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Bodies with microstructure (I) (**)

A GIORGIO SESTINI per il suo 70° compleanno

Introduction

The goal of these lectures (given at the International Centre for Theoretical Physics during the Autumn Course in 1976) (1) is an exposition of some aspects of the theory of bodies affected by dislocations. In introducing the subject, we adopt concepts and notation from a paper by Noll [**3**]$_1$ (pp. 211-242) or [**3**]$_2$ (see also [**6**], ch. V), but the approach is different in that material properties are not discussed; the kinematics (including changes of reference) is examined first; and a comparison is made with the so called director theories.

Basic notation is as follows.

$\mathscr{E}$ three-dimensional Euclidean space of points $\boldsymbol{x}$, $\boldsymbol{y}$, etc.

$\boldsymbol{a}$, $\boldsymbol{b}$, etc., vectors; i.e., elements of the translation space $\mathscr{V}$ of $\mathscr{E}$.

$\boldsymbol{G}$, $\boldsymbol{A}$, etc., second order tensors; i.e., elements of Lin, the space of linear mappings of $\mathscr{V}$ into $\mathscr{V}$.

$\mathbf{1}$, identity in Lin.

$\boldsymbol{G}^T$, transpose of $\boldsymbol{G}$.

$\det \boldsymbol{G}$, determinant of $\boldsymbol{G}$.

$\operatorname{tr} \boldsymbol{G}$, trace of $\boldsymbol{G}$.

w, f, etc., third order tensors; i.e., elements of Lin, the space of linear mappings of $\mathscr{V}$ into Lin.

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(1) The original text was amended in many ways, however. I wish to thank C. Davini and P. Podio Guidugli for many critical remarks.

Lin$^+$, the subset of invertible members of Lin.

Orth, the group of $\boldsymbol{Q} \in$ Lin such that $\boldsymbol{Q}^T\boldsymbol{Q} = \mathbf{1}$.

Sym [Skw], the subspace of $\boldsymbol{G} \in$ Lin, such that $\boldsymbol{G} = \boldsymbol{G}^T$, $[\boldsymbol{G} = -\boldsymbol{G}^T]$.

The composition of fields is expressed by use of the sign $\circ$.

For the tensorial product of vectors and tensors we use the symbol $\otimes$. By scalar product $\boldsymbol{G}\cdot\boldsymbol{A}$ we mean $\operatorname{tr}(\boldsymbol{G}\boldsymbol{A}^T)$.

Notice that *Lin* can be considered also as the space of linear mappings from Lin into $\mathscr{V}$, starting with the convention

$$\mathsf{w}(\boldsymbol{a}\otimes\boldsymbol{b}) = (\mathsf{w}\boldsymbol{b})\,\boldsymbol{a}\,, \qquad \forall \boldsymbol{a}, \boldsymbol{b} \in \mathscr{V}\,. \tag{1.1}$$

For instance, using the Ricci commutator $\mathsf{e} \in \mathit{Lin}$, given any $\boldsymbol{a} \in \mathscr{V}$, one can define an $\boldsymbol{A} \in$ Skw such that

$$\boldsymbol{A} = \mathsf{e}\boldsymbol{a}\,, \tag{1.2}$$

and vice versa, given $\boldsymbol{A} \in$ Skw one can recover $\boldsymbol{a}$ such that

$$\boldsymbol{a} = \tfrac{1}{2}\,\mathsf{e}\boldsymbol{A}\,. \tag{1.3}$$

We remark also that

$$\mathsf{e}(\boldsymbol{a}\otimes\boldsymbol{b}) = \boldsymbol{a}\times\boldsymbol{b}\,. \tag{1.4}$$

Over *Lin* it is possible to define a major and minor notion of transpose, i.e., for any $\mathsf{h} \in \mathit{Lin}$ one can obtain the major transpose h^T and the minor right or left transpose ${}^t\mathsf{h}$ and h^t through the conditions

$$\boldsymbol{A}\cdot\mathsf{h}\boldsymbol{a} = \boldsymbol{a}\cdot\mathsf{h}^T\boldsymbol{A}^T\,, \qquad \mathsf{h}\boldsymbol{a} = ({}^t\mathsf{h}\boldsymbol{a})^T\,, \qquad \mathsf{h}\boldsymbol{A} = \mathsf{h}^t\boldsymbol{A}\,, \qquad \forall \boldsymbol{a} \in \mathscr{V}\,,\ \boldsymbol{A} \in \mathrm{Lin}\,. \tag{1.5}$$

Correspondingly, there is a major property of symmetry (or antisymmetry) when

$$\mathsf{h} = \mathsf{h}^T\,, \qquad (\mathsf{h} = -\,\mathsf{h}^T) \tag{1.6}$$

and a minor property of left (or right) symmetry (or antisymmetry) when

$${}^t\mathsf{h} = \mathsf{h} \quad ({}^t\mathsf{h} = -\,\mathsf{h})\,, \qquad \mathsf{h}^t = \mathsf{h} \quad (\mathsf{h} = -\,\mathsf{h}^t)\,. \tag{1.7}$$

Many elementary properties follow; for instance: if h is right antisymmetric, than there exists a unique $\boldsymbol{A} \in$ Lin such that

$$\boldsymbol{A}\boldsymbol{u} = \mathsf{e}(\mathsf{h}^T\boldsymbol{u})\,, \qquad \mathsf{h}^T\boldsymbol{u} = \tfrac{1}{2}\,\mathsf{e}(\boldsymbol{A}\boldsymbol{u})\,, \qquad \forall \boldsymbol{u} \in \mathscr{V}\,. \tag{1.8}$$

We introduce several smooth scalar, vector and tensor fields, e.g. $\psi(x^*)$, $\boldsymbol{d}(\boldsymbol{x}^*)$, $\boldsymbol{K}(\boldsymbol{x}^*)$, etc. defined over open connected sets B^* of $\mathscr{E}$ (or over their closures), together with the gradients $\nabla\psi$, $\nabla\boldsymbol{d}$, $\nabla^2\boldsymbol{d}$, etc.

We will consider also one-to-one smooth mappings of open connected sets onto open connected sets, say

$$\boldsymbol{x} = \boldsymbol{a}(\boldsymbol{x}^*)\,, \qquad \boldsymbol{x}^* = \boldsymbol{a}^{(i)}(\boldsymbol{x})\;;$$
$$\boldsymbol{x}^* \in B^* = \text{domain}\ \boldsymbol{a} = \text{range}\ \boldsymbol{a}^{(i)}\,, \qquad \boldsymbol{x} \in B = \text{domain}\ \boldsymbol{a}^{(i)} = \text{range}\ \boldsymbol{a}\,.$$

Thus we can think of ψ, $\boldsymbol{d}$, $\boldsymbol{K}$, for instance, also as a fields on B, e.g. $\boldsymbol{d}(\boldsymbol{a}^{(i)}(\boldsymbol{x}))$; with an abuse of notation we write simply $\boldsymbol{d}(\boldsymbol{x})$; correspondingly we can introduce the gradient and we use for it the notation grad

$$\nabla\psi = (\nabla\boldsymbol{a})^T(\text{grad}\,\psi)\,,\ \nabla\boldsymbol{d} = (\text{grad}\,\boldsymbol{d})\,\nabla\boldsymbol{a}\;;\ (\nabla\boldsymbol{K})\,\boldsymbol{u} = (\text{grad}\,\boldsymbol{K})(\nabla\boldsymbol{a}\,\boldsymbol{u})\,, \qquad \forall\boldsymbol{u} \in \mathscr{V}\,.$$

Finally we need to introduce a whole class of smooth mappings of B^* into $\mathscr{E}$ and to work with fields over the ranges of each of those mappings; if γ is a symbol which specifies one of the mappings of the class, then we will call ∇_γ the respective gradient operator (so that we could use $\nabla_{\boldsymbol{a}}$ as an alternative symbol for the operator grad).

