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Equivalences, congruences and decompositions in semigroups (**)

A GIORGIO SESTINI per il suo 70° compleanno

Introduction

In **1** an equivalence relation ϱ_a is considered in a semigroup S which is useful in different lines (magnifying elements, topological semigroups, etc.) and we study some basic properties of this equivalence. In **2** we assume that S has a subsemigroup $\overline{S}$ with given property and we show that ϱ_a is a congruence relation and we introduce a quotient semigroup of S.

In **3** necessary and sufficient conditions are given in order that $\overline{S}$ be a subsemigroup of S with prescribed property.

Remark. $K(S)$ will denote a minimal ideal of S, $E(S)$ — the set of all idempotent elements of S. Moreover, if A, B are subsemigroups of S, then $A \subset B$ means that A is a proper subset of B.

1. - Let S be a semigroup.

Definition 1.1. Let $a \in S$. We define a relation ϱ_a by

$$x \varrho_a y \Leftrightarrow ax = ay \qquad (x, y \in S) .$$

ϱ_a is an equivalence relation.

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Let $C(a, x) = \{y \in S \mid ax = ay\}$ the equivalence class of x. The equivalence ϱ_a defines a partition $\pi_a(S)$ of S where the parts of $\pi_a(S)$ are the elements of the quotient set $S/\varrho_a = \{C(a, x) \mid x \in S\}$.

Theorem 1.2.

(i) $C(a, x) \subseteq C(sa, x)$, $\forall s \in S$.

(ii) *If a is a left cancellable element of S, then every class $C(a, x)$ consists of a single element.*

(iii) *If S is left simple, then $\pi_a(S) = \pi_b(S)$ for all $a, b \in S$.*

(iv) *If $ax = bx$ holds for all $x \in S$, then $\pi_a(S) = \pi_b(S)$.*

Proof. (i) is evident.

(ii) $ax = ay$ implies $x = y$, thus $C(a, x) = \{x\}$, $\forall x \in S$.

(iii) It holds $Sa = S$ for all $a \in S$. Let $y \in C(a, x)$. Then for any element b of S there is an element $s \in S$ such that $b = sa$. Hence $C(a, x) \subseteq C(b, x)$ by (i). The converse inclusion can be obtained similarly, and thus $C(a, x) = C(b, x)$ for each $x \in S$, that is $\pi_a(S) = \pi_b(S)$.

(iv) Let $y \in C(a, x)$, i.e. $ay = ax$. But $ax = bx$ and $ay = by$, whence $bx = by$, $y \in C(b, x)$ and $C(a, x) \subseteq C(b, x)$. Similarly, $C(b, x) \subseteq C(a, x)$ and we get $C(a, x) = C(b, x)$ for all $x \in S$. Thus Theorem 1.2. is proved.

Remarks. (a) In general, $\pi_a(S) = \pi_b(S)$ does not imply $ax = bx$, $\forall x \in S$.

(b) If S is a left zero semigroup, then $ax = ay = a$, $\forall y \in S$ and thus $C(a, x) = S$, $\forall a, x \in S$, that is $\pi_a(S)$ has a single class ($\forall a \in S$).

(c) Let a be a left magnifying element of S, i.e. $aM = S$ holds for a proper subset M of S. Then every class $C(a, x)$ of $\pi_a(S)$ contains at least one element of M. Indeed, there is an element $m \in M$ such that $ax = am$, whence $m \in C(a, x)$. Choosing an element $\overline{m}_i$ in $C(a, x_i)$ $(i \in I)$, then $\overline{M} = \{\overline{m}_i; i \in I\}$ is a minimal subset of S having the property $a\overline{M} = S$ (cfr. also [2]).

Theorem 1.3. *Let S be a semigroup, $e \in E(S)$ such that Se is a minimal left ideal of S. If s is an element of S such that $es = ese$, then $\pi_e(S) = \pi_{es}(S)$.*

Proof. We have to show that $esx = esy$ $(x, y \in S)$ implies $ex = ey$ and conversely. Let $esx = esy$. Since Se is a minimal left ideal of S, eSe is the maximal subgroup of S containing e. Denote $(ese)^{-1}$ the inverse of ese in eSe. Then $ex = (ese)^{-1} esex = (ese)^{-1} esx = (ese)^{-1} esy = (ese)^{-1} esey = ey$.

