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# $P$-algebras and combinatory notation (**)

A Giorgio Sestini per il suo 70° compleanno

## 0. - Introduction

**0.0.** - Motivation. The domain of a (homogeneous) operation is built from the carrier set by set-theoretic constructions which usually are cartesian products or (set-theoretic) exponentiation. An elementary question thus arises: what happens when we change these constructions?

We consider the case of the power-set construction, namely when the domain of an operation is the set of all subsets of the carrier set. A somehow similar case is outlined in [7], but here we look at almost all the elementary results of Universal Algebra. Namely we consider congruences, homomorphisms, direct products, polynomials and equationality together with their usual relationships.

The power-set case is interesting for two reasons. First, because there are well-known operations of this kind, e.g. complete lattices and choice functions. Thus, we have a chance to check the meaning of the results sought by applying them to these well-known operations. Second, because the power-set of the carrier set can be thought of as the set of characteristic functions of the subsets of the carrier set. Hence, we have a special case of another set-theoretic construction for the domain of an operation, namely exponentiation of an (auxiliary) set to the carrier set.

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**0.1.** – Results. The answer to the question in **0.0** is « nothing happens, as far as the power-set case is concerned ». In fact, all the ordinary definitions can be easily extended and the corresponding theorems hold. Moreover, their application to well-known cases yields sound statements. For instance, choice functions cannot be equationally characterized, whereas complete lattices are characterized by a « single law » (generalized idempotency) which replaces the two-laws characterization (generalized associativity and simple idempotency) of [**1**].

In the power-set case we also add some straightforward result about the recognizability of polynomials (e.g. about recognizable sets and congruences over the algebra of polynomials). We do not give results about regularity both because it seems technically too close to the corresponding results for ordinary algebra as in [**9**] and because all this topic would need a preliminary investigation about its practical applications and its relationship with other topics (a regular expression here should denote a set of « nested possibilities »).

As far as further extensions are concerned, we simply outline two cases. The former involves the power-set construction together with ordinary constructions. This yields no surprises. The latter involves exponentiation with the carrier set at the exponent. Here, we find systems richer than the ordinary ones.

A final remark. Due to our notational conventions as in **0.2**, many definitions and theorems are slightly reworded with respect to ordinary formulations (e.g. see [**4**]). For instance, most our theorems simply involve the use of the combinator $\boldsymbol{C}$ in its « functional interpretation » [**2**] rather than the usual longer formulations.

However, conciseness or elegance are minor motivations for this rewording. Rather, we try to suggest that at least in the Universal Algebra field a Combinatory Logic approach could face the categorical one. If this conjecture should be true, a more developed Combinatory Logic will analytically match Category Theory (in the same way as Analytic Geometry and Synthetic Geometry do).

**0.2.** – Notation. Set notation is as in [**3**], except for the empty set which here is denoted by $\emptyset$. Moreover $B^A$ denotes the set of functions from $A$ into $B$ and $r \cdot s$ denotes the composition of relations $s$ and $r$ (usual reversed notation).

Functional notation for application follows [**2**], e.g. $abc$ denotes the value at $c$ of function $ab$ (where $ab$ is the value at $b$ of function $a$), whereas $a(bc)$ denotes the value at $bc$ of function $a$. To say $f \in B^A$ we also write $f\colon A \to B$. Moreover, as usual, $f\colon A \Vdash\!\!\to B$, $f\colon A \twoheadrightarrow B$ and $f\colon A \Vdash\!\!\twoheadrightarrow B$ denote an injection, a surjection and a bijection respectively.

To handle functions, we also use « combinators » as described in the table below. Some of these, indeed, are classical combinators (perhaps partly « typed » [**5**]) with their functional interpretation ($\boldsymbol{C}$ and a case of $\boldsymbol{B}_X$). Others do not, because they act on sets which need not to be functions. (Anyway, when we speak of combinators, we do not understand the classical combinators with their abstract reduction properties, but our extended ones with the given interpretations).

$\boldsymbol{C}f$ For $f : A \to C^B$, it is the function $\boldsymbol{C}f\colon B \to C^A$ such that $\boldsymbol{C}fba = fab$ for all $a \in A$ and $b \in B$. (Clearly, $\boldsymbol{C}$ is one to one whatever $A$, $B$ and $C$ are).