2. - Transplacements and placements

Let us now introduce the class $\mathscr{D}$ of *complete transplacements* as the set of couples $(\boldsymbol{a}, \boldsymbol{G})$, where $\boldsymbol{a}$ is chosen within a class $\mathscr{D}_1$, and $\boldsymbol{G}$ within a class $\mathscr{D}_2$, with the following properties:

(i) The members of $\mathscr{D}_1$ (called *apparent transplacements*) are invertible mappings (of class C^2 at least), whose domains and ranges are open connected subsets of $\mathscr{E}$; the value of $\boldsymbol{F} = \nabla\boldsymbol{a}$, the gradient of $\boldsymbol{a}$, at any element of the domain of $\boldsymbol{a}$ is a member of Lin^+.

(ii) The members of $\mathscr{D}_2$ (called *reference transplacements*) are mappings (of class C^1 at least) from open connected subsets of $\mathscr{E}$ into Lin^+.

(iii) In any couple $(\boldsymbol{a}, \boldsymbol{G}) \in \mathscr{D}$ the domains of $\boldsymbol{a}$ and $\boldsymbol{G}$ are the same.

(iv) If $(\boldsymbol{a}^{(1)}, \boldsymbol{G}^{(1)})$ and $(\boldsymbol{a}^{(2)}, \boldsymbol{G}^{(2)})$ belong to $\mathscr{D}$ and if the range of $\boldsymbol{a}^{(1)}$ coincides with the domain of $\boldsymbol{a}^{(2)}$, then $\mathscr{D}$ contains also the composition

$$(\boldsymbol{a}, \boldsymbol{G}) = (\boldsymbol{a}^{(2)}, \boldsymbol{G}^{(2)})\circ(\boldsymbol{a}^{(1)}, \boldsymbol{G}^{(1)})\,,$$

which is defined as follows: $\boldsymbol{a}$ is a simply $\boldsymbol{a}^{(2)}\circ\boldsymbol{a}^{(1)}$, whereas $\boldsymbol{G}$ is specified pointwise through the formula

$$\boldsymbol{G}(\boldsymbol{x}) = \big(\boldsymbol{G}^{(2)}(\boldsymbol{a}^{(1)}(\boldsymbol{x}))\big)\,\boldsymbol{G}^{(1)}(\boldsymbol{x})\,, \qquad \forall\boldsymbol{x} \in \text{domain}\ \boldsymbol{a}^{(1)}\,.$$

(v) If $(\boldsymbol{a}, \boldsymbol{G})$ belongs to $\mathscr{D}$, then also the inverse transplacement $(\boldsymbol{a}^{(i)}, \boldsymbol{G}^{(i)})$ belongs to $\mathscr{D}$; here $\boldsymbol{a}^{(i)}$ is the inverse mapping of $\boldsymbol{a}$ and the second member is the mappings from range $\boldsymbol{a}$ into Lin^+ defined pointwise by

$$\boldsymbol{G}^{(i)}(\boldsymbol{x}) = \big(\boldsymbol{G}(\boldsymbol{a}^{(i)}(\boldsymbol{x}))\big)^{-1}, \qquad \forall \boldsymbol{x} \in \text{range}\, \boldsymbol{a}\,.$$

(vi) $\mathscr{D}$ contains all complete rigid transplacements, i.e., couples $(\boldsymbol{a}^{(R)}, \boldsymbol{Q})$, where $\boldsymbol{a}^{(R)}$ is an isometry between open connected subsets of $\mathscr{E}$ and $\boldsymbol{Q}$ has a constant value on domain $\boldsymbol{a}^{(R)}$

$$\boldsymbol{Q} = \nabla \boldsymbol{a}^{(R)} \in \mathrm{Orth}\,.$$

Remark. If $\boldsymbol{a}, \boldsymbol{b} \in \mathscr{D}_1$ and domain $\boldsymbol{a}$ = domain $\boldsymbol{b}$, then, often, $\mathscr{D}$ is ample enough to contain $(\boldsymbol{a}, \nabla \boldsymbol{b})$, in particular to contain $(\boldsymbol{a}, \boldsymbol{F})$, $\boldsymbol{F} = \nabla \boldsymbol{a}$.

The latter property is true at least for $\boldsymbol{a}$ within an appropriate subset of $\mathscr{D}_1$, as condition (vi) above shows.

In general, however, $\boldsymbol{G}$ is not the gradient of a mapping $\boldsymbol{b}$.

We can now specify the notion of a *continuous body with* (affine) *microstructure*. $\mathscr{B}$ is a set of particles X with a number of properties; first of all, $\mathscr{B}$ is a smooth continuous body in the usual sense, i.e., a class $\mathscr{P}_1$ of mappings π exists, $\pi\colon \mathscr{B} \to \mathscr{E}$ (called *apparent placements*) such that:

(i) Every $\pi \in \mathscr{P}_1$ is one-to-one and its range $B = \pi(\mathscr{B})$ is an open connected subset of $\mathscr{E}$, which is called the region occupied by $\mathscr{B}$ in the (apparent) *placement* π. The spatial point $\boldsymbol{x} = \pi(X)$ is called the *place* of the particle X in the placement π.

(ii) If $\pi^{(1)}, \pi^{(2)} \in \mathscr{P}_1$, then there exists a member $\boldsymbol{a}^{(1,2)}$ of $\mathscr{D}_1$ such that

$$\boldsymbol{a}^{(1,2)} = \pi^{(2)} \circ (\pi^{(1)})^{-1}\,;$$

$\boldsymbol{a}^{(1,2)}$ is the apparent transplacement from $\pi^{(1)}(\mathscr{B})$ into $\pi^{(2)}(\mathscr{B})$.

(iii) If $\pi \in \mathscr{P}_1$, $\boldsymbol{a} \in \mathscr{D}_1$ and the range of π coincides with the domain of $\boldsymbol{a}$, then $\boldsymbol{a} \circ \pi \in \mathscr{P}_1$.

With this partial structure of $\mathscr{B}$ we can define *local placements* λ_X at each particle as equivalence classes of a partition $\mathscr{C}_X$ within placements $\pi \in \mathscr{P}_1$, induced by the following relation of equivalence at X. Let $\pi^{(1)}, \pi^{(2)} \in \mathscr{P}_1$, then $\pi^{(1)} \sim \pi^{(2)}$ if

$$\nabla\big(\pi^{(2)} \circ (\pi^{(1)})^{-1}\big)\big|_{\pi^{(1)}(X)} = \boldsymbol{1}\,.$$

We can also define a *reference* for $\mathscr{B}$ as a function Λ which associates with each particle a local placement; a very special case is a *homogeneous reference*, i.e. a reference which associates with each $X \in \mathscr{B}$ the equivalence class deduced from a unique placement. The set of all references on $\mathscr{B}$ will be indicated by $\mathscr{R}$.