Conversely, let $ex = ey$. Then $esx = es(ex) = es(ey) = (ese)y = esy$.

Theorem 1.3. is completely proved.

The converse of Theorem 1.3. holds if S is right reductive, i.e. $ax = bx$ $(\forall x \in S)$ implies $a = b$ $(a, b \in S)$.

Theorem 1.4. *Let S be a right reductive semigroup, $e \in E(S)$. If s is an element of S such that $\varrho_e \subseteq \varrho_{es}$, then $es = ese$.*

Proof. By hypothesis, $ex = ey$ implies $esx = esy$ $(x, y \in S)$. Hence for each $x \in S$ we have $ex = e(ex)$, i.e. $(es)x = (es)ex = (ese)x$. Since S is right reductive, we get $es = ese$.

Theorem 1.3. and Theorem 1.4. imply the following

Theorem 1.5. *If S is a right reductive semigroup and $e \in E(S)$ such that Se is a minimal left ideal of S, then the following conditions are equivalent*

(i) $es = ese$;

(ii) $\pi_e(S) = \pi_{es}(S)$ $\quad (s \in S)$.

The next result is known (see [1], theorem 1.17), we prove it for the sake of completeness.

Theorem 1.6. *Let $K(S)$ be a completely simple minimal ideal of S. If $e \in E(K(S))$, the following are equivalent*

(i) $es \in Se$,

(ii) $es = ese$,

(iii) $Ls \subseteq L$, *where* $L = Se$ *is a minimal left ideal,*

(iv) $fs \in Sf$ *for all* $f \in E(L) = E(K(S)) \cap L$.

Proof. (*i*) $\Rightarrow$ (ii). (i) implies that there is an element $v \in S$ such that $es = ve$. Thus $ese = (ve)e = ve = es$.

(ii) $\Rightarrow$ (iii). Since $es = ese$, we get $Ls = Ses = Sese \subseteq Se = L$.

(iii) $\Rightarrow$ (iv). If $f \in E(L)$, then $L = Sf$ and $fs \in Ls \subseteq L = Sf$. Finally, (iv) implies (i) evidently.

By Theorem 1.3., any of conditions (i)-(iv) of Theorem 1.6. implies $\pi_e(S) = \pi_{es}(S)$. If S is right reductive, then $C(e, x) \subseteq C(es, x)$, $\forall x \in S$ implies (i)-(iv) of Theorem 1.6. by Theorem 1.4.

Theorem 1.7. *Let S be a right reductive semigroup containing a completely simple minimal ideal $K(S)$. If $e \in E(K(S))$ and $s \in S$ the following are equivalent:*

(i) $es \in Se$,

(ii) $es = ese$,

(iii) $Ls \subseteq L$, *where* $L = Se$ *is a minimal left ideal,*

(iv) $fs \in Sf$ *for all* $f \in E(L)$,

(v) $\pi_e(S) = \pi_{es}(S)$.

Proof. By Theorems 1.5. and 1.6.

Theorem 1.8. *Let $K(S)$ be a completely simple minimal ideal of a semigroup S. Let $e \in E(K(S))$ and thus $L = Se$ is a minimal left ideal. Then $L = K(S)$ implies $\pi_e(S) = \pi_{es}(S)$, $\forall s \in S$. Conversely, if S is right reductive and $\varrho_e \subseteq \varrho_{es}$, $\forall s \in S$, then $L = Se = K(S)$.*

Proof. If $L = K(S)$, then L is a right ideal and $Ls \subseteq L$. By Theorems 1.6. and 1.3. we obtain $\pi_e(S) = \pi_{es}(S)$, $\forall s \in S$. Conversely, if $C(e, x) \subseteq C(es, x)$, $\forall x, s \in S$ and S is right reductive, then Theorems 1.4. and 1.6. imply $Ls \subseteq L$, $\forall s \in S$, that is, L $(= Se)$ is a right ideal of S. But L is minimal, and hence it follows that $L = K(S)$.

