conditions. They gave the solution of an integral formulation of the integrodifferential problem by using the Neumann series; but they did not show that the series is a solution of the integrodifferential problem.

Bowden, Williams and others [**2**], [**7**], dealt with similar problems by using the Case normal mode expansion method.

The semigroup theory is used by Hintz [**4**], and by Mika and Stankiewicz [**6**]. They used the well known properties of the semigroup generated by the linear streaming operator in order to show that a function, whose physical meaning is clear, is the solution of the nonhomogeneous transport problem.

In the sequel we give another proof of existence and uniqueness of the solution of a transport problem by using semigroup techniques. Our proof differs from those of the quoted authors, because it does not require the introduction of functions that are formally suggested by physical meaning, but it requires only the introduction of functions belonging to a wide class having several properties. Finally, we give some examples, the last of them having a clear physical meaning.

2. - The mathematical model of a transport problem

We consider a slab of thickness $2a$, in which monoenergetic neutrons diffuse. We assume that the macroscopic cross section Σ and the multiplication factor γ are constant. Scattering and fission are assumed to be isotropic.

The integrodifferential equation that describes the evolution of the system is the following

$$\frac{\partial N(x, y, t)}{\partial t} = -yv\frac{\partial N(x, y, t)}{\partial x} - v\Sigma N(x, y, t) + v\frac{\gamma}{2}\int_{-1}^{1} N(x, y', t)\,\mathrm{d}y'; \tag{1}$$

$N(x, y, t)$ is the neutron density at time t, in the point of abscissa x, $-a \leqslant x \leqslant a$, with velocity making an angle with the x axis whose cosine equals y. The modulus of the velocity is v, a constant.

The neutron distribution at $t = 0$ is a known function of $x \in [-a, a]$ and $y \in [-1, 1]$

$$N(x, y, 0) = N_0(x, y)\,. \tag{2}$$

At any time t, the neutron distribution of particles entering the slab through the boundary planes are known function of y and t

$$N(-a, y, t) = f(y, t)\,, \qquad y \in (0, 1]\,, \tag{3}$$

$$N(a, y, t) = g(y, t)\,, \qquad y \in [-1, 0)\,. \tag{4}$$

Moreover we assume

$$f(y, t) = g(y, t) = 0\,, \qquad t < 0\,, \tag{5}$$

that is, no particle enters the system before a certain time that we may assume to be $t = 0$.

Though eq. (1) is linear with respect to the unknown function N, nevertheless problem (1)-(4) is nonlinear owing to (3) and (4). Indeed, if N and N_1 are solutions of (1), (3), and (4), then their sum $N + N_1$ satisfies (1), but does not satisfies (3) and (4). We recall that the initial condition has no role in the definition of the operator generating the semigroup.

We may introduce a corresponding *linear* nonhomogeneous problem in the following way. Let $V = V(x, y, t)$ be a known function satisfying the boundary conditions (3) and (4); let $w = w(x, y, t)$ be a solution of (1)-(4). Obviously we assume that V is endowed with properties such that we can perform all the operations needed in the sequel.

We put $u = w - V$, and we get

$$\frac{\partial u}{\partial t} = -yv\frac{\partial u}{\partial x} - v\Sigma u + v\frac{\gamma}{2}\int_{-1}^{1} u(x, y', t)\,dy' + \Big\{-\frac{\partial V}{\partial t} - yv\frac{\partial V}{\partial x} - v\Sigma V + v\frac{\gamma}{2}\int_{-1}^{1} V(x, y', t)\,dy'\Big\}\,, \tag{6}$$

$$u(x, y, 0) = N_0(x, y) - V(x, y, 0)\,, \tag{7}$$

$$u(-a, y, t) = 0\,, \qquad y \in (0, 1]\,, \tag{8}$$

$$u(a, y, t) = 0\,, \qquad y \in [-1, 0)\,; \tag{9}$$

since V is a known function, the problem (1)-(4) becomes the problem (6)-(9); now, eq. (6) is a linear nonhomogeneous equation with respect to the unknown function $u = u(x, y, t)$, and with a source term

$$s\,(x, y, t) = -\frac{\partial V}{\partial t} - yv\frac{\partial V}{\partial x} - v\Sigma V + v\frac{\gamma}{2}\int_{-1}^{1} V(x, y', t)\,dy'. \tag{10}$$

Moreover the boundary conditions are zero for $t \geq 0$, that is the time interval in which we look for a solution.

Now we formulate the problem in a suitable space of functions.