$\boldsymbol{B}_X e$ For $e \subseteq A \times B$, it is the relation $\boldsymbol{B}_X e \subseteq A^X \times B^X$ defined componentwise from $e$, i.e. for all $a\colon X \to A$ and $b\colon X \to B$, $\langle a, b\rangle \in \boldsymbol{B}_X e$ iff $\langle ax, bx\rangle \in e$ for all $x \in X$. In particular, for $f\colon A \to B$, $\boldsymbol{B}_X f\colon A^X \to B^X$ and, for $g\colon X \to A$, $\boldsymbol{B}_X fg = f\cdot g$.

$e^\uparrow$ For $e \subseteq A \times B$, it is the function $e^\uparrow\colon PA \to PB$ such that $e^\uparrow A' = \{b \mid \langle a, b\rangle \in e$ and $a \in A'\}$ for all $A' \subseteq A$.

$F^\downarrow$ For $F \subseteq B^A$, it is the function $F^\downarrow\colon A \to PB$ such that $F^\downarrow a = \{b \mid \langle a, b\rangle \in$ $\in f \in F\}$ for all $a \in A$.

The above arrows act as « postfix » combinators, namely $ab^\uparrow$ means $(ab)^\uparrow$ and not $a(b^\uparrow)$. Similarly, composition has lower priority than application, namely $ab\cdot cd$ means $(ab)\cdot(cd)$.

Finally, the product of a family $A\colon I \to PB$ of sets is denoted by $\prod_I A_i$. Thus, this is a (bastard) $\lambda$-notation with type. (Others will be introduced in the following sections).

**0.3.** – Notions assumed. We assume some familiarity with set-theory [**6**], with lattice theory (complete lattices only) [**1**] and also with the rules related to the combinators introduced, in particular with the following ones. (Quantifications are understood).

$$(h\cdot k)^\uparrow = h^\uparrow \cdot k^\uparrow, \tag{1}$$

$$\boldsymbol{C}(\boldsymbol{C}f) = f, \tag{2}$$

$$\boldsymbol{C}(a\cdot b)c = \boldsymbol{C}ac\cdot b, \tag{3}$$

$$\boldsymbol{C}a^\uparrow b^\downarrow c = ac^\uparrow b. \tag{4}$$

Also some Universal Algebra is assumed.

## 1. - *P*-operations

This section contains the basic properties of P-operations, namely those concerning homomorphisms, congruences and direct products. It also introduces F-operations which are similar to ordinary operations and will be helpful later on.

**1.0.** – Definition. A *P-operation* on the set $A$ is a function $f\colon PA \to A$.

**1.1.** – Definition. A *congruence* of $f$ as above is an equivalence $e$ over $A$ such that, for all $A', A'' \subseteq A$,

$$e^{\dagger}A' = e^{\dagger}A'' \quad \text{implies} \quad \langle fA', fA''\rangle \in e\,. \tag{5}$$

The set of congruences of $f$ is denoted by $\Theta f$.

**1.2.** – Theorem. *The above set $\Theta f$ is closed under intersections.* (*Thus it forms a complete lattice ordered by inclusion*).

Proof. Let $E \subseteq \Theta f$. For all $e \in E$, $\cap E^{\dagger}A' = \cap E^{\dagger}A''$ implies $e^{\dagger}A' = e^{\dagger}A''$, because $\cap E \subseteq e$. Hence using (5) for all $e \in E$, we get $\langle fA', fA''\rangle \in \cap E$.

**1.3.** – Definition. Given P-operations $f\colon PA \to A$ and $g\colon PB \to B$, a *homomorphism* of $f$ into $g$ is a function $h\colon A \to B$ such that

$$h\cdot f = g\cdot h^{\dagger}\,. \tag{6}$$

As usual, when $h$ is onto, $g$ is called a *homomorphic image* of $f$. The set of homomorphisms of $f$ into $g$ is denoted by $Hfg$. Clearly, by (1), $Hff$ forms a monoid under composition (the *endomorphism monoid* of $f$) with the identity as unit.

**1.4.** – Theorem. (Notation as above) *Given $f$, $h$ is a homomorphism for some $g$, if and only if $h$ induces a congruence of $f$.*

Proof. (« If ») Since $h^{\dagger}A' = h^{\dagger}A''$ implies the hypothesis in (5), equation (6) can be used to define $g$ on $h^{\dagger}A$.

