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Cauchy type free boundary problems
for nonlinear parabolic equations (**)  

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

1.1. - The Cauchy type free boundary problem (problem (C))

It is well known that in many branches of applied mathematics (such as optimal stopping time problems, biomechanics, fluid flow in porous media, etc.) the following class of parabolic free boundary problems is encountered.

Problem (C). Find a triple \((T, s, u)\) such that

(i) \(T > 0, s(t)\) is continuous and positive for any \(t\) in \([0, T]\),

(ii) \(u(x, t)\) is continuous in the closure of the domain \(D_T = \{(x, t): 0 < x < s(t), 0 < t < T\}\), \(u(x, t)\) is continuous for \(0 < x < s(t), 0 < t < T\), \(u_{xx}, u_t\) are continuous in \(D_T\),

(iii) the following equations are satisfied

\[
(1.1) \quad a(x, t, u, u_x, s) u_{xx} - u_t = q(x, t, u, u_x, \varepsilon) \quad \text{in} \quad D_T,
\]

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\[(1.2) \quad s(0) = b > 0, \quad (1.3) \quad u(x, 0) = h(x), \quad 0 < x < b, \]
\[(1.4) \quad u(0, t) = \varphi(t), \quad (1.5) \quad u(s(t), t) = f(s(t), t), \quad 0 < t < T, \]
\[(1.6) \quad u_x(s(t), t) = g(s(t), t), \quad 0 < t < T, \]

where \( h(x) \), \( \varphi(t) \), \( f(x, t) \), \( g(x, t) \) are prescribed functions for \( x \geq 0, t \geq 0 \) and the coefficients \( a, q \) are given functions of their arguments \( (a > 0) \).

**Remark 1.1.** The condition \((1.4)\) on \( x = 0 \) can be replaced by

\[(1.4)’ \quad u_x(0, t) = \psi(t), \]

with the substitution of \((ii)\) with

\[(ii)’ \quad u(x, t) \text{ satisfies } (ii) \text{ and } u_x \text{ is continuous also at } x = 0. \]

Problem (C) in which the Cauchy data for \( u \) are assigned on \( x = s(t) \), differs substantially from Stefan-like problems where the value of \( u \) is prescribed on the free boundary \( x = s(t) \) and an explicit relationship between \( s(t) \) and \( u_x(s(t), t) \) is given. Nevertheless, if the parabolic equation \((1.1)\) reduces to

\[ u_{xx} - u_t = g(x, t), \]

Problem (C) can be transformed into a free boundary problem of the type studied in [2], provided that one of the following conditions is fulfilled

\[(1.7) \quad f_x(x, t) - g(x, t) = 0, \quad (x > 0, t > 0), \]
\[(1.7)’ \quad f_x(x, t) - g(x, t) = 0, \quad f_{xx}(x, t) - f_t(x, t) \neq g(x, t), \quad (x > 0, t > 0), \]

(all the terms appearing in \((1.7)\), \((1.7)’\) are supposed to be continuous).

Contradicting the above assumptions may yield non uniqueness or nonexistence or noncontinuous dependence of the solution upon the data (see [1] and [5]).

We shall see that for the nonlinear case a similar analysis can be performed. However the free boundary problems we shall lead to consider differ from the type studied in [2], in the fact that higher order derivatives (e.g. \( u_{xx}, u_{xxx} \) appear in the free boundary conditions: such problem will be referred to as *Stefan type problems of order 2, 3, etc.*
1.2. - Classes of regular solutions to problem (C)

We define the following classes of solutions of Problem (C) possessing higher regularity.

Definition 1.2. A solution \((T, s, u)\) is said to belong to the class \(\mathcal{S}_1\) if, besides (i)-(iii),

(iv) \(s(t)\) is continuously differentiable in \((0, T)\),
(v) \(u_{xx}, u_t\) are continuous up to \(x = s(t), t > 0\).

Definition 1.3. A solution \((T, s, u)\) is said to belong to the class \(\mathcal{S}_2\) if, besides (i)-(v),

(vi) \(u_x\) is continuous up to \(x = s(t), t > 0\),
(vii) \(u_{xxx}, u_{xt}\) are continuous for \(0 < x < s(t), 0 < t < T\).

Remark 1.4. In dealing with Problem (C) we always assume that

(A) \(h, q, a\) and \(q\) are continuous functions of their arguments, \(f(x, t)\) is continuous for \(x > 0, t > 0\), and \(g(x, t)\) is continuous for \(x > 0, t > 0\). Moreover the following conditions are to be satisfied

\[
(1.8) \quad h(0) = q(0), \quad h(b) = f(b, 0)
\]

(only the second condition is required when \((1.4)'s\) is considered).

In dealing with solutions belonging to \(\mathcal{S}_1\) we also require that

(A1) the function \(f\) is continuously differentiable for \(t > 0\).

Finally, when solutions in \(\mathcal{S}_2\) are considered, to (A), (A1) we add the requirement

(A2) \(g\) is continuous for \(t > 0\) and

\[
(1.9) \quad h'(b) = g(b, 0); \quad \text{moreover } g \text{ has to be continuously differentiable for } t > 0 \, (1).
\]

\((1)\) Conditions (A), (A1), (A2) are suggested by the definitions of Problems (C), (C1), (C2) and by Definitions 1.2, 1.3.
1.3. - First case of equivalence of problem (C) with a Stefan type problem of higher order

Assume that \((T, s, u)\) is a solution of problem (C) belonging to \(\mathcal{S}_1\) and suppose (A), (A1) are satisfied.