We put $T = [0, +\infty)$, and $D = [-a, a] \times [-1, 1]$; let $E = L^p(D)$ be the complex space of the classes of equivalence of measurable functions that have summable p-th powers, and let $F = C(T, E)$ be the space of the continuous functions with domain T and range in E.

We define a linear operator J with domain $D(J) = E$ and range $R(J) \subset E$, $(Ju)(x, y) = \frac{1}{2} \int_{-1}^{1} u(x, y')\, dy'$; J is a bounded operator with norm $\|J\| \leq 1$.

We define a linear operator A with $D(A) \subset E$, and $R(A) \subset E$, where

$$D(A) = \left\{u \in E : y \frac{\partial u}{\partial x} \in E;\ u(-a, y) = 0, \text{ for a.e. } y \in (0, 1];\right. \tag{11}$$

$$\left. u(a, y) = 0, \text{ for a.e. } y \in [-1, 0)\right\},$$

and such that

$$(Au)(x, y) = -yv \frac{\partial u}{\partial x}(x, y) - v\Sigma u(x, y) + v\gamma (Ju)(x, y); \tag{12}$$

in (11) and (12), the derivative is a generalized derivative, see [**8**]; this amounts to say that u is an absolutely continuous function with respect to $x \in [-a, a]$ for a.e. $y \in [-1, 1]$.

A is the generator of a strongly continuous semigroup of bounded linear operators $Z(t)$, $t \geq 0$, see [**11**], [**10**], [**9**].

We define a function $s \in F$, by means of a function $V \in F$ (endowed with suitable properties) by the formula

$$s(t) = -\frac{dV(t)}{dt} - yv \frac{\partial V(t)}{\partial x} - v\Sigma V(t) + v\gamma J V(t) . \tag{13}$$

In (13), the symbols of the derivatives have the following meanings (a) $dV(t)/dt$ is an element of E such that $\lim_{h \to 0} \|h^{-1}[V(t + h) - V(t)] - dV(t)/dt\|_E = 0$; (b) $\partial V(t)/\partial x$ is, at each $t \in T$, the generalized derivative of $V(t) \in E$.

Finally, we rewrite problem (6)-(9) in the abstract form

$$\frac{du(t)}{dt} = Au(t) + s(t) , \tag{14}$$

$$\lim_{t \to 0^+} u(t) = u_0 \in D(A) . \tag{15}$$

By a solution of the abstract Cauchy problem (14), (15), we mean a function $u \in F$, that has a strongly continuous derivative for any $t > 0$ (in the sense recalled above, sub (a)), and such that (14) is satisfied for any $t > 0$, and that $\lim_{t \to 0^+} \|u(t) - u_0\|_E = 0$.

In order that s be well defined by (13), V needs to satisfy the following hypothesis

(i) $V(t) \in E$, $\forall t \in T$.

(ii) $V: t \to V(t)$ is strongly differentiable, $\forall t \in T$.

(iii) $V(t) = V(x, y; t)$ has a generalized derivative $\partial V(t)/\partial x \in E$, $\forall t \in T$.

Remark. If $V(t) \in E$, then $JV(t) \in E$.

Remark 2. If $\partial V/\partial x \in E$, then $y(\partial V/\partial x) \in E$.

The preceding hypothesis make sure that the function s is defined for $t \in T$. In order that s be continuous with respect to t, we assume

(iv) $t \to \mathrm{d}V(t)/\mathrm{d}t$ is a continuous function for $t \in T$; in other words, $V \in C^1(T, E)$.

(v) $t \to y(\partial V(t)/\partial x)$ is a continuous function for $t \in T$.

Remark 3. If $V \in C^1(T, E)$, then $JV \in C^1(T, E)$.

Remark 4. Because of (iii), $V(x, y; t)$ is absolutely continuous with respect to $x \in [-a, a]$, for a.e. $y \in [-1, 1]$, $\forall t \in T$; therefore the following limits exist

(16) $$\lim_{x \to -a^+} V(x, y; t) = V(-a, y; t), \qquad \text{for a.e. } y \in (0, 1],$$

(17) $$\lim_{x \to a^-} V(x, y; t) = V(a, y; t), \qquad \text{for a.e. } y \in [-1, 0).$$

We remark that the initial datum is the difference between the initial particle distribution N_0 and the value $V(0)$.

It is well known that the Cauchy problem (14), (15) has one and only one solution if the linear operator A is the generator of a strongly continuous semigroup of bounded operators, and if the function $s \in C^1(T, E)$ (we disregard other sufficient conditions), see [**1**], [**5**]. Obviously, we need other hypothesis

(vi) $V \in C^2(T, E)$, (vii) $y(\partial V/\partial x) \in C^1(T, E)$.