We are now in the position to specify the complete structure of $\mathscr{B}$; it is characterized by a class $\mathscr{P}$ of mappings (π, Λ), called *complete placements* of $\mathscr{B}$, where $\pi \in \mathscr{P}_1$ and $\Lambda \in \mathscr{R}$ belongs to a class $\mathscr{P}_2$ with the following properties:

(i) Any Λ is smooth in the sense that any placement π of $\mathscr{P}_1$ admits of a smooth relative gradient $\boldsymbol{H} = \nabla_\Lambda \pi$ with respect to any Λ in $\mathscr{P}_2$; $\boldsymbol{H}$ is the field over B which is defined pointwise as follows. Let $\boldsymbol{x}$ be the place of the particle X in the placement π; let λ_X be the local placement associated with X by Λ; let γ be a placement belonging to the equivalence class λ_X such that $\gamma(X) = \pi(X) = \boldsymbol{x}$ (2). Then

$$\boldsymbol{H}(\boldsymbol{x}) = \nabla_\Lambda \pi \,|_{\boldsymbol{x}} = \nabla_\gamma(\pi \circ \gamma^{-1})\,|_{\boldsymbol{x}}\,. \tag{2.1}$$

Here ∇_γ is the gradient operator acting on fields defined on range γ, as mentioned at the end of **1**. Notice that in general a different γ needs be chosen for each particle, so that in $\nabla_\gamma(\pi \circ \gamma^{-1})$ operator, operand and domain vary with X; on the other hand one is interested only in the value of $\nabla_\gamma(\pi \circ \gamma^{-1})$ at $\boldsymbol{x}$.

(ii) If $(\pi^{(1)}, \Lambda^{(1)})$ and $(\pi^{(2)}, \Lambda^{(2)})$ belong to $\mathscr{P}$, then there exists a member $(\boldsymbol{a}^{(1,2)}, \boldsymbol{G}^{(1,2)})$ of $\mathscr{D}$ such that: (a) $\boldsymbol{a}^{(1,2)}$ is the apparent transplacement from $\pi^{(1)}$ into $\pi^{(2)}$ and (b) for all $\boldsymbol{x} \in \mathscr{B}$ the local placement $\lambda_X^{(2)}$ i.e. the value of $\Lambda^{(2)}$ at X, can be obtained from $\lambda_X^{(1)}$ (the value of $\Lambda^{(1)}$ at X) as the class of placements $\gamma^{(2)}$ for which

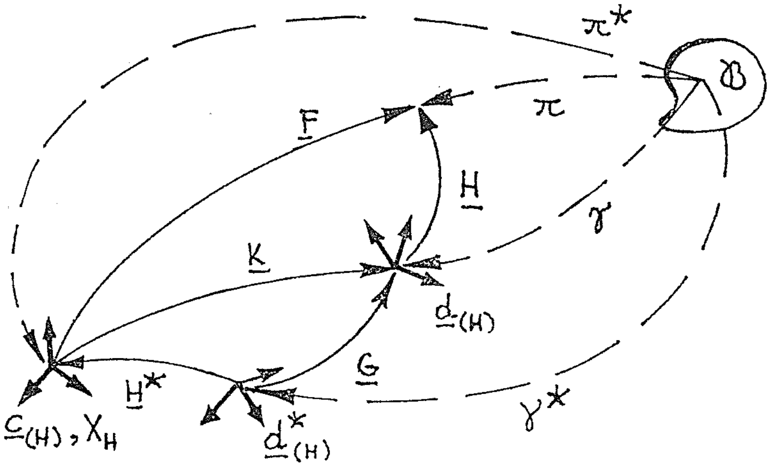
$$\boldsymbol{G}^{(1,2)}(\boldsymbol{x}) = \nabla\big(\gamma^{(2)} \circ (\gamma^{(1)})^{-1}\,|_x\big)\,, \qquad \forall \boldsymbol{x} \in \text{range}\, \pi^{(1)} \tag{2.2}$$

for any choice of $\gamma^{(1)} \in \lambda_x^{(1)}$ such that $\gamma^{(1)}(X) = \pi^{(1)}(X) = \boldsymbol{x}$ (see remark in footnote (2)). $(\boldsymbol{a}^{(1,2)}, \boldsymbol{G}^{(1,2)})$ is the complete transplacement from $(\pi^{(1)}, \Lambda^{(1)})$ into $(\pi^{(2)}, \Lambda^{(2)})$.

(iii) If $(\pi^{(1)}, \Lambda^{(1)}) \in \mathscr{P}$, $(\boldsymbol{a}, \boldsymbol{G}) \in \mathscr{D}$ and range $\pi^{(1)}$= domain $\boldsymbol{a}$, then $\mathscr{P}$ contains the composition $(\pi^{(2)}, \Lambda^{(2)}) = (\boldsymbol{a}, \boldsymbol{G}) \circ (\pi^{(1)}, \Lambda^{(1)})$, defined as follows $\pi^{(2)} = \boldsymbol{a} \circ \pi^{(1)}$, and $\Lambda^{(2)}$ has values $\lambda_X^{(2)}$ with the property that $\boldsymbol{G}(\boldsymbol{x}) = \nabla_{\gamma^{(1)}}(\gamma^{(2)}$

(2) Such a placement exists by property (iii) of apparent placements and property (vi) of complete transplacements. This remark is invoked repeatedly later.

$\circ(\gamma^{(1)})^{-1})|_x$ if $\gamma^{(2)}[\gamma^{(1)}]$ is any placement belonging to $\lambda_X^{(2)}[\lambda_X^{(1)}]$ and $\gamma^{(1)}(X) = \pi^{(1)}(X) = \boldsymbol{x}$. The composition is called the complete placement obtained from $(\pi^{(1)}, \Lambda^{(1)})$ by the transplacement $(\boldsymbol{a}, \boldsymbol{G})$.



3. - Directors

A particular complete placement (π^*, Λ^*), called here *primary placement*, is often preferred either for greater ease in the description of the material properties of a body or for other reasons. Then any other complete placement (π, Λ) is conveniently specified by (π^*, Λ^*) and the complete transplacement $(\boldsymbol{a}, \boldsymbol{G})$ from (π^*, Λ^*).

We can also identify a particle X with its primary place $\boldsymbol{x}^* = \pi^*(X)$; correspondingly we specify from now on the notation introduced in **2** as follows: we use the symbol ∇ for the gradient operator on fields defined on $B^* = \text{range}\, \pi^*$; we interpret $(\boldsymbol{a}, \boldsymbol{G})$ as the complete transplacement and $\boldsymbol{F}$ as the apparent transplacement gradient $\nabla \boldsymbol{a}$ from the placement (π^*, Λ^*) into a generic placement (π, Λ).