From (1.5), (1.6) we get

\[
(f_x - g)\dot{s} + f_t - u_t = 0.
\]

If (1.7) is satisfied, we can write (1.8) in the form

\[
\dot{s} = A_1[s(t), t, u(s(t), t), u_x(s(t), t), u_{xx}(s(t), t)], \quad (0 < t < T),
\]

with

\[
A_1 = [f_x(s(t), t) - g(s(t), t)]^{-1} \{s[s(t), t, u(s(t), t), u_x(s(t), t), s(t)]
\]

\[
- u_{xx}(s(t), t) - g[s(t), t, u(s(t), t), u_x(s(t), t), s(t)] - f_t(s(t), t),
\]

where \(u(s(t), t)\) and \(u_x(s(t), t)\) can be replaced by \(f(s(t), t)\) and \(g(s(t), t)\), respectively.

Now we state the following

**Problem (C1).** Find a triple \((T, s, u)\) satisfying (i), (ii), (iv), (v) and equations (1.1), (1.2), (1.3), (1.4) [or (1.4)' (2)], (1.6) and

\[(1.11)' \quad \dot{s}(t) = A_1[s(t), t, u(s(t), t), g(s(t), t), u_{xx}(s(t), t)], \quad (0 < t < T).\]

We have the following

**Proposition 1.5.** If conditions (A), (A1) and (1.7) are satisfied, any solution \((T, s, u)\) of Problem (C1) solves Problem (C) and belongs to \(\mathcal{S}_1\).

**Proof.** It suffices to show that (1.5) is satisfied. To this purpose note that

\[
\frac{d}{dt} u(s(t), t) = [g\dot{s} + u_t]_{x=s(t)},
\]

owing to (1.6), whence

\[
\frac{d}{dt} u(s(t), t) = \frac{d}{dt} f(s(t), t),
\]

(2) In that case condition (ii) is modified according to Remark 1.1.
because (1.11)' and (1.12) imply (1.10). Recalling the assumptions on \( f \), the proof is completed.

Since the converse of Proposition 1.5 has been already proved, we have that looking for solutions of (C) in \( \mathcal{S}_1 \) is equivalent, under the above assumptions, to look for solutions to (C1).

### 1.4. - Second case of equivalence of problem (C) with a Stefan type problem of higher order

Now let us assume that \((T, u, u) \in \mathcal{S}_2\) solves Problem (C) and that (\(\Lambda\)), (A1), (A2), are satisfied.

We want to study the case in which instead of (1.7) we have

\[
f_x(x, t) = g(x, t) .
\]

This condition implies the differentiability of \( f_x \); we assume that

\[
a(x, t, f(x, t), p, s) f_{xx}(x, t) - f_x(x, t) = g(x, t, f(x, t), p, s) ,
\]

for any \( x > 0, \ t \geq 0, s > 0, p \in (-\infty, +\infty) \).

Note that

\[
u_x(s(t), t) = f_x(s(t), t) \quad (0 < t < T),
\]

\[
[(af_{xx} - f_t - q)s - au_{xt} + af_{xt}]_{u=\eta(t)} = 0 \quad (0 < t < T) .
\]

Thus

\[
\dot{s}(t) = \Lambda_2[s(t), t, u(s(t), t), u_x(s(t), t), u_{xt}(s(t), t)] ,
\]

where

\[
\Lambda_2 = \{a[s(t), t, u(s(t), t), u_x(s(t), t), s(t)]f_{xx}(s(t), t)
\]

\[
- f_x(s(t), t) - g[s(t), t, u(s(t), t), u_x(s(t), t), s(t)]\}^{-1}
\]

\[
a[s(t), t, u(s(t), t), u_x(s(t), t), s(t)]\{u_{xt}(s(t), t) - f_x(s(t), t)\} ,
\]

where \( u(s(t), t) \) and \( u_x(s(t), t) \) can be replaced by \( f(s(t), t) \) and \( g(s(t), t) \) respectively.
We state the following

**Problem (C2).** Find a triple \((T, s, u)\) satisfying (i), (ii), (iv), (v), (vi), (vii) and equations (1.1)-(1.5) [or (1.4) (a)] and

\[
(1.17') \quad \dot{s}(t) = A_2[s(t), t, f(s(t), t), u_\alpha(s(t), t), u_{\alpha\ell}(s(t), t)] .
\]

We have the following

**Proposition 1.6.** If \((A), (A1), (A2), (1.13)\) and (1.14) are satisfied, any solution of (C2) solves (C) in the class \(\mathcal{S}_2\).

**Proof.** It suffices to prove (1.6), i.e.

\[
(1.19) \quad u_\alpha(s(t), t) = f_x(s(t), t) .
\]

It is easy to show that

\[
\frac{d}{dt} u_\alpha(s(t), t) = [a^{-1}(u_\ell - j)]_{x=s(t)} \dot{s}(t) + \frac{d}{dt} f_x(s(t), t) ,
\]

where \([a]_{x=s(t)}\) must be considered as a known positive function of \(t\).

On the other hand, differentiating (1.5) we get

\[
(u_\alpha - f_x)_{x=s(t)} \dot{s}(t) + (u_\ell - f_\ell)_{x=s(t)} = 0 .
\]

Therefore the difference \(X = (u_\alpha - f_x)_{x=s(t)}\) satisfies the following linear differential equation

\[
\dot{X}(t) = - [a^{-1}]_{x=s(t)} \hat{s}^2 X(t) \quad (0 < t < T) ,
\]

with zero initial value. This implies (1.19).

Since the converse of Proposition 1.6 has been already proved, the above stated assumptions guarantee that to find a solution of Problem (C) belonging to \(\mathcal{S}_2\) is equivalent to solve (C2).

It is immediately seen that (C2) is equivalent to

**Problem (C2').** Same as Problem (C2) with (1.5) replaced by (1.15).

(\footnote{See Remark 1.1.})
1.5. - Further remarks and principal results

To conclude this section, let us consider the case

\begin{equation}
(1.20) \quad f_x(x, t) = g(x, t), \quad a(x, t, u, p, s) f_x(x, t) - f_t(x, t) = q(x, t, u, p, s),
\end{equation}

for any $x > 0$, $t > 0$ and any $u, p \in (-\infty, +\infty)$, $s > 0$.