Remark 5. The assumptions (i)-(vii) are not independent; for instance, (vi) implies several of the preceding ones.

Now we can state the following

Proposition. Under the hypothesis (i)-(vii), the abstract Cauchy problem (14), (15), has one and only one strongly continuously differentiable solution u, given by $u(t) = Z(t)\,u_0 + \int_0^t Z(t-t')\,s(t')\,dt'$, and it is such that $u(t) \in L^p(D)$, $\forall t \in T$. Moreover, the function $t \to w(t) = u(t) + V(t)$ has a strongly continuous derivative, and it is such that the following limits exist for any $t \in T$: $\lim_{x \to -a^+} w(x, y; t) = V(-a, y; t)$, for a.e. $y \in (0, 1]$, $\lim_{x \to a^-} w(x, y; t) = V(a, y; t)$, for a.e. $y \in [-1, 0)$.

Thus the following definition seems suitable.

Definition. Let $f(t) = f(y; t) \in L^p((0, 1])$, $\forall t \in T$, and $g(t) = g(y; t) \in L^p([-1, 0))$, $\forall t \in T$; let V be chosen as in the Proposition and such that $V(-a, y; t) = f(y; t)$, and that $V(a, y; t) = g(y; t)$. Then we define w to be the abstract solution of the problem (1)-(4).

Indeed, w has a strongly continuous derivative with respect to $t \in T$, it satisfies $dw/dt = -yv(\partial w/\partial x) - v\Sigma w + v\gamma J w$, and the boundary conditions: $w(-a, y; t) = f(y; t)$ for a.e. $y \in (0, 1]$, $\forall t \in T$, $w(a, y; t) = g(y; t)$ for a.e. $y \in [-1, 0)$, $\forall t \in T$, and the initial datum $w(x, y; 0) = u_0(x, y) + V(x, y; 0) = N_0(x, y)$.

3. - Some examples

Let us assume that the boundary conditions may be written as the product of two functions of y and t respectively

$$f(y; t) = \alpha(y)\,P(t), \qquad y \in (0, 1],\ t \in T, \tag{18}$$

$$g(y; t) = \beta(y)\,Q(t), \qquad y \in [-1, 0),\ t \in T. \tag{19}$$

If $\alpha \in L^p((0,1])$, $\beta \in L^p([-1,0))$, and $P, Q \in C^2(T, \boldsymbol{C})$, then V, defined by

$$V(x, y; t) = \alpha(y) P(t), \qquad x \in [-a, a],\ y \in (0, 1],\ t \in T, \tag{20}$$

$$V(x, y; t) = \beta(y) Q(t), \qquad x \in [-a, a],\ y \in [-1, 0),\ t \in T, \tag{21}$$

satisfies assumptions (i)-(vii)—that are sufficient conditions in order to make sure the existence and the uniqueness of the solution of the problem (14), (15)—as it is easy to show.

Then,

$$s(x, y; t) = -\alpha(y) P'(t) - v\Sigma\alpha(y) P(t) + \tfrac{1}{2} v\gamma \cdot \left\{P(t) \int_0^1 \alpha(y')\, dy' + Q(t) \int_{-1}^0 \beta(y')\, dy'\right\}, \qquad y \in (0, 1], \tag{22}$$

$$s(x, y; t) = -\beta(y) Q'(t) - v\Sigma\beta(y) Q(t) + \tfrac{1}{2} v\gamma \cdot \left\{P(t) \int_0^1 \alpha(y')\, dy' + Q(t) \int_{-1}^0 \beta(y')\, dy'\right\}, \qquad y \in [-1, 0). \tag{23}$$

We may also define a function V, by

$$V(x, y; t) = \bar{\alpha}(x, y) P(t), \qquad x \in [-a, a],\ y \in (0, 1],\ t \in T, \tag{24}$$

$$V(x, y; t) = \bar{\beta}(x, y) Q(t), \qquad x \in [-a, a],\ y \in [-1, 0),\ t \in T, \tag{25}$$

provided that $\bar{\alpha} \in L^p([-a, a] \times (0, 1])$ and $\bar{\beta} \in L^p([-a, a] \times [-1, 0))$, that $P, Q \in C^2(T, \boldsymbol{C})$, that $\bar{\alpha}$ and $\bar{\beta}$ have generalized derivatives with respect to x such that $\partial\bar{\alpha}/\partial x \in L^p([-a, a] \times (0, 1])$ and $\partial\bar{\beta}/\partial x \in L^p([-a, a] \times [-1, 0))$, and $\bar{\alpha}(-a, y) = \alpha(y)$ for a.e. $y \in (0, 1]$ and $\bar{\beta}(a, y) = \beta(y)$ for a.e. $y \in [-1, 0)$.