(« Only if ») Let $e$ be the equivalence induced by $h$. Assume $e^{\dagger}A' = e^{\dagger}A''$. Then, $h^{\dagger}A' = h^{\dagger}A''$. Therefore, by (6), $h(fA') = h(fA'')$ and we get (5).

**1.5.** – Definition. Given $f$ and $g$ as in 1.3, $f$ is a *suboperation* of $g$ if $f \subseteq g$ (i.e. if $h$ is the identity). Clearly, given $f$, $g$ and $h$, there always exists a $g'$ which is a homomorphic image of $f$ and a suboperation of $g$.

**1.6.** – Definition. The *direct product* of a family $g \in \prod_J A_j^{PA_j}$ of P-operations is the P-operation $\mathbf{\Pi}_J g_j \colon PB \to B$, where $B = \prod_J A_j$, such that, for all $D \subseteq B$ and $j \in J$

$$(7) \qquad \mathbf{\Pi}_J g_j Dj = gj(D^{\downarrow}j)\,.$$

**1.7.** – Theorem. *Given P-operation $f$ and family $g$ as above, consider the function $C'$ such that, for each family of homomorphisms $h \in \prod_J Hfg_j$, $C'h = \mathbf{C}h$ (in other words $C'$ is a typed $\mathbf{C}$). Then*

$$(8) \qquad C' \colon \prod_J Hfg_j \Vdash\!\!\twoheadrightarrow Hf\,\mathbf{\Pi}_J g_j\,.$$

(Trivially, $C'$ can also be defined as the relation such that

$$(9) \qquad (h, k) \in C' \quad \text{iff} \quad \boldsymbol{p}j \cdot k = hj \quad \text{for all } j \in J\,,$$

where $\boldsymbol{p}j$ is the $j$-th projection, i.e. $\boldsymbol{p}jb = bj$ for all $j \in J$ and $b \in B$, see also $\mathbf{C}^*$ in [**2**]. Thus by (9), (8) yields the customary universality property of direct products).

Proof. Since $C'$ is one to one by definition, we only have to check the codomain. This is to say that $\mathbf{C}h \cdot f = (\mathbf{\Pi}_J g_j) \cdot \mathbf{C}h^{\dagger}$ is equivalent to $hj \cdot f = gj \cdot hj^{\dagger}$ for all $j \in J$. This follows from (7) by easy combinatory passages. (Use (2), (3), (4), (7) and a bit of applications; shake well). For instance apply $\boldsymbol{p}j \cdot \mathbf{C}$ to both sides of the former equality. The left-hand side of the former equality becomes the left-hand side of the latter by (2) and (3), while right-sides are equal because of (7), which by (4) for $D = \mathbf{C}h^{\dagger}A'$ implies $\mathbf{C}(\mathbf{\Pi}_J g_j)j \cdot \mathbf{C}h^{\dagger} = gj \cdot hj^{\dagger}$ for all $j \in J$.

**1.8.** – Comment. If we say that a function $f \colon A^R \to A$ is an «F-operation» on $A$, then all definitions and theorems of this paper can be restated in terms of F-operations. (When set $R$ is an ordinal these are ordinary homogeneous operations of universal algebras [4].)

We will need to make reference to the F-case. Thus we will use the postfix « -F » in order to mention the definitions and theorems corresponding to

F-operations. For instance, **1.7**-F will denote the theorem on the universality of a direct product of a family of F-operations.

The same convention is assumed for the formulae. These are obtained by a proper replacing of combinators ($\boldsymbol{B}_R$ replaces $\uparrow$ and $\boldsymbol{C}$ replaces $\downarrow$). For instance (5) becomes: for all $a'$, $a'': R \to A$

$$(5\text{-F}) \qquad e\cdot a' = e\cdot a'' \quad \text{implies} \quad \langle fa', fa''\rangle \in e\,.$$

We also have

$$(6\text{-F}) \qquad h\cdot f = g\cdot \boldsymbol{B}_R h$$

and

$$(7\text{-F}) \qquad \prod_J{}^{\!\!\urcorner} gj\, dj = gj(\boldsymbol{C}dj) \quad \text{for all } d: R \to B\,.$$

(Note that (4) disappears because of (3) and (2) while (8) and (9) remain as they are.)