Then in addition to (1.15) and (1.19), from (1.16) we have $u_{xt} = f_{xt}$ on the free boundary for any solution of Problem (C) such that $u_{xt}$ is continuous up to the free boundary.

If we assume that $f$ is infinitely differentiable and that $a, g, u$ are also infinitely differentiable in a neighborhood of the free boundary, it is easy to show that the derivatives of any order of $u$ at the free boundary coincide with the corresponding derivatives of $f$. This means that when (1.20) is satisfied we cannot find any relationship between $\dot{s}$ and some derivative of $u$.

On the other hand, as it has been pointed out for the linear case, Problem (C) is not well posed under (1.20). Hence we are clearly motivated to restrict our analysis to Problems (C1) and (C2).

In § 2 of the present paper we shall prove the well-posedness of Problem (C1), confining ourselves to the boundary condition (1.4) for the sake of brevity and setting

\begin{equation}
(1.6)' \quad u_x(s(t), t) = 0,
\end{equation}

with obvious redefinition of the differential operator and of the data. For the sake of generality, we study the free boundary condition

\begin{equation}
(1.21) \quad \dot{s}(t) = \lambda_1[(s(t), t, u(s(t), t), u_{xx}(s(t), t))],
\end{equation}

with $\lambda_1$ prescribed independently of the other data and coefficients.

In § 3 we shall prove the well-posedness of problem (C2)', where we set

\begin{equation}
(1.15) \quad u_x(s(t), t) = 0,
\end{equation}

also in this case redefining the data and the coefficients. We shall study the more general free boundary condition

\begin{equation}
(1.22) \quad \dot{s} = \lambda_2[s(t), t, u(s(t), t), u_x(s(t), t), u_{xxx}(s(t), t)].
\end{equation}
2. - Problem (C1)

A simplified version of Problem (C1), namely with the coefficient a depending on \(u\) only, with no source term in the differential equation and with \(u_x\) entering the free boundary condition in a linear way, has been studied in [2] with reference to a problem of liquid flow in porous media. In [2] the boundary condition at \(x = 0\) was assumed of the type (1.4).

In this section a different approach will be used to deal with the more general scheme introduced in § 1.

For the sake of conciseness only the boundary condition (1.4) will be considered.

2.1. - Assumptions and notation of spaces and norms

For the notation of spaces and norms we refer to [2], Sec. 2.

Here we add the spaces \(C_{j,k}(\mathcal{D})\) \((j = 1, 2, 3; k = 0, 1)\)—where \(\mathcal{D}\) is a domain of \(\mathbb{R}^2\)—whose elements are the functions \(u(x, t)\) having bounded continuous partial derivatives in \(\mathcal{D}\) up to the \(j\)-th order w.r.t. \(x\) and to the \(k\)-th order w.r.t. \(t\), with the norms \(\|\cdot\|_{C_{j,k}(\mathcal{D})}\) defined as usual.

Concerning the data and the coefficient in Problem (C1), we shall assume

\((\alpha)\) \(h \in H_{2+\alpha}[0, b], \quad \psi \in H_{1+\alpha}[0, T], \quad h'(0) = \psi(0), \quad h'(b) = 0\) for some given \(\bar{T} > 0, \quad \alpha \in (0, 1)\);

\((\beta)\) \(a(x, t, u, p, s)\) is twice continuously differentiable (although w.r.t. \(t\) it is enough to require the Hölder continuity of \(a\) and of its derivatives w.r.t. the other arguments); for some continuous function \(\mu(\xi, \eta)\), nondecreasing w.r.t. \(\xi, \eta > 0\),

\[0 < \mu^{-1}(|u|, |p|) < a(x, t, u, p, s) \leq \mu(|u|, |p|),\]

\[\forall x > 0, \quad u, p \in (-\infty, -\infty), \quad t \in (0, \bar{T}), \quad s \geq 0;\]

\((\gamma)\) \(q(x, t, u, p, s)\) has the same differentiability properties as \(a\);

\((\delta)\) \(|\lambda_1(x, t, u, p)| \leq \nu(|u|, |p|)\), where \(\nu\) is as \(\mu\) and \(\lambda_1\) is continuously differentiable.
2.2. - An auxiliary free-boundary problem

Set \( \mathbb{R}^3 = [0, 3b/2] \times [0, T] \) and let \( \Omega(\gamma, \bar{T}) \), \( \gamma \in (0, \alpha] \), be the set of the functions \( U(x, t) \) defined in \( \mathbb{R}^3 \) such that \( U \in C_{1,\gamma} \cap C_{1,\delta}, \) \( U(x, 0) = h^a(x), \) \( U(x, t) = \psi(t) \), where \( h^a(x) \) is a smooth extension of \( h(x) \).

For any \( U \in \Omega(\gamma, \bar{T}) \) consider the following problem

\[
\begin{align*}
(2.1) \quad & a(x, t, U, V, S) V_{xx} - V_t = Q(x, t, U, U_x, V_x, S) \\
& \text{in } D_T = \{(x, t); 0 < x < S(t), 0 < t < T\}, \\
(2.2) \quad & S(0) = b, \quad (2.3) \quad V(x, 0) = h'(x) \quad (0 < x < b), \\
(2.4) \quad & V(0, t) = \psi(t), \quad (2.5) \quad V(S(t), t) = 0 \quad (0 < t < T), \\
(2.6) \quad & \dot{S}(t) = \lambda(S(t), t, U(S(t), t), V(S(t), t)) \quad (0 < t < T),
\end{align*}
\]

where \( Q(x, t, U, U_x, V_x, S) = q_x + q_a U_x + q_b V_x - a_a U_x V_x - a_u U_x S - a_v V_x S \); the arguments of \( a, q \) and of their derivatives are \( x, t, U, U_xS \).