We mantain the notation $\boldsymbol{H}$ for the field introduced by (2.1) within the definition of complete placement and we introduce in a similar way the fields $\boldsymbol{H}^*$, $\boldsymbol{K}$:

$$(3.1) \qquad \boldsymbol{H}^* = \nabla_{\Lambda^*}\pi^* = \nabla\big(\pi^*\circ(\gamma^*)^{-1}\big)\,, \qquad \boldsymbol{K}^{-1} = \nabla_\Lambda \pi^* = \nabla(\pi^*\circ\gamma^{-1})\,;$$

we remark also that, with an appropriate choice of γ, γ^* within λ_X and λ_X^*, respectively,

$$G = \nabla(\gamma \circ (\gamma^*)^{-1}) \text{ and hence } \nabla_{A*}\pi = \nabla(\pi \circ (\gamma^*)^{-1}) = HG ,$$

and

$$K = H^{-1}F = GH^{*-1} . \tag{3.2}$$

In accordance with the conventions of **1**, we denote by $\nabla_A \varphi$ the gradient of any scalar field φ on B^* with respect to the reference in any placement (π, A) and, to distinguish it from the gradient with respect to the primary reference, we use for the latter the notation $\nabla_{A*}\varphi$

$$(\nabla_A \varphi)\cdot u = (\nabla\varphi)\cdot(K^{-1}u) , \qquad (\nabla_{A*}\varphi)\cdot u = (\nabla\varphi)\cdot(H^* u) , \qquad \forall u \in \mathscr{V} . \tag{3.3}$$

We operate with ∇_A, ∇_{A*} also on vector or tensor fields defined on B^*; for instance

$$(\nabla_A G)\, u = (\nabla G)(K^{-1}u) , \qquad \forall u \in \mathscr{V} . \tag{3.4}$$

We presume also that a coordinate system (for instance, a Cartesian frame) is available and denote with X^R $(R = 1, 2, 3)$ $[x^i$ $(i = 1, 2, 3)]$ the coordinates of a particle in π^* [in any π].

We can then introduce the *directors* for an alternative description of the structure of our body ([**4**], sect. 34); to this end in any complete placement and at any particle X, choose a placement γ which belongs to the equivalence class λ_X, consider the composition $\gamma \circ (\pi^*)^{-1}$ and define the vectors

$$d_{(R)} = \frac{\partial[\gamma \circ (\pi^*)^{-1}]}{\partial X^R} , \qquad R = 1, 2, 3 . \tag{3.5}$$

These vectors are the directors at the particle X within the complete placement considered; their definition involves a choice of γ but in fact they depend only on the choice of the coordinates and on λ_X.

We will use the notation $d^*_{(R)}$ for the directors in the primary placement:

$$d^*_{(R)} = \frac{\partial[\gamma^* \circ (\pi^*)^{-1}]}{\partial X^R} , \qquad R = 1, 2, 3; \quad \gamma^* \in \lambda^*_X . \tag{3.6}$$

Remark. The choice of directors within one placement has a large degree of arbitrariness. We could choose different coordinates $\tilde{X}^S$ in writing

(3.5). If

$$X^R = \psi^R(\tilde{X}^S) \qquad \text{and} \qquad A^R_S = \frac{\partial \psi^R}{\partial \tilde{X}^S}, \tag{3.7}$$

then a new set of directors $\tilde{\boldsymbol{d}}_{(S)}$ can be evidenced

$$\tilde{\boldsymbol{d}}_{(S)} = \sum_{1}^{3}{}_R\, \boldsymbol{d}_{(R)} A^R_S . \tag{3.8}$$

One may wonder to what use objects which are affected by such arbitrariness can be put within the theory. All relations deduced with their help must be tested for indifference with respect to changes of the type (3.8); thus an element of complexity is introduced which may appear to be quite unnecessary.

The signal advantage of the introduction of directors is in that it allows us to imagine a connection between the kinematics of bodies with microstructure and that of bodies made up of complex particles (say, dumbbell or tethrahedral), where directors are not simply cumbersome tools but have rather an immediate physical significance and their use in the theory is therefore fully legitimate.

Actually a reference on $\mathscr{B}$ can be defined using directors rather than local placements, once the apparent placement is known. In fact, when π is known and the field of directors $\boldsymbol{d}_{(R)}$ ($R = 1, 2, 3$) is given on B, one can determine, as we shall see, the field $\boldsymbol{H}$ on B and hence one element γ of λ_X for each X.

Let us start with some technical remarks: we use below also the set of reciprocal directors $\boldsymbol{d}^{(R)}$ which are defined by the relations

$$\boldsymbol{d}^{(R)} \cdot \boldsymbol{d}_{(S)} = \delta^R_S, \qquad \sum_{1}^{3}{}_R\, \boldsymbol{d}^{(R)} \otimes \boldsymbol{d}_{(R)} = \mathbf{1}, \tag{3.9}$$

and transform as a consequence of a change of coordinates as follows

$$\tilde{\boldsymbol{d}}^{(R)} = \sum_{1}^{3}{}_S\, B^R_S \boldsymbol{d}^{(S)} \qquad \text{where } B^R_S = (A^S_R)^{-1} .$$

Then if one calls $\boldsymbol{c}_{(R)}$ the base vectors of the coordinates, one obtains from (3.5), (2.1), (3.1), (3.2)

$$\boldsymbol{d}_{(R)} = \big(\nabla(\gamma \circ (\pi^*)^{-1})\big)\, \boldsymbol{c}_{(R)} = \big(\nabla(\gamma \circ \pi^{-1} \circ a)\big)\, \boldsymbol{c}_{(R)} = \boldsymbol{H}^{-1} \boldsymbol{F} \boldsymbol{c}_{(R)} = \boldsymbol{K} \boldsymbol{c}_{(R)}, \tag{3.10}$$

and in particular (see also (3.6), (3.1), (3.2))

$$\boldsymbol{d}^*_{(R)} = \boldsymbol{H}^{*-1} \boldsymbol{c}_{(R)}, \qquad \boldsymbol{d}_{(R)} = \boldsymbol{G} \boldsymbol{d}^*_{(R)} = \boldsymbol{G} \boldsymbol{H}^{*-1} \boldsymbol{c}_{(R)} . \tag{3.11}$$

In terms of K, H and G, the reciprocal directors are given by

$$d^{(R)} = (K^{-1})^T c^{(R)}\,, \qquad d^{*(R)} = H^{*T} c^{(R)}\,, \qquad d^{(R)} = (G^{-1})^T d^{*(R)}\,.$$

Vice versa, K, H, H^* can be specified in terms of $d_{(R)}$, $d^*_{(R)}$, their reciprocal directors and of the base vectors:

$$(3.12)\qquad \begin{aligned} K &= \sum_1^3{}_R\, d_{(R)} \otimes c^{(R)}\,, & K^{-1} &= \sum_1^3{}_R\, c_{(R)} \otimes d^{(R)}\,,\\ H &= \sum_1^3{}_R\, (F c_{(R)}) \otimes d^{(R)}\,, & H^* &= \sum_1^3{}_R\, c_{(R)} \otimes d^{*(R)}\,. \end{aligned}$$

If one adopts an intrinsic point of view (i.e., if one does not want to rely on coordinates) then the following relations become relevant

$$(3.13)\qquad \begin{aligned} &\sum_1^3{}_{RS}\, (c^{(R)}\cdot c^{(S)})\, d_{(R)} \otimes d_{(S)} = KK^T\,, \qquad G = \sum_1^3{}_R\, d_{(R)} \otimes d^{*(R)}\,,\\ &\sum_1^3{}_{RS}\, (c^{(R)}\cdot c^{(S)})\, d^*_{R} \otimes d^*_{(S)} = H^{*-1}(H^{*-1})^T\,. \end{aligned}$$

It follows that the directors alone determine completely G whereas in K and $(H^*)^{-1}$ an orthogonal right factor is left unspecified even when the metric of the coordinate system is given; such indetermination is obviously in connection with an arbitrariness of the choice of the coordinate frame, which affects equally F.