## 2. - *P*-algebras

What said for single P-operations can be restated for each element of a family of P-operations. Hence, definitions and theorems of section **1** extend over such families (or « P-algebras ») simply by adding a quantification (over the index of the family). Thus, most of this section (from **2.1** to **2.7**) reduces to a reference table. We only add the notion of a variety of P-algebras and an outline of further extensions.

**2.0.** – Definition. A *P-algebra* on $A$ with *alphabet* $\Sigma$ is a function $\alpha: \Sigma \to A^{PA}$. (In **2.0**-F we also need a « rank » function or « species » $r$ with domain $\Sigma$ in order to write $\alpha \in \prod_\Sigma A^{A^{r\sigma}}$ for an F-algebra).

**2.1.** – Definition. *Congruence* of a P-algebra $\alpha$. Set of these denoted by $\Theta\alpha$.

**2.2.** – Corollary. *Intersection closure and lattice of congruences.* (Note that $\Theta\alpha = \bigcap_\Sigma \Theta\alpha_\sigma$.)

**2.3.** – Definition. *Homomorphism* between P-algebras $\alpha: \Sigma \to A^{PA}$ and $\beta: \Sigma \to B^{PB}$. *Homomorphic image. Endomorphisms.* Set of homomorphisms denoted by $H\alpha\beta$.

**2.4.** – Corollary. *Correspondence among congruences and homomorphisms.*

**2.5.** – Definition. *Subalgebras* of P-algebras.

**2.6.** – Definition. *Direct product* of a family of P-algebras.

**2.7.** – Corollary. *Universality property of the direct product.*

**2.8.** – Definition. A *variety* of P-algebras is a class of P-algebras with the same alphabet which is closed under homomorphic images, subalgebras and direct products.

**2.9.** – Comment. Since all results of this paper hold both for F-algebras and for P-algebras, one can trivially mix them and get homogeneous algebras with operations of both types (or also of the ordinary type $f\colon A^n \to A$, $n$ being a natural number). Moreover, composition of types also is possible, e.g. one could possibly get an operation $f\colon P(A\times A) \to A$ taking relations (or graphs) as arguments.

The trick for handling composed types is still combinator replacement. For instance, let $f\colon P(A^R) \to A$ be a « $\boldsymbol{BP}$·F-operation », then we have homomorphisms defined by

$$h\cdot f = g\cdot \boldsymbol{B}_R h^\dagger \tag{6-$\boldsymbol{BP}$·F}$$

Note that these extensions depend on the corresponding extensions of (1), e.g. (1-$\boldsymbol{BP}$·F) is $\boldsymbol{B}_R(h\cdot k)^\dagger = \boldsymbol{B}_R h^\dagger \cdot \boldsymbol{B}_R k^\dagger$. (Postfix conventions as in **1.8.**)

### 3. - Polynomials

Symbolic polynomials (« terms ») form a P-algebra which is related with the algebra of polynomials (the « extension ») of a given P-algebra in the same way as for the ordinary case. Also the representation theorem for the endomorphism monoid of the term algebra continue to hold.

**3.0.** – Definition. Given sets $\Sigma$ and $X$, the « set » $T$ of the *P-terms* over $\Sigma$ generated by $X$ is (recursively) defined by

$$X \subseteq T\,,$$

$$\Sigma\times PT \subseteq T\,. \tag{10}$$

(Note that in (10-F) we have $\bigcup_{\Sigma}\{\sigma\}\times T^{r\sigma} \subseteq T$). The *depth lt* of a *P*-term $t\in T$

is the length of its recursive generation, i.e. it is the ordinal given by the function $l$ defined by: $lx = 0$ for all $x \in X$ and $l\langle\sigma, D\rangle = 1 + \cup(l^{\uparrow} D)$ for all $\sigma \in \Sigma$ and $D \in PT$. Depth recursion will be used in most of the definitions and theorems of this section. (Here, « recursion » is transfinite recursion).

**3.1.** – Definition. The *term P-algebra* over $\Sigma$ generated by $X$ is the P-algebra $\tau\colon \Sigma \to T^{PT}$ defined by the « operation » of making pairs, i.e. $\tau\sigma D = \langle\sigma, D\rangle$ for all $\sigma \in \Sigma$ and $D \in PT$.