2. – The equivalence relation ϱ_a defined in **1** will be a congruence relation under certain conditions.

Suppose that a semigroup S has an element x_0 such that $x_0S = \bar{S} \subset S$, and (a) $\bar{S}x_0 = \bar{S}$; (b) $ss' = ss''$ implies $s' = s''$ for all $s, s', s'' \in \bar{S}$. Let us consider the classes $C(x_0, y)$ of the relation ϱ_{x_0}. Let us fix an element y_i $(i \in I)$ in every class. Then $S = \bigcup_{i \in I} C(x_0, y_i)$, where $C(x_0, y_i) \cap C(x_0, y_j) = \phi$ $(i \neq j)$.

Theorem 2.1. *Every class $C(x_0, y_i)$ contains at most one element of $\bar{S}$.*

Proof. If $s_1, s_2 \in \bar{S}$ and $x_0s_1 = x_0s_2$, then $x_0^2s_1 = x_0^2s_2$, and in view of (b), $s_1 = s_2$ follows $(x_0^2 \in \bar{S})$.

Theorem 2.2. *If $\bar{S}$ is a finite or a right simple semigroup, then every class $C(x_0, y_i)$ contains exactly one element of $\bar{S}$.*

Proof. If $\bar{S}$ is finite, then $x_0\bar{S} = \bar{S}$. For if s_1, s_2 are different elements of $\bar{S}$, then $x_0s_1 \neq x_0s_2$ by Theorem 2.1., whence $x_0\bar{S} = \bar{S}$ because of $|\bar{S}| = |x_0\bar{S}|$. Thus every class $C(x_0, y_i)$ contains exactly one element of $\bar{S}$. If $\bar{S}$ is right simple then $x_0\bar{S} = \bar{S}$. For a class $C(x_0, y_i)$ we have $x_0y_i \in \bar{S}$. Hence $x_0^2\bar{S} = \bar{S}$ and there is an element $s \in \bar{S}$ such that $x_0^2s = x_0y_i$, that is $x_0(x_0s) = x_0y_i$, whence $x_0s \in C(x_0, y_i)$. But $x_0s \in \bar{S}$.

Theorem 2.3. *$C(x_0, y_i) = C(s, y_i)$ for all $s \in \bar{S}$ $(i \in I)$.*

Proof. Let $x_0y_i = s_1$ $(s_1 \in \bar{S})$; $x \in C(x_0, y_i)$ if and only if $x_0x = s_1$. Let $s_2 \in \bar{S}$. For any element x of $C(s_2x_0, y_i)$ it holds $s_2x_0x = s_2s_1$. Hence it follows that $x_0x = s_1$, i.e., $x \in C(x_0, y_i)$. Thus $C(s_2x_0, y_i) = C(x_0, y_i)$, where $s_2 \in S$. But $\bar{S}x_0 = \bar{S}$ by condition (a) and $C(s, y_i) = C(x_0, y_i)$.

Evidently, if $y_i \neq y_j$ (that is, $y_j \in C(x_0, y_i)$, $i, j \in I$) then $C(s, y_i) \neq C(s', y_j)$ $(s, s' \in \bar{S})$. For if $C(s, y_i) = C(s', y_j)$ then it follows that $C(x_0, y_i) = C(x_0, y_j)$ which is a contradiction. Thus the classes $C(s, y_i)$, $s \in \bar{S}$ are different when y_i runs over different ϱ_{x_0} equivalence classes.

By Theorem 2.3. $C(s, y_i)$ is a function of y_i but it is independent from s, we can write $C(y_i)$ instead of $C(s, y_i)$.