But it is interesting to remark finally that H is not correspondingly affected; because in the relation $H = FK^{-1}$, the indetermined factors cancel out.

In conclusion, the quantities introduced later may be of use in both theories, though their importance may be quite different in the two contexts.

4. - Geometric preliminaries

A number of important objects can be defined in terms of directors: first of all a metric tensor

$$(4.1)\qquad \gamma_{RS} = d_{(R)}\cdot d_{(S)} = (Kc_{(R)})\cdot(Kc_{(S)}) = (K^T K)\cdot(c_{(S)} \otimes c_{(R)})\,;$$

the quantities γ are therefore, in our interpretation of directors, the components of $K^T K$ on the coordinate frame.

Secondly the *linear connection* (see [**2**]$_1$, for instance)

$$\Gamma^T_{RS} = \frac{\partial \boldsymbol{d}_{(S)}}{\partial X^R} \cdot \boldsymbol{d}^{(T)} \, . \tag{4.2}$$

The quantities Γ^T_{RS} can be interpreted as the anholonomic components of a third order tensor: the *wryness* $\mathbf{w}$

$$\mathbf{w} = \sum_1^3{}_{RST} \Gamma^T_{RS} \boldsymbol{d}_{(T)} \otimes \boldsymbol{d}^{(S)} \otimes \boldsymbol{d}^{(R)} \, . \tag{4.3}$$

Directly $\mathbf{w}$ can be expressed by the relation

$$\mathbf{w} = \sum_1^3{}_{RS} \frac{\partial \boldsymbol{d}_{(R)}}{\partial X^S} \otimes \boldsymbol{d}^{(S)} \otimes \boldsymbol{d}^{(R)} \, ,$$

or, more compactly, by

$$\mathbf{w} = \sum_1^3{}_R \nabla_A \boldsymbol{d}_{(R)} \otimes \boldsymbol{d}^{(R)} \, . \tag{4.4}$$

In fact, from (3.4) one obtains [3]

$$\sum_1^3{}_S \frac{\partial \boldsymbol{d}_{(R)}}{\partial X^S} \otimes \boldsymbol{d}^{(S)} = \nabla_A \boldsymbol{d}_{(R)} \, . \tag{4.5}$$

One can express the wryness also in terms of the tensor $\boldsymbol{K}$ and its gradient. From (4.4), by substitution of (3.10), (3.3) one has, for any choice of $\boldsymbol{u}$, $\boldsymbol{v}$ in $\mathscr{V}$

$$(\mathbf{w}\boldsymbol{u})\boldsymbol{v} = \sum_1^3{}_R \big(\nabla(\boldsymbol{K}\boldsymbol{c}_{(R)})\big)(\boldsymbol{K}^{-1}\boldsymbol{v})\big(\boldsymbol{K}^{-1T}\boldsymbol{c}_{(R)}\big)\cdot\boldsymbol{u}\,\big) = \sum_1^3{}_R \Big(\big((\nabla\boldsymbol{K})(\boldsymbol{K}^{-1}\boldsymbol{v})\big)\,\boldsymbol{c}_{(R)}\Big)\Big(\big(\boldsymbol{K}^{-1T}\boldsymbol{c}_{(R)}\big)\cdot\boldsymbol{u}\Big).$$

On the other hand, for any $\boldsymbol{A}$, $\boldsymbol{B} \in \mathrm{Lin}$,

$$\sum_1^3{}_R (\boldsymbol{A}\boldsymbol{c}_{(R)}) \otimes (\boldsymbol{B}\boldsymbol{c}_{(R)}) = \boldsymbol{A}\boldsymbol{B}^T \, ; \tag{4.6}$$

hence

$$(\mathbf{w}\boldsymbol{u})\boldsymbol{v} = \big((\nabla\boldsymbol{K})(\boldsymbol{K}^{-1}\boldsymbol{v})\big)\;(\boldsymbol{K}^{-1}\boldsymbol{u}) \, , \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V} \, . \tag{4.7}$$

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(3) Choose in (3.4) $\boldsymbol{u} = \boldsymbol{d}_{(S)}$ and write $\boldsymbol{d}^{(R)}$ in place of $\boldsymbol{G}$.

Remark 1. If a vector field $\boldsymbol{b}$, defined on B^*, exists, such that $\boldsymbol{K} = \nabla \boldsymbol{b}$, then

$$\mathbf{w} = \mathbf{w}^t . \tag{4.8}$$

Vice versa, if (4.8) applies in a simply connected open neighbourhood of a point in B^*, then one can define in such neighbourhood a vector field $\boldsymbol{b}$, such that $\boldsymbol{K} = \nabla \boldsymbol{b}$.

Remark 2. If a vector field $\boldsymbol{b}$ and a tensor field $\boldsymbol{A}$, defined on B^*, exist, such that $\boldsymbol{K} = \boldsymbol{A}\boldsymbol{B}$, with $\boldsymbol{B} = \nabla \boldsymbol{b}$, then

$$\operatorname{skw}(\mathbf{w}^T \boldsymbol{u}) = \operatorname{skw}(\tilde{\mathbf{w}}^T \boldsymbol{u}) , \qquad \forall \boldsymbol{u} \in \mathscr{V} ,$$

if one puts

$$(\tilde{\mathbf{w}}\boldsymbol{u})\boldsymbol{v} = \big((\nabla_B \boldsymbol{A})(\boldsymbol{A}^{-1}\boldsymbol{v})\big)(\boldsymbol{A}^{-1}\boldsymbol{u}) , \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V} .$$

In particular (see formula (3.2)), while

$$(\mathbf{w}\boldsymbol{u})\boldsymbol{v} = \big((\operatorname{grad} \boldsymbol{H}^{-1})(\boldsymbol{H}\boldsymbol{u})\big)(\boldsymbol{H}\boldsymbol{v}) + \boldsymbol{H}^{-1}\big((\nabla \boldsymbol{F})(\boldsymbol{K}^{-1}\boldsymbol{u})\big)(\boldsymbol{K}^{-1}\boldsymbol{v}) , \ \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V} , \tag{4.9}$$

the skew part of $\mathbf{w}^T \boldsymbol{u}$ can be expressed in terms of the quantities appearing in the first addendum in the right-hand side of (4.9) only.