As a third example, we define formally V by

$$V(x, y; t) = f\left(y, t - \frac{x + a}{yv}\right) \exp\left[-\Sigma(x + a)/y\right], \qquad x \in [-a, a],\quad y \in (0, 1], \quad t \in \boldsymbol{R}, \tag{26}$$

$$V(x, y; t) = g\left(y, t - \frac{x - a}{yv}\right) \exp\left[-\Sigma(x - a)/y\right], \qquad x \in [-a, a],\quad y \in [-1, 0), \quad t \in \boldsymbol{R}, \tag{27}$$

and we put f and g equal to zero if the second independent variable is negative.

V formally satisfies the boundary conditions and the equation

$$\frac{\partial V}{\partial t} + yv\frac{\partial V}{\partial x} + v\Sigma V = 0 . \tag{28}$$

Hence, it gives the following source term

$$s(x, y; t) = v\gamma J V(x, y; t) . \tag{29}$$

Remark 6. V describes the distribution inside the slab of the particles entering through the boundary planes, i.e., the distribution of the particles having not yet undergone any collisions.

Now we look for sufficient conditions on f and g in order that $s = v\gamma JV \in C^1(T, E)$. Sufficient conditions are those giving a $V \in C^1(T, E)$—so that also $JV \in C^1(T, E)$—and such that V is a solution belonging to $C(T, E)$ of the abstract equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} + yv\frac{\partial V}{\partial x} + v\Sigma V = 0 . \tag{30}$$

Since $V \in C^1(T, E)$ implies $\mathrm{d}V/\mathrm{d}t \in C(T, E)$, we need only sufficient conditions in order that the function $t \in T \to y(\partial V/\partial x)$ belongs to $C(T, E)$.

To meet the case, we make the following assumptions:

(i) $f\colon (\nu, \tau) \to f(\nu, \tau)$ is defined for $(\nu, \tau) \in (0, 1]\times \boldsymbol{R}$, $f(\nu, \tau) = 0$, if $\tau < 0$; $g\colon (\nu, \tau) \to g(\nu, \tau)$ is defined for $(\nu, \tau) \in [-1, 0)\times \boldsymbol{R}$, $g(\nu, \tau) = 0$, if $\tau < 0$.

(ii)
$$\int_0^1 \mathrm{d}\nu \int_0^t |f(\nu, \tau)|^p \mathrm{d}\tau < +\infty , \qquad \forall t \in \boldsymbol{R};$$
$$\int_{-1}^0 \mathrm{d}\nu \int_0^t |g(\nu, \tau)|^p \mathrm{d}\tau < +\infty , \qquad \forall t \in \boldsymbol{R}.$$

(iii) $\forall t > 0$, f has a generalized derivative $\partial f(\nu, \tau)/\partial \tau$ in the set $(0, 1]\times[0, t]$, belonging to L^p, i.e., such that

$$\int_{-1}^1 \mathrm{d}\nu \int_0^t \left|\frac{\partial f(\nu, \tau)}{\partial \tau}\right|^p \mathrm{d}\tau < +\infty;$$

$\forall t > 0$, g has a generalized derivative $\partial g(\nu, \tau)/\partial \tau$ in the set $[-1, 0)\times[0, t]$, belonging to L^p, i.e., such that

$$\int_{-1}^0 \mathrm{d}\nu \int_0^t \left|\frac{\partial g(\nu, \tau)}{\partial \tau}\right|^p \mathrm{d}\tau < +\infty .$$

Assumptions (i), (ii) make sure that V, defined by (26), and (27), belongs to E, $\forall t \in \boldsymbol{R}$. Indeed, we have

$$v \int_0^1 \mathrm{d}\nu \int_0^t |f(\nu, \tau)|^p \mathrm{d}\tau \geqslant \int_0^1 \mathrm{d}\nu \int_0^t |f(\nu, \tau)|^p \nu v \,\mathrm{d}\tau \geqslant \int_0^1 \mathrm{d}\nu \int_{t-2a/\nu v}^t |f(\nu, \tau)|^p \nu v \,\mathrm{d}\tau$$

$$\geqslant \int_{-a}^a \mathrm{d}x \int_0^1 |f(y, t - \frac{x+a}{yv})|^p \mathrm{d}y \geqslant \int_{-a}^a \mathrm{d}x \int_0^1 |f(y, t - \frac{x+a}{yv}) \exp[-\Sigma(x+a)/y-]|^p \mathrm{d}y ;$$

a similar result holds for the function g; thus we obtain by summing,

$$\int_{-a}^a \mathrm{d}x \int_{-1}^1 |V(x, y; t)|^p \mathrm{d}y \leqslant v \int_0^1 \mathrm{d}\nu \int_0^t |f(\nu, \tau)|^p \mathrm{d}\tau + v \int_{-1}^0 \mathrm{d}\nu \int_0^t |g(\nu, \tau)|^p \mathrm{d}\tau < +\infty .$$