Theorem 2.4. *There exists $y_k \in S$ $(k \in I)$ such that $\forall x \in C(y_i)$ and $\forall y \in C(y_j)$ $(i, j \in I)$ it holds $xy \in C(y_k)$.*

Proof. We have $x_0x = x_0y_i = s_i \in \bar{S}$ and $y \in C(y_j)$ implies $y \in C(s_i, y_j)$, that is $s_iy = s_iy_j$. In this case $x_0(xy) = s_iy = s_iy_j = x_0(y_iy_j)$, i.e. $xy \in C(x_0, y_iy_j) = C(x_0, y_k) = C(s_i, y_k) = C(y_k)$ $(k \in I)$, that is $y_k\varrho_{x_0}(y_iy_j)$.

Corollary 2.5. *ϱ_{x_0} is a congruence relation on S,* i.e. *$S/\varrho_{x_0} = \{C(y_i)\}_{i \in I}$ is a quotient semigroup $\bar{C}$ with property $C(y_i)C(y_j) = C(y_k)$, where $C(y_k) = C(y_iy_j)$ $(i, j, k \in I)$.*

Theorem 2.6. *Let C^* be a subset of $\bar{C}$ consisting of classes $C(y_i)$ which have an element of $\bar{S}$. Then $C^* \cong \bar{S}$.*

Proof. By Theorem 2.1. the class $C(y_i)$ $(i \in I)$ has at most one element of $\bar{S}$. If $s_i \in C(y_i)$ and $s_i \in \bar{S}$, then $C(y_i) = C(s_i)$. The mapping $\varphi\colon C^* \to \bar{S}$, $\varphi(C(s_i)) = s_i$ is an isomorphism, because of $C(s_i)C(s_j) = C(s_is_j)$ by Theorem 2.5., and C^* is a subsemigroup of $\bar{C}$.

Theorem 2.7. *$C^* = \bar{C}$ if and only if $x_0\bar{S} = \bar{S}$ (i.e. $x_0\bar{S} \subset \bar{S}$ implies $C^* \subset \bar{C}$).*

Proof. $C^* = \bar{C}$ if and only if every class $C(y_i)$ has an element $s_i \in \bar{S}$. When y_i runs over the different classes $C(y_i)$, $x_0 y_i$ describes $x_0 S = \bar{S}$. Thus $x_0 \bar{S} = x_0 S = \bar{S}$. Hence it follows that $C^* \subset \bar{C}$ if $x_0 \bar{S} \subset \bar{S}$. Conversely, if $x_0 \bar{S} = x_0 S = \bar{S}$, then by Theorem 2.2. $C^* = \bar{C}$.

Remark. We can obtain analogous theorems if $Sx_0 = \bar{S} \subset S$ and (a′) $x_0 \bar{S} = \bar{S}$, (b′) $s's = s''s \Rightarrow s' = s''$, $\forall s, s', s'' \in \bar{S}$ hold instead of (a), (b).

3. – We shall give necessary and sufficient conditions for the existence of subsemigroups $\bar{S} \subset S$ satisfying conditions (a) and (b) of **2**. We start from the following decomposition [**3**]

$$S = \bigcup_{i=0}^{5} S_i \,, \tag{1}$$

where

$$(2)_0 \qquad S_0 = \{a \in S;\ aS \subset S \text{ and } \exists x \in S - \{0\} \text{ so that } ax = 0\} \,,$$

$$(2)_1 \qquad S_1 = \{a \in S;\ aS = S \text{ and } \exists y \in S - \{0\} \text{ so that } ay = 0\} \,,$$

$$(2)_2 \qquad S_2 = \{a \in S - (S_0 \cup S_1);\ aS \subset S \text{ and } \exists x_1, x_2 \in S \,, \text{ so that } x_1 \neq x_2,\ ax_1 = ax_2\} \,,$$

$$(2)_3 \qquad S_3 = \{a \in S - (S_0 \cup S_1);\ aS = S \text{ and } \exists y_1, y_2 \in S \,, \text{ so that } y_1 \neq y_2,\ ay_1 = ay_2\} \,,$$

$$(2)_4 \qquad S_4 = \{a \in S - \bigcup_{i=0}^{3} S_i;\ aS \subset S\} \,,$$

$$(2)_5 \qquad S_5 = \{a \in S - \bigcup_{i=0}^{3} S_i;\ aS = S\} \,.$$

The sets S_i $(i = 0, 1, 2, 3, 4, 5)$ are disjoint subsemigroups of S and the following relations hold