Remark 1 shows that one can take as a measure of the discrepancy of the reference from a homogeneous reference, the *inhomogeneity* defined as follows

$$\mathbf{s} = \tfrac{1}{2}(\mathbf{w} - \mathbf{w}^t) ; \tag{4.10}$$

or, explicity,

$$2(\mathbf{s}\boldsymbol{u})\boldsymbol{v} = \big((\nabla \boldsymbol{K})(\boldsymbol{K}^{-1}\boldsymbol{v})\big)(\boldsymbol{K}^{-1}\boldsymbol{u}) - \big((\nabla \boldsymbol{K})(\boldsymbol{K}^{-1}\boldsymbol{u})\big)(\boldsymbol{K}^{-1}\boldsymbol{v}) , \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V} . \tag{4.11}$$

From (4.9) one has also

$$2(\mathbf{s}\boldsymbol{u})\boldsymbol{v} = \big((\operatorname{grad} \boldsymbol{H}^{-1})(\boldsymbol{H}\boldsymbol{u})\big)(\boldsymbol{H}\boldsymbol{v}) - \big((\operatorname{grad} \boldsymbol{H}^{-1})(\boldsymbol{H}\boldsymbol{v})\big)(\boldsymbol{H}\boldsymbol{u}) , \ \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V} , \tag{4.12}$$

and it is easy to check that

$$2(\mathbf{s}\boldsymbol{u})\boldsymbol{v} = \boldsymbol{H}^{-1}\big(((\nabla_A \boldsymbol{H})\boldsymbol{v})\boldsymbol{u} - ((\nabla_A \boldsymbol{H})\boldsymbol{u})\boldsymbol{v}\big) , \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V} . \tag{4.13}$$

Notice that, to compare these formulae with those of [**3**]$_1$ (pp. 211-242), Noll's $\boldsymbol{F}$ is our $\boldsymbol{H}$, and there is a factor $\frac{1}{2}$ in our definition of $\boldsymbol{s}$.

5. - Dislocations

Because $\boldsymbol{s}$ satisfies the minor right antisymmetry condition, it is possible (see formulae (1.8)) to introduce a second order tensor $\boldsymbol{A}$ which is called *dislocation density* and is such that

$$\boldsymbol{A}\boldsymbol{u} = \mathsf{e}(\boldsymbol{s}^T\boldsymbol{u}) , \qquad \forall \boldsymbol{u} \in \mathscr{V} ; \tag{5.1}$$

vice versa, given $\boldsymbol{A}$, it is possible to express $\boldsymbol{s}$

$$\boldsymbol{s}^T\boldsymbol{u} = \tfrac{1}{2}\,\mathsf{e}(\boldsymbol{A}\boldsymbol{u}) . \tag{5.2}$$

The tensor $\boldsymbol{A}$ can be used to form the *Burgers vector* $\boldsymbol{b}$ relative to any plane of unit normal $\boldsymbol{n}$

$$\boldsymbol{b} = \boldsymbol{A}^T\boldsymbol{n} . \tag{5.3}$$

Example 1. Suppose that $\boldsymbol{c}_{(1)}$, $\boldsymbol{c}_{(2)}$, $\boldsymbol{c}_{(3)}$ are unit vectors in an orthogonal Cartesian frame and

$$\boldsymbol{d}_{(1)} = \varphi(X_2)\,\boldsymbol{c}_{(1)} , \qquad \boldsymbol{d}_{(2)} = \boldsymbol{c}_{(2)} , \qquad \boldsymbol{d}_{(3)} = \boldsymbol{c}_{(3)} ,$$

with $\varphi \neq 0$. Then all quantities Γ^L_{HK} vanish except at most Γ^1_{21} which equals $\varphi'\varphi^{-1}$; it follows that all Cartesian components of $\boldsymbol{s}$ vanish except at most $s_1{}^{12} = -\frac{1}{2}\varphi'\varphi^{-1}$ and $s_1{}^{21} = \frac{1}{2}\varphi'\varphi^{-1}$.

Again, all cartesian components of $\boldsymbol{A}$ vanish except at most $A_{31} = \varphi'\varphi^{-1}$. Hence

$$\boldsymbol{A} = \varphi'\varphi^{-1}\boldsymbol{c}_{(3)} \otimes \boldsymbol{c}_{(1)} . \tag{5.4}$$

If $\varphi = \exp[\alpha X_2]$, we have a homogeneous field of *edge dislocations*, where the Burgers vector is $\alpha\boldsymbol{c}_{(1)}$ and the edge unit vector is $\boldsymbol{c}_{(3)}$.

Example 2. Suppose

$$\boldsymbol{d}_{(1)} = \boldsymbol{c}_{(1)} + \varphi(X_2)\,\boldsymbol{c}_{(3)} , \qquad \boldsymbol{d}_{(2)} = \boldsymbol{c}_{(2)} , \qquad \boldsymbol{d}_{(3)} = \boldsymbol{c}_{(3)}$$

and hence, $\boldsymbol{d}^{(1)} = \boldsymbol{c}_{(1)}$, $\qquad \boldsymbol{d}^{(2)} = \boldsymbol{c}_{(2)}$, $\qquad \boldsymbol{d}^{(3)} = -\varphi(X_2)\,\boldsymbol{c}_{(1)} + \boldsymbol{c}_{(3)}$.

All Cartesian components of $\boldsymbol{s}$ vanish except, at most, $s_3{}^{21} = \varphi'(X_2)$ and $s_3{}^{12} = -\varphi'(X_2)$.

All Cartesian components of $\boldsymbol{A}$ vanish except, at most, $A_{33} = \varphi'$

$$\boldsymbol{A} = \varphi' \boldsymbol{c}_{(3)} \otimes \boldsymbol{c}_{(3)} . \tag{5.5}$$

If $\varphi = \alpha X_2$, we have a homogeneous field of *screw dislocations*, where the Burgers vector is $\alpha \boldsymbol{c}_{(3)}$ and the screw unit vector is $\boldsymbol{c}_{(3)}$.

In general one obtains from (5.1), (4.10), (4.3) that $\boldsymbol{A}$ can be expressed as a sum of three dyads

$$\boldsymbol{A} = \sum_1^3{}_L \boldsymbol{f}^{(L)} \otimes \boldsymbol{d}_{(L)} \qquad \text{with} \qquad \boldsymbol{f}^{(L)} = \sum_1^3{}_{HK} \Gamma^L_{HK} \boldsymbol{d}^{(K)} \times \boldsymbol{d}^{(H)} , \tag{5.6}$$

so that the Burgers vector relative to the plane of unit normal $\boldsymbol{d}^{(M)}/|\boldsymbol{d}^{(M)}|$, is given by

$$\boldsymbol{b}^{(M)} = \boldsymbol{f}^{(M)}/|\boldsymbol{d}^{(M)}| . \tag{5.7}$$

This vector has a screw component

$$b_s^{(M)} = \boldsymbol{b}^{(M)} \cdot \frac{\boldsymbol{d}^{(M)}}{|\boldsymbol{d}^{(M)}|} = -(\det \gamma_{RS})^{-\frac{1}{2}}(|\boldsymbol{d}^{(M)}|)^{-2}(\Gamma^M_{M+1,M+2} - \Gamma^M_{M+2,M+1}) , \tag{5.8}$$

and two edge components $b_{e1}^{(M)}$ and $b_{e2}^{(M)}$:

$$\begin{aligned} b_{e1}^{(M)} &= \boldsymbol{b}^{(M)} \cdot \frac{\boldsymbol{d}_{(M+1)}}{|\boldsymbol{d}_{(M+1)}|} \\ &= -(\det \gamma_{RS})^{-\frac{1}{2}}(|\boldsymbol{d}_{(M)}|\,|\boldsymbol{d}_{(M+1)}|)^{-1} \sum_1^3{}_R (\Gamma^M_{R,R+1} - \Gamma^M_{R+1,R}) \gamma_{R+2,M+1} , \\ b_{e2}^{(M)} &= \boldsymbol{b}^{(M)} \cdot \frac{\boldsymbol{d}_{(M+2)}}{|\boldsymbol{d}_{(M+2)}|} \\ &= -(\det \gamma_{RS})^{-\frac{1}{2}}(|\boldsymbol{d}_{(M)}|\,|\boldsymbol{d}_{(M+2)}|)^{-1} \sum_1^3{}_R (\Gamma^M_{R,R+1} - \Gamma^M_{R+1,R}) \gamma_{R+2,M+2} . \end{aligned} \tag{5.9}$$

Certain local properties of the dislocation density can be easily expressed in terms of the spectral properties of $\boldsymbol{A}$. For instance

I. *In each point of* B *there is a plane such that the Burgers vector relative to it is either null or of the screw type.*

In fact $\boldsymbol{A}$ has at least one real (perhaps null) eigenvalue.