With slight modifications to the preceding calculations, we get also $\| V(x, y; t+h) - V(x, y; t)\| \to 0$ if $h \to 0$; then $V \in C(T, E)$.

Let us consider the function $W\colon t \to W(t)$ defined by

$$W(x, y; t) = \frac{\partial f}{\partial t}(y, t - \frac{x+a}{yv}) \exp[-\Sigma(x+a)/y] , \tag{31}$$

$$x \in [-a, a],\; y \in (0, 1],\; t \in \boldsymbol{R} ,$$

$$W(x, y; t) = \frac{\partial g}{\partial t}(y, t - \frac{x-a}{yv}) \exp[-\Sigma(x-a)/y] , \tag{32}$$

$$x \in [-a, a],\; y \in [-1, 0),\; t \in \boldsymbol{R} .$$

The derivatives are generalized derivatives. Just as we did above for V, we have now $W \in C(T, E)$, owing to assumption (iii). Moreover, it can be shown that $W(t)$ is the strong derivative of $V(t)$. Thus, we have $V \in C^1(T, E)$.

Finally, we remark that the generalized derivative $\partial V/\partial x$ exists, and that the function $t \to y(\partial V/\partial x)$ is well defined and it is $y(\partial V/\partial x) = -(1/v)W - \Sigma V$, and therefore we have $y(\partial V/\partial x) \in C(T, E)$. Then it is verified that V is a solution of (30).

Remark 7. In the last example, assumptions (i)-(v) are sufficient in order that the source term $s \in C^1(E, T)$. This is a consequence of the definition of V by (26), and (27).

4. - Final comments

In this work, we give an example of a method suitable to associate a linear nonhomogeneous differential equation in a Banach space to a problem of evolution which is formally linear, but it has nonhomogeneous boundary conditions. These are transformed in a source term. So doing, we get a linear (additive and homogeneous) operator A, which is the generator of a strongly continuous semigroup of bounded linear operators $Z(t)$, $t \geqslant 0$; we are then able to state that there exists one and only one solution of an abstract Cauchy problem. Special attention is devoted to describe a set of properties that a suitable function V must have in order to obtain a strongly continuoulsy differentiable source term.

The method here described has a wide range of applicability. We limit ourselves to deal with a problem of transport in a slab. The considered particular case show that the method needs mathematical formulations much simpler than those used in [**4**] and [**6**].

References

[**1**] A. BELLENI MORANTE, *Applied semigroups and evolution equations*, Oxford University Press, Oxford 1979.

[**2**] R. L. BOWDEN and C. D. WILLIAMS, J. Mathematical Phys. **5** (1964), 1527-1540.

[**3**] K. M. CASE and P. F. ZWEIFEL: $[\cdot]_1$ J. Mathematical Phys. **4** (1963), 1376-1385; $[\cdot]_2$ *Linear transport theory*, Addison-Wesley, Reading 1967.

[**4**] G. R. HINTZ, J. Mathematical Phys. **12** (1971), 1100-1112.

[**5**] T. KATO, *Perturbation theory for linear operators*, Springer Verlag, Berlin 1976.

[**6**] J. MIKA and R. STANKIEWICZ, Transport Theory Statist. Phys. **2** (1972), 55-67.

[**7**] P. A. NEWMAN and R. L. BOWDEN, J. Mathematical Phys. **11** (1970), 2445-2458.

[**8**] V. I. SMIRNOV, *A course of higher analysis*, V, Pergamon Press, Oxford 1964.

[**9**] A. SUHADOLC, J. Math. Anal. Appl. **35** (1971), 1-13.

[**10**] I. VIDAV, J. Math. Anal. Appl. **22** (1968), 144-155.

[**11**] G. WING, *An introduction to transport theory*, J. Wiley, New York 1962.

Sommario

Si descrive come le condizioni al contorno in un problema di evoluzione possono essere trasformate in termine di sorgente, nell'ambito della teoria delle equazioni differenziali negli spazi di Banach.

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