The free boundary problem (2.1)-(2.6) has been studied in [2]. The assumptions made in § 2.1 are sufficient to ensure the existence of a unique solution \( (T, S, V) \) of this problem for any \( U \) in \( \Omega(\gamma, T) \).

2.3. - Estimates on \((T, S, V)\)

Let us introduce the following subset of \( \Omega(\gamma, T) \): \( B(K, \gamma, K_1, T) = \{ U \in \Omega(\gamma, T) : \| U \|_{c_{1,0}} < K, \| U \|_{c_{1,\gamma}} < K_1 \} \), with suitably large \( K, K_1 \).

For any \( U \in B(K, \gamma, K_1, T) \) the following estimates hold true:

\[
\begin{align*}
(2.8) \quad & T > T_0(K, K_1, \gamma), \quad (2.9) \quad \| V \|_{c_{1,0}} < N_1(K), \\
(2.10) \quad & \| V \|_{c_{1,\beta(K)}} < N_2(K), \quad (2.11) \quad \| V \|_{c_{1,\beta(K)}} < N_3(K),
\end{align*}
\]

\(^{(4)}\) Remark that in [2] we assumed that the leading coefficient, the source term and the function appearing in the free boundary condition were continuously differentiable (see assumptions \((B)-(D)\), § 3). Actually, the assumption \( h \in H^{2+\alpha} \) and thm. 5.1 p. 561 of [3] allow us to replace the differentiability w.r.t. \( t \) by Hölder continuity.
(2.12) \[ \| V \|_{C^2_{x_1}}(D_{x_8}^T) \leq N_t(K, K_1, \gamma, \| U \|_{C^1_{x_1}}) \tau^{-1}, \quad D_{x_8}^T = D_{x_8} \cap \{(x, t) : t > r\}, \]

\[ b/2 < S(t) < (3/2)b, \quad t \in (0, T_0), \]

\[ \| S \|_{C^0_{x_3}} < N_s(K), \quad (2.15) \| \hat{S} \|_{L^2_{\beta(K)^2}} < N_s(K, K_1, \gamma), \]

where the constants \( \beta \in (0, 1) \), \( T_0 \) and \( N_t \) depend also on the data and on the coefficients.

### 2.4. Definition of the operator \( \mathcal{F} \)

Let us define

\[ \bar{U}(x, t) = \int_0^x V(\xi, t) d\xi + \int_0^t \{a(0, \tau, U(0, \tau), \psi(\tau), S(\tau)) \} V_x(0, \tau) \]

\[ - a(0, \tau, U(0, \tau), \psi(\tau), S(\tau)) \} d\tau + h(0), \quad (x, t) \in D_{x_8} \]

and extend it smoothly to \( D_{x_8}^{(b)} \) in a fixed way. Because of the estimates of § 2.3 we have

\[ \bar{U} \in \Omega(\beta, T_0). \]

The equation

\[ \bar{U} = \mathcal{F} U \]

defines the operator \( \mathcal{F} : B(K, \gamma, K_1, T_0) \to \Omega(\beta, T_0). \)

### 2.5. Solutions of (C1) associated to fixed points of \( \mathcal{F} \)

If \( u \in B(K, \gamma, K_1, T_0) \) is a fixed point of \( \mathcal{F} \) the corresponding solution \((T_0, s, v)\) of (2.1)-(2.6) generates the solution \((T_0, s, u)\) of Problem (C1).

As a matter of fact, it is

\[ u_x = v, \quad (x, t) \in D_{x_8} \]

and equation (2.1) can be written

\[ v_x = \frac{\partial}{\partial x} \left[ a(x, t, u, u_x, s, v_x) - g(x, t, u, u_x, s) \right]. \]
From (2.16), (2.17) and (2.19), (2.20) it is easy to derive equation (1.1) for \( u \). Also conditions (1.2), (1.3), (1.4), (1.6), (1.9) are easily obtained.

Conversely, it is obvious that if \((T_o, s, u)\) is a solution of Problem (C.1), then \( u \) (with the above mentioned extension) is a fixed point of \( \mathcal{F} \).

2.6. - Estimates on \( \mathcal{F}U \)

From § 2.3 and from (2.16) we infer the following estimates on \( \mathcal{F}U \)

\[
\| \mathcal{F}U \|_{c_{1+\beta}} < N_7(K), \quad \| \mathcal{F}U \|_{c_{1,0}} < N_8(K).
\]

Moreover, estimates of type (2.11), (2.12) apply to \( \| \mathcal{F}U \|_{c_{2t,\beta}} \) and to \( \| \mathcal{F}U \|_{c_{2,1}} \).

We need a more careful estimate of \( \| \mathcal{F}U \|_{c_{2,0}(\mathcal{P})} = \| \mathcal{F}U \|_{c_{2,0}(\mathcal{P})} \),
\[ \forall \mathcal{U} \in (0, T_o). \]

To this purpose, we look for an estimate of \( \| \mathcal{U} \|_{c_{1,0}(\mathcal{P})} \), \( t \in (0, T_o) \).

The transformation

\[
x(S(t)y, t) = y, \quad U(S(t)y, t) = \hat{U}(y, t), \quad V(S(t)y, t) = \hat{V}(y, t),
\]

\[
Q[S(t)y, t, U(S(t)y, t), U_x(S(t)y, t), V_x(S(t)y, t), S(t)] = \hat{Q}(y, t, \hat{U}, \hat{V}, S)
\]
carries (2.1), (2.3), (2.4), (2.5) into

\[
S^{-1}a(Sy, t, \hat{U}, \hat{V}, S)\hat{V}_{yy} + y\dot{S}S^{-1}\hat{V}_y - \hat{V}_t = \hat{Q} \quad (y, t) \in (0, 1) \times (0, T_o),
\]

(2.24) \[ \hat{V}(y, 0) = h'(by) \quad y \in (0, 1), \]

(2.25) \[ \hat{V}(0, t) = \varphi(t), \quad \hat{V}(1, t) = 0 \quad t \in (0, T_o). \]