II. *If* $\operatorname{sym}\boldsymbol{A}$ *is definite there is no plane such that the Burgers vector relative to it is non null and purely of the edge type.*

In fact for $\boldsymbol{b}$ relative to $\boldsymbol{n}$ it should be, otherwise,

$$\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{n}=0 \qquad \text{or} \qquad \boldsymbol{n}\cdot(\operatorname{sym}\boldsymbol{A})\boldsymbol{n}=0\,.$$

III. *If* $\operatorname{sym}\boldsymbol{A}$ *is indefinite there is (at least) a simple infinity of planes such that the relative Burgers vector is either null or purely of the edge type.*

Remarks 1 and 3 suggest simple examples of alternative elementary interpretations of the properties of $\boldsymbol{A}$. For instance, if α is any real number and $\boldsymbol{c}$, $\boldsymbol{d}$ are orthogonal unit vectors, then the special case

$$\boldsymbol{A}=\alpha(\boldsymbol{c}\otimes\boldsymbol{c}-\boldsymbol{d}\otimes\boldsymbol{d})=\frac{\alpha}{2}\big((\boldsymbol{c}+\boldsymbol{d})\otimes(\boldsymbol{c}-\boldsymbol{d})+(\boldsymbol{c}-\boldsymbol{d})\otimes(\boldsymbol{c}+\boldsymbol{d})\big)$$

can be interpreted as the superposition of dislocation densities either of the screw type, or of the edge type $[\mathbf{2}]_2$.

Nevertheless one could use the spectral properties of $\boldsymbol{A}$ systematically to propose a formal classification of cases.

6. - Strain

Now we are ready to compare placements, using for this purpose the objects defined in the previous sections. Our task is mainly that of finding conditions under which two placements differ at most by a rigid transplacement. One of the placements will be the primary placement and all quantities relating to it will again be marked with an asterisk.

Obviously relevant quantities are the transplacement gradient $\boldsymbol{F}=\nabla\boldsymbol{a}$, and $\boldsymbol{G}$. The corresponding strain tensors are the classical tensor

$$\boldsymbol{E}=\tfrac{1}{2}(\boldsymbol{F}^T\boldsymbol{F}-\mathbf{1})\,, \tag{6.1}$$

and a new one

$$\boldsymbol{B}=\tfrac{1}{2}(\boldsymbol{G}^T\boldsymbol{F}-\mathbf{1})\,. \tag{6.2}$$

As an alternative to $\boldsymbol{B}$ one could use the symmetric tensor

$$E^{[G]} = \tfrac{1}{2}(G^T G - 1) \tag{6.3}$$

and a second tensor

$$E^{[R]} = \tfrac{1}{2}\big((R^{[G]})^T R^{[F]} - 1\big) , \tag{6.4}$$

which involves the orthogonal tensors appearing in the polar decompositions

$$F = R^{[F]} U^{[F]} , \qquad G = R^{[G]} U^{[G]} , \tag{6.5$_I$}$$

where the *right stretch tensors* $U^{[F]}$ and $U^{[G]}$ are given by

$$U^{[F]} = (F^T F)^{\frac{1}{2}} , \qquad U^{[G]} = (G^T G)^{\frac{1}{2}} . \tag{6.5$_{II}$}$$

Notice that, whereas a rigid transplacement imposed upon $\mathscr{B}$ affects equally F and G: $F \to QF$, $G \to QG$ (where Q is the orthogonal tensor associated with the transplacement), there is no effect on either E, B, $E^{[G]}$, $E^{[R]}$.

In fact, *a necessary and sufficient condition for the rigidity of the complete transplacement* (a, G) *is that* E *and* B, *or alternatively* E *and both* $E^{[G]}$, $E^{[R]}$ *be zero over* B^*.

Proof. If E vanishes, then the apparent transplacement is rigid and F is orthogonal; furthermore, if $B = 0$, then G coincides with F.

If, alternatively, the apparent transplacement is rigid and $E^{[R]} = E^{[G]} = 0$, then $R^{[G]} = F$ and finally also $G = F$.

Sometimes, instead of F and G, the tensors H and K are used and we remark that, for them, the rules of transformation as consequence of a rigid transplacement are

$$K \to QK , \qquad K \to QHQ^T .$$

Polar decompositions for K and H

$$K = R^{[K]} U^{[K]} , \quad H = R^{[H]} U^{[H]} \tag{6.6}$$

lead to strain tensors

$$E^{[K]} = \tfrac{1}{2}\big((U^{[K]})^2 - 1\big) , \qquad E^{[H]} = \tfrac{1}{2}\big((U^{[H]})^2 - 1\big) , \tag{6.7}$$

which are related to $\boldsymbol{E}$

(6.8) $$\boldsymbol{E} = \boldsymbol{K}^T \boldsymbol{E}^{(H)} \boldsymbol{K} + \boldsymbol{E}^{(K)} .$$

However, the vanishing of $\boldsymbol{E}^{[H]}$ and $\boldsymbol{E}^{[K]}$ on B assure the rigidity of the apparent transplacement, but not necessarily that of the complete transplacement even if $\boldsymbol{H}^* = \mathbf{1}$.

If we return now to formula (4.7) and express the quantities appearing in it in terms of $\boldsymbol{G}$ and of elements of the primary placement we obtain the relation

$$(\mathbf{w}\boldsymbol{u})\boldsymbol{v} = (\nabla_{A*}\boldsymbol{G})(\boldsymbol{G}^{-1}\boldsymbol{u})(\boldsymbol{G}^{-1}\boldsymbol{v}) + \boldsymbol{G}\big(\mathbf{w}^*(\boldsymbol{g}^{-1}\boldsymbol{u})(\boldsymbol{g}^{-1}\boldsymbol{v})\big) , \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V} .$$

This formula suggests the introduction of the following tensor $\mathbf{f}$, the *strain of orientation* (see [**5**], sect. 61)

(6.9) $$(\mathbf{f}\boldsymbol{u})\boldsymbol{v} = \boldsymbol{G}^{-1}\big(\mathbf{w}(\boldsymbol{G}\boldsymbol{u})\big)(\boldsymbol{G}\boldsymbol{v}) - (\mathbf{w}\boldsymbol{u})\boldsymbol{v} , \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V} .$$

Because

(6.10) $$\mathbf{f}\boldsymbol{u} = \boldsymbol{G}^{-1}\big((\nabla_{A*}\boldsymbol{G})\boldsymbol{u}\big) = \boldsymbol{G}^{-1}\big((\nabla\boldsymbol{G})\boldsymbol{H}^*\boldsymbol{u}\big) , \qquad \forall \boldsymbol{u} \in \mathscr{V} ,$$

the vanishing of $\mathbf{f}$ *over* B^* *implies that* $\boldsymbol{G}$ *is a constant tensor there. If furthermore* $\mathbf{E} = \mathbf{0}$ *in* B^* *and* $\mathbf{B}$ *vanishes at least in one place, then the complete transplacement is rigid.*