**3.2.** – Definition. The *extension* of P-algebra $\alpha\colon \Sigma \to A^{PA}$ from $\Sigma$ to $T$ (as above) is the F-algebra $\bar{\alpha}\colon T \to A^{A^X}$, defined by

$$\bar{\alpha} x a = ax \quad \text{for all} \quad a\colon X \to A \quad \text{and} \quad x \in X,$$

$$(11) \qquad \bar{\alpha}\langle\sigma, D\rangle a = \alpha\sigma(\bar{\alpha}^{\uparrow} D^{\downarrow} a) \quad \text{for all } \langle\sigma, D\rangle \in T \text{ and } a\colon X \to A.$$

**3.3.** – Theorem. *Given $\alpha$ and $\tau$ as above, $C\bar{\alpha}\colon A^X \Vdash\!\!\twoheadrightarrow H\tau\alpha$.*

Proof. From **3.2**, $C\bar{\alpha}\colon A^X \to A^T$. Moreover, for all $a\colon X \to A$, $C\bar{\alpha}a$ is a homomorphism because of (11) and (4), namely $C\bar{\alpha}a(\tau\sigma D) = \bar{\alpha}\langle\sigma, D\rangle a = \alpha\sigma(\bar{\alpha}^{\uparrow} D^{\downarrow} a) = \alpha\sigma(C\bar{\alpha}a^{\uparrow} D)$ for all $D \in PT$.

Conversely, if $h \in H\tau\alpha$, take $a\colon X \to A$ being its restriction to $X$. Trivially, $C\bar{\alpha}ax = hx$ for all $x \in X$. Moreover, if $C\bar{\alpha}a^{\uparrow}D = h^{\uparrow}D$, we get $C\bar{\alpha}a\langle\sigma, D\rangle = \alpha\sigma(h^{\uparrow}D) = h(\tau\sigma D) = h\langle\sigma, D\rangle$ for all $\sigma \in \Sigma$. Hence, by (transfinite) induction we get $C\bar{\alpha}a = h$, while the uniqueness of the $a$ chosen is trivial.

**3.4.** – Definition. Given $\alpha$ and $\tau$ as above and a family $a\colon X \to A$, the homomorphic image of $\tau$ under $C\bar{\alpha}a$ will be called the subalgebra of $\alpha$ *connected* with $a$.

**3.5.** – Corollary. *With the above notation, $C\bar{\alpha}\colon A^X \Vdash\!\!\twoheadrightarrow H\bar{\tau}\bar{\alpha}$, where $H$ denotes homomorphisms between F-algebras. (Thus, $\bar{\alpha}(\bar{\tau}tt')a = \bar{\alpha}t(C\bar{\alpha}a \cdot t')$ for all $t \in T$, $t'\colon X \to T$ and $a\colon X \to A$).*

Proof. Routine (e.g. induction on the depths in the alphabet of $\bar{\tau}$).

**3.6.** – Definition. Given $\tau$ as above, its *r-c product* (see **3.8**) is the operation $\circ\colon T^X \times T^X \to T^X$ defined in infix notation by $t'' \circ t' = C\bar{\tau}t'' \cdot t'$ for all $t', t''\colon X \to T$ (namely $(t'' \circ t')x = \bar{\tau}(t'x)t''$). By the next theorem, this defines a monoid (see [9] for the F-case).

**3.7.** – Theorem. *$C\bar{\tau}$ is an isomorphism between the r-c product and the (composition of the) endomorphism monoid of $\tau$.*

Proof. From **3.3**, $C\bar{\tau}: T^X \Vdash\!\!\!\twoheadrightarrow H\tau\tau$. Thus, we only have to prove $C\bar{\tau}(t' \circ t'') = C\bar{\tau}t' \cdot C\bar{\tau}t''$ for all $t', t'': X \to T$, which follows from **3.5**. (Set $\alpha = \tau$ and abstract the alphabet letter.)

**3.8.** – Comment. «r-c» should mean rows by columns. (When definition **3.6**-F is extended to free algebras, one can apply it to vector spaces and get matrix multiplication «rows by columns».) The most natural definition would have been columns by rows, i.e. $t' \circ t'' = C\bar{\tau}t'' \cdot t'$. However, an r-c product fits our (reversed) notation for composition better, as seen in («representation») Theorem **3.7**.

**3.9.** – Comment. «Set» $T$ in **3.0** is a proper class. Though this yields no big troubles till **3.3**, starting from **3.4** we need some care. For instance, $A^T = \emptyset$, when common set theory [**6**] is used. As usual, we can mend the situation either by introducing a universe [**6**] to make $T$ a set (e.g. $PA$ becomes the set of subsets of $A$ *which are in the universe*) or by allowing classes to be members of (hyper) sets or of (hyper) classes [**3**].