We split \( \hat{V} \) into the sum

\[ \hat{V} = \hat{V}_1 + \hat{V}_2, \]

where \( \hat{V}_1 \) solves

\[
L\hat{V}_1 = S^{-1}a(Sy, t, \hat{U}, \hat{V}, S)\hat{V}_{yy} - \hat{V}_1 = \hat{Q} - y\dot{S}S^{-1}\hat{V}_y
\]

with zero initial and boundary conditions, and \( \hat{V}_2 \) solves

\[ L\hat{V}_2 = 0 \]

and satisfies (2.24)-(2.26). It is well known that \( \hat{V}_1 \) and \( \hat{V}_2 \) exist.
Representing $\hat{\mathcal{P}}_1$ by means of the Green function in $(0,1) \times (0, T_a)$ of the parabolic operator $L$ and using standard estimates (see e.g. [3], p. 413), as a consequence of (2.9), (2.10), (2.13), (2.14) we get

$$
(2.30) \quad \sup_{0 < \tau < T} |\hat{\mathcal{P}}_1(y, \tau)| < N_s(K, K_1) t \quad \quad (2.30)' \quad \sup_{0 < \tau < T} |\hat{\mathcal{P}}_1(y, \tau)| < N_s(K, K) t. 
$$

Concerning $\hat{\mathcal{P}}_2$, it is obviously dominated as follows

$$
(2.31) \quad \sup_{0 < \tau < T} |\hat{\mathcal{P}}_2(y, \tau)| < b\|h^0\|_{c_1(0,b)} + \|\Psi\|_{c_1(0,\tau)} t,
$$

and an estimate for $\hat{\mathcal{P}}_{2,0}$ is (see [2])

$$
(2.31)' \quad \sup_{0 < \tau < T} |\hat{\mathcal{P}}_{2,0}(y, \tau)| < b\|h\|_{c_1(0,b)} + N_{10}(K, K_1) t. 
$$

As a consequence of (2.27), (2.30), (2.30)', (2.31), (2.31)' we get

$$
(2.32) \quad \|V\|_{c_1(0,b)} < (1 + b)\|h^0\|_{c_1(0,b)} + N_{11}(K, K_1) t. 
$$

From (2.16) and (2.32) we deduce the desired estimate

$$
(2.33) \quad \|\mathcal{F}U\|_{c_2,0(c(0,b))} < c(b)\|h\|_{c_1(0,b)} + N_{12}(K, K_1) t. \quad (\ast). 
$$

Now, we choose the parameters entering $\mathcal{B}$ as follows

$$
(2.34) \quad K = \overline{K} = 2c(b)\|h\|_{c_1(0,b)}, \quad \gamma = \overline{\gamma} = \beta(\overline{K}), \quad K_1 = \overline{K}_1 = N_s(\overline{K})
$$

and

$$
(2.35) \quad T = \overline{T} = \min\{T_0, [c(b)\|h\|_{c_1(0,b)} / N_{12}(\overline{K}, \overline{K}_1)]^{\frac{1}{2}}\}. 
$$

We find

$$
(2.36) \quad \|\mathcal{F}U\|_{c_2,0(c(0,b)^{\frac{1}{2}})} < \overline{K}, 
$$

$$
(2.36)' \quad \|\mathcal{F}U\|_{c_2,0(c(0,b)^{\frac{1}{2}})} < \overline{K}_1, \quad \forall U \in B(\overline{K}, \overline{\gamma}, \overline{K}_1, \overline{T}).
$$

Therefore

$$
(2.37) \quad \mathcal{F}: B(\overline{K}, \overline{\gamma}, \overline{K}_1, \overline{T}) \rightarrow B(\overline{K}, \overline{\gamma}, \overline{K}_1, \overline{T}).
$$

(\ast) An alternative estimate of the type $c(b)\|h\|_{c_1(0,b)} + N(K)^{\frac{1}{2}}$ follows immediately from (2.11).
2.7. Continuity of $\mathcal{F}$

Take $U^{(1)}$, $U^{(2)} \in B(\overline{K}, \overline{K}, \overline{T})$ and let $(\overline{T}, S^{(1)}, V^{(1)}), (\overline{T}, S^{(2)}, V^{(2)})$ be the corresponding solutions of (2.1)-(2.6).

Set $V^{(i)} = 0$ for $x > S^{(i)}(t)$, $i = 1, 2$, and define

\begin{align*}
\delta(t) &= S^{(1)}(t) - S^{(2)}(t), \quad (2.38) \quad W(x, t) = V^{(1)}(x, t) - V^{(2)}(x, t). \\
(2.39)
\end{align*}

From (2.16) we have

\begin{align*}
\|\mathcal{F} U^{(1)} - \mathcal{F} U^{(2)}\|_{c_{1,0}(a(t))} \\
< N\{\sup_{t \in I} |\delta|_{c(0, t)} + t[\|\delta\|_{c(0, t)} + \|W_x(0, \cdot)\|_{c(0, t)} + \|U^{(1)}(0, \cdot) - U^{(2)}(0, \cdot)\|_{c(0, t)}]\}.
\end{align*}

(2.40)

The function

\begin{equation}
\tilde{W}(y, t) = W(ys(t), t)
\end{equation}

vanishes for $y = 0, y = 1, t = 0$, and satisfies the equation

\begin{equation}
L^{(1)} \tilde{W} = \tilde{\Theta} \quad (y, t) \in (0, 1) \times (0, \overline{T}),
\end{equation}

where $L^{(1)}$ is the operator defined in (2.28) with $S = S^{(1)}, \overline{U} = \overline{U}^{(1)}, \overline{V} = \overline{V}^{(1)}$, while the source term $\tilde{\Theta}$ can be dominated by

\begin{align*}
Nt^{-1}[\|\overline{U}^{(1)} - \overline{U}^{(2)}\|_{c_{1,0}(a(t) \times (0, t))} + \|\tilde{W}\|_{c(0, t) \times (0, t)} + \|\delta\|_{c(0, t)}]\}
= N\{\|\delta\|_{c(0, t)} + \|\tilde{W}\|\}.
\end{align*}

(2.41)

owing to (2.9)-(2.15).