$$(3)_1 \qquad S_5 S_i \subseteq S_i \,, \quad S_i S_5 \subseteq S_i \quad (0 \leqslant i \leqslant 5) \,,$$

$$(3)_2 \qquad S_4 S_3 \subseteq S_2 \,, \quad S_4 S_2 \subseteq S_2 \,, \quad S_4 S_1 \subseteq S_0 \,,$$

$$(3)_3 \qquad S_4 S_0 \subseteq S_0 \,, \quad S_2 S_3 \subseteq S_2 \,, \quad S_0 S_1 \subseteq S_0 \,.$$

We obtain a similar decomposition

$$S = \bigcup_{i=0}^{5} D_i\,, \tag{4}$$

if in $(2)_i$ $(i = 0, \ldots, 5)$ the multiplication by a is on the right.

The result of this § **3** is the first step in this field of research.

Any semigroup with at most one O annihilator has a unique decomposition (1) as well as one of type (4). Let $x_0 S = \bar{S} \subset S$. Let us consider the decompositions (1) and (4) of $\bar{S}$

$$\bar{S} = \bigcup_{i=0}^{5} \bar{S}_i = \bigcup_{i=0}^{5} \bar{D}_i\,. \tag{5}$$

It is easy to see that property (a) holds if and only if $x_0^2 \in \bar{D}_1 \cup \bar{D}_3 \cup \bar{D}_5$. For $x_0^2 \in \bar{S}$ and $\bar{S}x_0 = x_0 S x_0 \subseteq x_0 S = \bar{S}$.

On the other hand, if $x_0^2 \in \bar{D}_1 \cup \bar{D}_3 \cup \bar{D}_5$, $\bar{S} = \bar{S}x_0^2 \subseteq \bar{S}x_0$ whence $\bar{S}x_0 = \bar{S}$ and (a) holds. Conversely, if (a) holds, then $\bar{S}x_0^2 = \bar{S}$ and $x_0^2 \in \bar{D}_1 \cup \bar{D}_3 \cup \bar{D}_5$. If $\bar{S} = \bar{S}_4 \cup \bar{S}_5$ then $\bar{s}\bar{s}_1 = \bar{s}\bar{s}_2$ implies $\bar{s}_1 = \bar{s}_2$ for all $\bar{s} \in \bar{S}$, $\bar{s}_1, \bar{s}_2 \in \bar{S}$ (property (b)). Conversely, if $\bar{S}$ has the property (b), then for every element $\bar{s} \in \bar{S}$ we have $\bar{s} \in \bar{S}_4 \cup \bar{S}_5$, that is $\bar{S} = \bar{S}_4 \cup \bar{S}_5$. Therefore we obtain the following

Theorem 3.1. *The semigroup $x_0 S = \bar{S} \subset S$ has properties* (a), (b) *if and only if $x_0^2 \in \bar{D}_1 \cup \bar{D}_3 \cup \bar{D}_5$ and $\bar{S} = \bar{S}_4 \cup \bar{S}_5$.*

Remark. The decomposition (5) of $\bar{S}$ isn't independent on the decomposition (1) and (4) of S. This problem will be discussed later on.

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Sunto

Si studiano (**1**) *certe proprietà generali, in un semigruppo* S, *della relazione di equivalenza* ϱ_a $(a \in S)$ *definita da* $x\varrho_a y \Leftrightarrow ax = ay$ $(x, y \in S)$. *Se in* S *esiste un sottosemigruppo proprio* $\bar{S}$ *con certe proprietà,* ϱ_a *risulta una congruenza; si studia il semigruppo quoziente* S/ϱ_a (**2**). *Infine in* **3** *si determina una condizione necessaria e sufficiente affinché in* S *esista un sottosemigruppo* $\bar{S}$ *con le proprietà richieste, ricorrendo alla decomposizione di Szép di un semigruppo.*

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