## 4. - Recognizability and equationality

We first extend Nerode's theorem about the recognition of terms. Then we characterize the identities of a P-algebra by the r-c product and the varieties in terms of equational classes. Finally we consider further extensions (of P-algebras) which have a richer system of homomorphisms.

**4.0.** – Definition. (Notation as in **3**) The equivalence over $T$ *recognized* by $\alpha$ *starting* with (or with *initial family*) $a: X \to A$ is the equivalence induced by $C\bar{\alpha}a$. A union $L$ of blocks of the corresponding partition will be called a *set recognized by* $\alpha$ starting with $a$.

**4.1.** – Corollary. *An equivalence is recognizable if and only if it is a congruence of $\tau$. (Moreover given $L$ as in **4.0**, we can find a finite $\alpha$ recognizing it iff the corresponding partition is finite.)*

Proof. See **3.3** and **2.4**.

**4.2.** – Corollary. *Let $\beta$ and $\alpha$ recognize $e'$ and $e''$ starting with $b$ and $a$ respectively. Then, there exists a homomorphism $h$ from the subalgebra connected with $b$ into $\alpha$, such that $h\cdot b = a$, if and only if $e' \subseteq e''$.*

Proof. (« Only if ») Trivial. (« If ») Build $h$ by $\boldsymbol{C}\bar{\alpha}a = h\cdot \boldsymbol{C}\bar{\beta}\,b$.

**4.3.** – Definition. Given $\alpha$ and $T$, the *set* of *identities* $I\alpha$ over $X$ *satisfied by* $\alpha$ is the equivalence induced by $\bar{\alpha}$ on $T$. In other words $I\alpha$ is the intersection of all equivalences recognized by $\alpha$ (for any initial family $a: X \to A$). Similarly, if $e \subseteq I\alpha$, we will say that $\alpha$ *satisfies* $e$. Thus, given a class $K$ of P-algebras over the same alphabet, the set $I_K = \cap\,(I^{\dagger} K)$ will be called the *set* of *identities* (over $X$) *satisfied by* $K$. (Dirty tricks as in **3.9** are allowed).

**4.4.** – Theorem. *A set of pairs $e \subseteq T\times T$ is a set of identities (of a single P-algebra or of a class) if and only if $\boldsymbol{B}_X e$ is a congruence of the r-c product.*

Proof.(« only if ») By **2.2**-F it is enough to prove the statement for a set of identities $I\alpha$ of a single P-algebra. By **1.4**-F, this is to say that $\boldsymbol{B}_X\bar{\alpha}$ determines a homomorphism from the r-c product or also that its composition with $\boldsymbol{C}$ is another homomorphism. Thus, we can show that the function $\alpha': T^X \to (A^X)^{A^X}$ defined by $\alpha' tax = (\boldsymbol{C}\bar{\alpha}a\cdot t)x = \bar{\alpha}(tx)a$ for all $t: X \to T$, $a: X \to A$ and $x \in X$ is a homomorphism from the r-c product into another (binary) operation.

If for the target operation we choose the commuted composition $\bullet$, namely $f\bullet g = g\cdot f$ for all $f, g: A^X \to A^X$ (see **3.8**), then **3.5** yields our desired result: $\alpha'(t'\bullet t'') = \alpha' t''\cdot \alpha' t'$ (apply $a$ and $x$ to both sides of this equality and set $t'' x = t$).

(« If ») Being $\boldsymbol{B}_X e$ a congruence (of the r-c product), a fortiori it is a « left » congruence, namely $e$ is a congruence of $\bar{\tau}$. By **2.4**, **3.3** and **3.5**, $e$ also is a congruence of $\tau$. Thus, by **2.4** we can define an $\alpha$ which is a homomorphic image of $\tau$ by some $h: T \twoheadrightarrow A$.