It is useful to introduce the notation

(6.11) $$\mathbf{g} = \tfrac{1}{2}(\mathbf{f} - \mathbf{f}^t)$$

and to notice that

(6.12) $$(\mathbf{g}\boldsymbol{u})\boldsymbol{v} = \boldsymbol{G}^{-1}\big(\mathbf{s}(\boldsymbol{G}\boldsymbol{u})\big)(\boldsymbol{G}\boldsymbol{v}) - (\mathbf{s}^*\boldsymbol{u})\boldsymbol{v} , \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V} .$$

Notice also from (6.9) that the anholonomic components of $\mathbf{f}$ on the directors in the primary placement are related with the differences of the relevant linear connections

(6.13) $$\boldsymbol{d}^{*(H)} \cdot (\mathbf{f}\boldsymbol{d}^*_{(L)})\, \boldsymbol{d}^*_{(K)} = \Gamma^H_{KL} - \Gamma^{*H}_{KL} ;$$

correspondingly

(6.14) $$\boldsymbol{d}^{*(H)} \cdot (\mathbf{g}\boldsymbol{d}^*_{(L)})\, \boldsymbol{d}^*_{(K)} = \Gamma^H_{[KL]} - \Gamma^{*H}_{[KL]} ,$$

with the usual convention regarding indices between square brackets.

7. - Kinematics

We consider now a *motion* of $\mathscr{B}$, i.e. a one-parameter family of complete placements; it can be specified by assigning the complete transplacement from the primary placement as a function of the parameter t over an interval $(0, d)$

$$\boldsymbol{x} = \boldsymbol{a}(X, t) , \qquad \boldsymbol{G} = \boldsymbol{G}(X, t) .$$

Beside the classical velocity gradient $\boldsymbol{V} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$, another field over B^* becomes then important: the *wrenching*

$$\boldsymbol{W} = \dot{\boldsymbol{G}}\boldsymbol{G}^{-1} . \tag{7.1}$$

If the complete transplacement is rigid then

$$\operatorname{sym} \boldsymbol{V} = \operatorname{sym} \boldsymbol{W} = \boldsymbol{0} , \qquad \operatorname{skw} \boldsymbol{V} = \operatorname{skw} \boldsymbol{W} ;$$

this remark suggests the interest of the decompositions

$$\boldsymbol{V} = \mathbf{e}\boldsymbol{\omega}^{(V)} + \operatorname{sym} \boldsymbol{V} , \qquad \boldsymbol{W} = \mathbf{e}\boldsymbol{\omega}^{(W)} + \operatorname{sym} \boldsymbol{W} , \tag{7.2}$$

where $\boldsymbol{\omega}^{(V)}$ and $\boldsymbol{\omega}^{(W)}$ are the vectors associated with $\operatorname{skw} \boldsymbol{V}$ and $\operatorname{skw} \boldsymbol{W}$, respectively. Notice, however, that $\boldsymbol{\omega}^{(V)} = \boldsymbol{\omega}^{(W)}$ does *not* necessarily imply $\boldsymbol{R}^{(F)} = \boldsymbol{R}^{(G)}$.

The gradient of $\boldsymbol{W}$ is simply related with the time-derivative of the strain of orientation

$$(\dot{\boldsymbol{f}}\boldsymbol{u})\boldsymbol{v} = \boldsymbol{G}^{-1}\big((\nabla_{A*} \boldsymbol{W})\boldsymbol{u}\big((\boldsymbol{G}\boldsymbol{v}\big) = \boldsymbol{G}^{-1}\big((\nabla \boldsymbol{W})(\boldsymbol{H}^*\boldsymbol{u})\big)(\boldsymbol{G}\, \boldsymbol{v}) , \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V} . \tag{7.3}$$

The question now arises of associating with a motion an appropriate measure of inertia; there is no unique answer to the question: any specific assumption seems to derive from constitutive ideas regarding the bodies under analysis.

In the classical theory of elasticity local placements, defined once for all and hence intended as invariable with time, provide simply the ground structure for the description of inhomogeneities (see $[\mathbf{3}]_1$); linear momentum ($\varrho\dot{\boldsymbol{x}}$ per unit volume; ϱ, mass density) is the only measure of inertial effects.

In other classical theories (continua of Maxwell type, for instance) the parameters characterizing local placements are taken as internal variables entering the constitutive equations; the measure of inertia, however, remains the linear momentum.

In some more recent theories (the theory of liquid crystals, for instance) the body is thought of as a continuous distribution of grains which can be subject to affine deformations independent of the deformation of the medium into which they are imbedded. Then, beyond the usual momentum, one accounts (see, for instance, [**1**]$_{1,2}$) also for a generalized moment of momentum per unit mass $\varrho\boldsymbol{S}$ of a molecular character, expressed with the use of a tensor $\boldsymbol{S}$ obtained as the product of the wrenching $\boldsymbol{W}$ by a Eulerian inertia tensor $\boldsymbol{I}$

$$\boldsymbol{S} = \boldsymbol{I}\boldsymbol{W}^T . \tag{7.4}$$

$\boldsymbol{I}$ satisfies an evolution equation

$$\dot{\boldsymbol{I}} = \operatorname{sym} \boldsymbol{S} , \tag{7.5}$$

which, in a sense, parallels the usual equation of continuity

$$\dot{\varrho} + \varrho \operatorname{div} \dot{\boldsymbol{x}} = 0 . \tag{7.6}$$

$\boldsymbol{S}$ must be added to the usual measure $\boldsymbol{x} \otimes \dot{\boldsymbol{x}}$ to form the total generalized moment of momentum per unit mass.

Correspondingly one must attribute to our body the kinetic energy per unit mass expressed by

$$\tau = \tfrac{1}{2}\left(\dot{\boldsymbol{x}}^2 + (\boldsymbol{I}\boldsymbol{W}^T)\cdot \boldsymbol{W}^T\right) . \tag{7.7}$$

One can construct also a refined version of, say, the theory of Maxwell continua where direct account is taken of the inertia associated with the movement of dislocations. The micrograins are assumed then to partecipate largely in the motion of the whole body, precisely in so far as their deformation is a compatible (holonomic) one. The excess deformation calls for an additional momentum, which can be presumed to be measured by an expression of the type $\boldsymbol{J}\mathring{\boldsymbol{A}}$ where $\boldsymbol{J}$ is an appropriate fourth order inertia tensor, and $\boldsymbol{A}$ is again the density of dislocations.

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Riassunto

Si trattano alcuni aspetti della teoria dei continui con struttura; in questa prima parte: le specifiche geometriche, le deformazioni e la cinematica. Nell'esposizione si segue essenzialmente lo schema di alcune lezioni tenute nell'autunno 1976 *presso il Centro Internazionale di Fisica Teorica* (*Trieste*), *anche se il testo è stato ampiamente riveduto. Dopo aver adattato definizioni introdotte da Noll in modo da poter trattare anche piazzamenti locali variabili nel tempo, si esplorano i legami delle quantità così definite con altre legate alla nozione di direttori. Si introducono poi misure appropriate di deformazione e di velocità di deformazione e se ne studiano alcune proprietà fondamentali.*

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