Here and in the following, constants $N$ depend also on the second order derivatives of $a$ and $q$ (except for $\partial^2 / \partial t^2$).

Using the techniques of [2], Sec. 4, we obtain

\begin{equation}
\|\tilde{W}\|_{c_{1,0}(a(t) \times (0, t))} \leq Nt^{-1}[\|\delta\|_{c(0, t)} + \|\overline{U}^{(1)} - \overline{U}^{(2)}\|_{c_{1,0}(a(t) \times (0, t))}] \forall t \in (0, \overline{T}).
\end{equation}

(2.42)

On the other hand, from theorem 2 of [2], (4)

\begin{equation}
\|\delta\|_{c(0, t)} \leq N[\|\delta\|_{c(0, t)} + \|\overline{U}^{(1)} - \overline{U}^{(2)}\|_{c_{1,0}(a(t))}]\}
\end{equation}

(4) This theorem can be easily extended to cover the case in which $s(t)$ appears in $a$ and $q$. 

\end{document}
which implies

\begin{equation}
\| \delta \|_{c_{1,0}(0, T_1)} < N \| U^{(1)} - U^{(2)} \|_{c_{1,0}(0, T_1)},
\end{equation}

for \( t \) in a suitable time interval \((0, T_1) \subset (0, T)\). The final estimate resulting from (2.40), (2.43) and (2.44) is

\begin{equation}
\| \mathcal{F} U^{(1)} - \mathcal{F} U^{(2)} \|_{c_{2,0}(0, T_1)} < N \| U^{(1)} - U^{(2)} \|_{c_{1,0}(0, T_1)} \forall t \in (0, T_1).
\end{equation}

2.8. - Existence and uniqueness theorem

From (2.45) it follows that there exists \( \mathcal{T} \in (0, T_1) \) such that \( \mathcal{T} \) is a contractive mapping of \( B(K, \tilde{\rho}, K_1, \mathcal{T}) \) into itself, with respect to the norm of \( C_{1,0}(K) \). Since \( \mathcal{F}B \) is closed w.r.t. this norm, and \( \mathcal{F}^2 B \subset \mathcal{F}B \) the results of 2.5 allow us to state the following theorem

**Theorem 2.1.** Under the assumptions listed in 2.1, Problem (C1) has a solution \((\tilde{\mathcal{T}}, s, u)\), which is unique in \((0, \tilde{T})\). Moreover \( s \in H_{1+\gamma}[0, \tilde{T}] \) and \( u \in C_{1+\gamma}(\tilde{T}) \).

2.9. - Continuous dependence

Let \( \Sigma \) be a set of data and coefficients such that \( b_0 > b > b_0 > 0, \| h \|_{\mathcal{H}_{2+\alpha}} \) and \( \| \psi \|_{\mathcal{H}_{2+\alpha}} \) are uniformly bounded and such that \( a, q, \lambda \) satisfy assumptions (a)-(d) also in a uniform way. For any element \( \sigma = \{ b, h, \psi, a, q, \lambda \} \in \Sigma \) the a priori estimates on \((T, s, u)\) derived above are uniform and we can define the norm

\begin{equation}
\Delta a = \sup_{0 < x < \min(1/2, \tilde{T})} | a^{(1)} - a^{(2)} |,
\end{equation}

and similarly the norms \( \Delta a_x, \Delta a_u, \Delta a_{\psi}, \Delta a_q, \Delta a, \Delta q_x, \Delta q_u, \Delta q, \Delta \lambda, \) for any pair

\[ \sigma_1 = \{ b^{(1)}, h^{(1)}, \cdot \cdot \cdot , a^{(1)}, q^{(1)}, \lambda^{(1)} \}, \quad \sigma_2 = \{ b^{(2)}, h^{(2)}, \cdot \cdot \cdot , a^{(2)}, q^{(2)}, \lambda^{(2)} \}. \]

Using techniques similar to those employed in § 2.7 it can be proved that for any \( U \in B(K, \tilde{\rho}, K, \tilde{T}) \), \( \mathcal{F}U \) depends continuously on \( \sigma \in \Sigma \) in a uniform
way, with respect to the distance

\[ \varrho(\sigma_1, \sigma_2) = |b^{(1)} - b^{(2)}| + \| \tilde{h}^{(1)} - \tilde{h}^{(2)} \|_{c_1} + |\psi^{(1)} - \psi^{(2)}|_{c_2} \]

\[ + \Delta a + \Delta a_x + \Delta a_\rho + \Delta q + \Delta q_x + \Delta q_\rho + \Delta \lambda \]

(set \( \tilde{h}^{(1)} = 0 \) for \( x > b^{(1)} \). More precisely

\[ \| \mathcal{F}_{\sigma_1} U - \mathcal{F}_{\sigma_2} U \|_{c_2,0(a_0^{(0)\rho})} < N \varrho(\sigma_1, \sigma_2). \]

As a consequence (see [4], p. 630), the following theorem is proved

**Theorem 2.2.** For any pair \( \sigma_1, \sigma_2 \in \Sigma \) the corresponding solutions \( (\hat{T}; s^{(1)}; u^{(1)}), (\hat{T}; s^{(2)}; u^{(2)}) \) satisfy the inequalities

\[ \| u^{(1)} - u^{(2)} \|_{c_2,0} < N \varrho(\sigma_1, \sigma_2), \]

\[ \| s^{(1)} - s^{(2)} \|_{c_1} < N \varrho(\sigma_1, \sigma_2). \]

Note that (2.50) follows from (2.49) and (2.44).