We only have to prove that $ht = hd$ iff $\bar{\alpha}t = \bar{\alpha}d$, since $\boldsymbol{B}_X e$ is induced by $\boldsymbol{B}_X h$. The « if » is trivial, since $h = \boldsymbol{C}\bar{\alpha}a$ for some $a: X \to A$. The converse, $ht = hd$ implies $\bar{\alpha}tb = \bar{\alpha}db$ for all $b: X \to A$, follows from the connectedness of $\alpha$ to $a$ (i.e. any $b$ is $\boldsymbol{C}\bar{\alpha}a\cdot t'$ for some $t': X \to T$) and from **3.5**. In fact, since $\boldsymbol{B}_X h$ induces a congruence, $ht = hd$ implies $h(\bar{\tau}tt') = h(\bar{\tau}dt')$ for all $t': X \to T$ (this is to say that $\boldsymbol{B}_X e$ is a « right » congruence).

**4.5.** – Definition. A class $K$ of P-algebras is *equational* if it contains all P-algebras which satisfy $I_K$ for all $X$ (as in **4.3**).

4.6. – Example. This perhaps is the easiest non-trivial set of identities $I_K$ defining a non-trivial equational class $K$. Take a singleton alphabet, $\Sigma = \{\wedge\}$, and « P-idempotency » (to define the identities). Namely, two terms over $X$ are identical simply when they have the same subset of generators

$$(t', t'') \in I_K \quad \text{iff} \quad \varphi t' = \varphi t'' , \tag{12}$$

where $\varphi\colon T \to PX$ (a « frontier » function) yields the subset $\varphi t$ of the $x$'s effectively used in (10) to build $t$. (Clearly, this defines a congruence of the r-c product. Moreover, P-algebra $\alpha$ of the « if » part in **4.4** is just union: $\alpha\colon \{\wedge\} \to PX^{P(PX)}$, $\alpha\wedge = \cup$ and $\varphi = h = C\bar{\alpha}a$, where $a\colon X \to PX$ is just the natural mapping of the identity, $ax = \{x\}$ for all $x \in X$).

Equational class $K$ is just the class of complete lattices. In fact, the single condition (12) is clearly equivalent to the pair of conditions used to characterize complete lattices [**1**]: $\wedge \cdot \wedge^{\dagger} = \wedge \cdot \cup$ and $\wedge\{a\} = a$ for all $a \in A$ (where $\wedge$ denotes any $\alpha\wedge$). (In **4.6**-F we substantially get the classes of semigroups and monoids depending on the choice of $r$.)

4.7. – Theorem. *A class of P-algebras is equational if and only if it is a variety.*

Proof. (« Only if ») Trivial. (« If » part) Let $K$ be a variety with alphabet $\Sigma$. In order to show that any $\alpha$ which satisfies $I_K$ (for all $X$) is in $K$, we first introduce an intermediate construction (usual free algebra construction).

Given $X$, let $J \subseteq \Theta_\tau$ be the set of congruences induced by $C\bar{\alpha}a$ for all $a\colon X \to A$ and all $\alpha \in K$. For each $j \in J$ let $hj$ be a homomorphism chosen among those inducing $j$ and let $\gamma j$ be the corresponding homomorphic image of $\tau$. Since $K$ contains its subalgebras, we thus have a family $\gamma\colon J \to K$ of P-algebras and a corresponding family $h$ of homomorphisms, $h \in \prod_J H\tau\gamma_j$.

Let $\beta\colon \Sigma \to B^{PB}$ be the direct product of $\gamma$ which again is in $K$. Thus, $I_K \subseteq I\beta$. Moreover, since $I_K = \cap J$ and since by **2.7**, $I\beta \subseteq j$ for all $j \in J$, then $I\beta \subseteq I_K$ and we conclude $I_K = I\beta$.

Taking the restriction $b$ of $Ch$ to $X$, we have an initial family $b\colon X \to B$, such that $\beta$ recognizes $I\beta = I_K$, and the corresponding subalgebra $\beta'\colon \Sigma \to B'^{PB'}$ connected with $b$. Thus $\beta' \in K$ for all $X$.

Now, consider any $\alpha\colon \Sigma \to A^{PA}$. Taking $X = A$ and $a\colon X \to A$ to be the identity, by **3.3** we have a homomorphism $C\bar{\alpha}a\colon T \twoheadrightarrow A$ of $\tau$ onto $\alpha$, which induces a congruence $e \supseteq I\alpha$. If $\alpha$ satisfies $I_K$, i.e. $I\alpha \supseteq I_K$, then $e \supseteq I\beta$. Thus, we can apply **4.2** and get a homomorphism from $\beta'$ onto $\alpha$. Hence $\alpha \in K$.

**4.8.** – Example. After giving an example of a variety in **4.6**, we now give an example of a class which is not a variety. Consider the class of choice functions, i.e. the functions $c\colon PA \to A$ such that $cA' \in A'$ for all nonempty $A' \subseteq A$. This can be thought of as a class of P-algebras over a singleton alphabet and clearly it is closed under homomorphic images and subalgebras.

On the contrary, direct products do not preserve the above membership property. Take a family of two identical choice functions with $A = \{0, 1\}$ and $cA = 1$. Then, the membership property for their direct product fails when one considers the diagonal set $B' = \{\langle 0, 1\rangle, \langle 1, 0\rangle\}$ (where pairs denote the corresponding binary families). Hence choice functions cannot be equationally characterized.

**4.9.** – Comment. After considering P-operations and F-operations, we should consider « $\boldsymbol{C}$F-operations », namely functions of the type $f\colon R^A \to A$. This seems more difficult than the P-case. A problem is that in (6) we need not to replace the (postfix) combinator $\uparrow$ by (another) single one.

When $R = \{0, 1\}$, for instance, we can still have a combinator $Y$ such that, for all $a\colon A \to R$ and $b \in B$, $Yhab = 1$ ff $aa' = 1$ for some $a'$ such that $ha' = b$. (This is what $\uparrow$ does, when thinking of $R^A$ as the set of characteristic functions in $PA$). But, we can also have a combinator $Z$ such that $Zhab = 1$ iff the same holds *for all* such $a'$.

However, this freedom cannot be too large. For instance, if $Y$ is the combinator chosen for $h \in Hfg$ and $Z$ is for $k \in Hgm$, then we must be able to choose a combinator $V$ for $k \cdot h$ such that $Zk \cdot Yh = V(k \cdot h)$. In fact, if we want to compose homomorphisms, we also need such an extension of (1).

Anyway, the real problem here is to find good reasons which legitimate a study of such $\boldsymbol{C}$F-operations. Two possible hints are the following ones.

First, there are well-known $\boldsymbol{C}$F-operations equipped with sound homomorphisms. One of these is the operation of taking the barycenter of a configuration (e.g. a configuration of weights an a square can be projected onto a configuration of weights on a side of the square while preserving the barycenter). Another one is maximizing a function(al) on a certain domain (consider the optimality principle of Dynamic Programming).

Second, the ability of handling $\boldsymbol{C}$F-operations can be preliminary to the study of more general « combinatory » operations. A combinatory operation of type $\boldsymbol{G}$ on a set $A$ could be defined as a function $f\colon \boldsymbol{G}A \to A$ where $\boldsymbol{G}$ is a « set-combinatory » term (which we will only exemplify here).

For instance, when $\boldsymbol{G} = FR$ we have an F-operation as in **1.8** ($F$ stands for set exponentiation), when $\boldsymbol{G} = P$ we have a P-operation, when $\boldsymbol{G} = \boldsymbol{C}FR$ we have a $\boldsymbol{C}$F-operation, when $\boldsymbol{G} = \boldsymbol{W}F$ we have an operation of the type

$f: A^A \to A$ and so on (see [**2**] for $\boldsymbol{W}$ and note that the above $f$ can also have *different* types as $FA$ or $\boldsymbol{C}FA$).

It could be interesting to see how much of ordinary Algebra can be extended to these combinatory operations (e.g. homomorphisms, polynomials, recognizability and equationality). Perhaps, one to one combinators will still be able to express the main theorems in the same way as $\boldsymbol{C}$ or $\boldsymbol{B}_X$ do in the present treatment. A (naive) formulation of such a conjecture is in [**8**].

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## Abstract

*P-algebras are families of P-operations (where a P-operation is just a function from the subsets of a set into its elements). P-algebras have almost all the properties of ordinary (homogeneous) algebras. In particular, (closed) varieties turn out to be equational classes. Complete lattices form a straight-forward example of such varieties (they are equationally defined by a single trivial property).*

*The notation introduced in order to handle P-algebras is an extension of set-theoretical notation close to Combinatory Logic. It expresses many constructs in definitions, statemnts and proofs in a standard way (e. g. by type assignments of a peculiar kind). Moreover, going from P-algebras to ordinary universal algebras or similar systems simply involves some easily defined notational transformations.*

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