**3. - Problem (C2)'**

The proof of well-posedness of Problem (C2)' follows the general scheme of §2, although many nontrivial modifications are needed. The main differences will be in the definition of the operator \( \mathcal{F} \) and in the choice of the functional spaces to be used.

**3.1. - Assumptions**

In addition to \((a)-(b)\) of §2.1 we shall assume

\[ \alpha' \quad h \in H_{s+\alpha}[0, b], \, \varphi \in H_{s+\alpha}[0, \hat{T}]; \]

\[ \beta' - \gamma' \quad a, q \text{ are independent of } s \text{ (and differentiable w.r.t. } t), \]

\[ \| a(0, 0, h(0), h'(0) - q(0, 0, h(0), h'(0)) = \varphi(0), \]

\[ \| a(b, 0, 0, 0) h'(b) - q(b, 0, 0, 0) = 0, \]
and satisfying standard growing conditions w.r.t. \( u, \ u_x \) (see e.g. thm. 5.2, p. 564 of [3]);

\[(\delta') \quad \lambda_z \text{ is continuously differentiable and} \]

\[|\lambda_z(x, t, u, p, \vartheta)| \lesssim \nu(|u|, |p|, |\vartheta|)\]

for some continuous nondecreasing function \( \nu \).

The independence of \( a \) and \( q \) of \( s \) has been introduced only for the sake of simplicity.

### 3.2. Auxiliary free-boundary problem

Let \( \Omega_t(\gamma, \bar{T}) \) be the set of the functions \( U \) defined in \( R_t^b \) such that

\[U \in C_{x+y}, \quad U(0, t) = \varphi(t), \quad U(x, 0) = h(x)\]

(with a smooth extension of \( h(x) \) for \( x > b \)).

For any \( U \in \Omega_t(\gamma, \bar{T}) \) we introduce the function

\[
\chi(x, t, U, U_x, Z, Z_x) = q_t + q_u Z + q_p Z_x - (a_t + a_u Z + a_p Z_x)(Z + q)/a,
\]

where the arguments of \( a, q \) and of their derivatives are \( x, t, U, U_x \).

Then we define a differentiable function \( \chi^a(x, t, U, U_x, Z, Z_x) \) such that

\[\chi^a = \chi \quad \text{for} \quad |Z| < Z_0,\]

\[|\chi^a(x, t, u, p, Z, 0)| \leq \sup_{|\vartheta| \leq x+1} |\chi(x, t, u, p, \vartheta, 0)|, \quad Z > Z_0 + 1,
\]

for some given \( Z_0 > 0 \), and we consider the following free boundary problem

\[a(x, t, U, U_x)Z_{xx} - Z_t = \chi^a(x, t, U, U_x, Z, Z_x) \quad \text{in} \quad D_T,
\]

\[S(0) = b,
\]

\[Z(x, 0) = a(x, 0, h, h')h^a(x) - q(x, 0, h, h'), \quad 0 < x < b,
\]

\[Z(0, t) = \varphi(t), \quad 0 < t < T,
\]
(3.9) \[ Z(S(t), t) = 0, \quad 0 < t < T, \]

(3.10) \[ \dot{S}(t) = \lambda_z(S(t), t, U(S(t), t), U_x(S(t), t), Z_x(S(t), t)), \quad 0 < t < T. \]

A unique solution \((T, S, Z)\) to (3.5)-(3.10) exists under the assumptions listed above (see [2]).

3.3. - Estimates on \((T, S, Z)\)

Let \(B_1(K, \gamma, K_1, T), \; T < \bar{T}\), be the subset of \(\Omega_1(\gamma, \bar{T})\) such that

(3.11) \[ \| U \|_{c_{2,0}(c_{0}^w)} < K, \quad \| U \|_{c_{1,0}(c_{0}^w)} < K_1. \]

If \(U \in B_1(K, \gamma, K_1, T)\), from [2] we have

(3.12) \[ T > T_0(K, \gamma, K_1, Z_0), \]

(3.13) \[ \| Z \|_{c_{1,0}(c_{0}^w)} < M_1(K, Z_0), \]

(3.14) \[ \| S \|_{c_{1,0}(c_{0}^w)} < M_2(K, \gamma, K_1, Z_0), \]

besides a nonuniform estimate of \(\| Z \|_{c_{2,1}}\) similar to (2.12). Like the constants \(N_1\) in § 2, also the constants \(M_i\) are obviously dependent on the data and the coefficients.

3.4. - Definition of the operator \(\mathcal{F}_1\)

For any \(U \in B_1(K, \gamma, K_1, T_0)\) we define

(3.15) \[ \mathcal{F}_1 U = \bar{U}, \]

where \(\bar{U}\) is the solution of the following nonlinear initial-boundary value problem

(3.16) \[ a(x, t, \bar{U}, \bar{U}_x) \bar{U}_{xx} - \bar{U}_t = q(x, t, \bar{U}, \bar{U}_x), \quad \text{in } D_{x_0}, \]

(3.17) \[ \bar{U}(x, 0) = h(x), \quad 0 < x < b, \]

(3.18) \[ \bar{U}(0, t) = \varphi(t), \quad 0 < t < T_0; \]

(3.19) \[ \bar{U}(S(t), t) = h(b) + \int_0^t \dot{S}(\tau) U_x(S(\tau), \tau) d\tau, \quad 0 < t < T_0. \]
Owing to the compatibility conditions assumed for \( h \) and \( \varphi \) at the points \((0,0), (b,0)\), this problem has a unique solution in \( C_{z+\beta} (\overline{D_{r_n}}) \) (Thm. 5.2, p. 564 of [3]). In (3.15) we mean that \( U \) is extended smoothly to \( R_{r_n}^{(b)} \) in a prescribed way. Therefore

\[
\mathcal{T}_1 : B_1(K, \gamma, \varepsilon K_1, T_0) \to \Omega_1(\beta, T_0).
\]

3.5. - Estimates on \( \mathcal{T} U \)

The norm \( \| \bar{U} \|_{c_{z+\beta}} \) is estimated in terms of the data and the coefficients and of \( \| S \|_{z+\beta}, \| U \|_{c_{z+\beta}} \), i.e.

\[
\| \bar{U} \|_{c_{z+\beta}} < M_z(K, \gamma, K_1, Z_0),
\]

where (3.14) has been used.

Estimate (3.21) implies

\[
\| \bar{U} \|_{c_{z+\beta}} < M_z(K, \gamma, K_1, Z_0) \beta_1(K),
\]

with \( M_z \) independent of \( K, \gamma, K_1, Z_0 \).

Moreover (see theorem 5.1, p. 561 of [3])

\[
\| \bar{U} \|_{c_{z+\beta}} < M_z(K).
\]

Taking e.g. \( \bar{K} = 2M_z, \bar{K}_1 = M_z(K), \bar{\gamma} = \beta(K) \) from (3.22), (3.23) we can find \( T_1 < T_2(\bar{K}, \bar{\gamma}, \bar{K}_1, Z_0) \) such that

\[
\| \bar{U} \|_{c_{z,\theta}(\varepsilon K_1)} < \bar{K}_1, \quad \| \bar{U} \|_{c_{z,\theta}(\varepsilon K_1)} < \bar{K}_1.
\]

At this point we remark that (3.3) and the maximum principle yield

\[
|Z(x, t)| < M_z(K) \left| 1 + \int_0^t \sup_{\varepsilon \in (0, \varepsilon(t))} |Z(x, \tau)|^2 \, d\tau \right|;
\]

hence for any sufficiently large \( Z_0 \) we can calculate a \( T_z(K) \) such that

\[
|Z(x, t)| < Z_0, \quad 0 < x < S(t), \quad 0 < t < T_z(K).
\]

Therefore, setting \( \bar{T} = \min \{ T_1, T_z(\bar{K}) \} \) we conclude that

\[
\mathcal{T}_1 : B_1(\bar{K}, \bar{\gamma}, \bar{K}_1, \bar{T}) \to B_1(\bar{K}, \bar{\gamma}, \bar{K}_1, \bar{T})
\]

and that \( \chi^* \) can be identified with \( \chi \) in (3.5).
3.6. - Solutions of (C2)' associated to fixed points of $\mathcal{F}_1$

Let $u$ be a fixed point of $\mathcal{F}_1$ and denote by $(T_0, s, z)$ the corresponding solution of (3.5)-(3.10).

From (3.19) we obtain $u_1(s(t), t) = 0$. Hence it is easy to see that the difference $z - u_1$ can be thought as the solution of a linear homogeneous parabolic equation with square summable coefficients and with zero initial and boundary data.

Therefore

$$u_1 = z \quad \text{in } D_{T_0},$$

which shows that $(T_0, s, u)$ solves (C2').

3.7. - Existence and uniqueness theorem

Using basically the same methods of Sec. 4 of [3], it can be seen that for the solutions $\hat{U}^{(1)}, \hat{U}^{(2)}$ of (3.16)-(3.19) corresponding to two respective elements $U^{(1)}, U^{(2)} \in B_1(\overline{K}, \overline{r}, \overline{R}, \overline{T})$, we have

$$\|\hat{U}^{(1)} - \hat{U}^{(2)}\|_{C_{1,0}(\overline{r})} \leq M(\overline{K}) t^\gamma \|U^{(1)} - U^{(2)}\|_{C_{1,0}(\overline{r})}, \quad \forall t \in (0, \overline{T}).$$

Thus the operator $\mathcal{F}_1$ is a contractive mapping w.r.t. the norm of $C_{1,0}(\overline{r})$ for some $\overline{T} \in (0, \overline{T}]$. The set $\mathcal{F}_1 B_1 \subset \mathcal{F}_1 B_1$ is closed w.r.t. such a norm and the existence of a unique solution $(\hat{T}, s, u)$ of (C2') is proved.

Theorem 3.1. Under the assumptions listed in § 3.1, Problem (C2)' has a solution $(\hat{T}, s, u)$, which is unique in $(0, \overline{T})$. Moreover $s \in H^{\gamma-\gamma} \left[0, \overline{T}\right]$ and $u_{xx} \in C_{1,0}(\overline{r})$.

3.8. - Continuous dependence

Let $\Sigma$ be the set whose elements $\sigma = \{b, h, \varphi, a, q, \lambda_2\}$ satisfy all the assumptions listed in 3.1 in a uniform way. For any $\sigma \in \Sigma$ we have a solution $(\hat{T}, s, u)$ of (C2') and a uniform estimate on the norm of $u$ in $C_{1,0}(\overline{r})$ is available. This allows us to define the quantities $\Delta a$, $\Delta q$, $\Delta a_t$, $\Delta q_t$, ..., $\Delta \lambda_2$ as in 2. Then we can state
Theorem 3.2. For any pair $\sigma_1$, $\sigma_2 \in \Sigma$ the corresponding solutions $(\hat{T}, s^{(1)}, u^{(1)}), (\hat{T}, s^{(2)}, u^{(2)})$ satisfy the inequalities

\begin{align}
\|u^{(1)} - u^{(2)}\|_{L^2(Q)} & \leq Ng(\sigma_1, \sigma_2), \\
\|s^{(1)} - s^{(2)}\|_{L^2(\partial)} & \leq Ng(\sigma_1, \sigma_2),
\end{align}

with

$$
\varrho(\sigma_1, \sigma_2) = |b_1 - b_2| + \|k^{(1)} - k^{(2)}\|_{C_0} + \|\varphi^{(1)} - \varphi^{(2)}\|_{C_0} \\
+ \Delta a + \Delta a_1 + \Delta a_u + \Delta q + \Delta q_a + \Delta q + \Delta q_a + \Delta q_s + \Delta \lambda_2.
$$

The proof of Theorem 3.2 is omitted for the sake of brevity.

References


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