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Equivalences,
congruences and decompositions in semigroups (**) 

A GIORGIO SESTINI per il suo 70º compleanno 

Introduction

In 1 an equivalence relation $\rho_a$ is considered in a semigroup $S$ which is useful in different lines (magnifying elements, topological semigroups, etc.) and we study some basic properties of this equivalence. In 2 we assume that $S$ has a subsemigroup $\overline{S}$ with given property and we show that $\rho_a$ is a congruence relation and we introduce a quotient semigroup of $S$.

In 3 necessary and sufficient conditions are given in order that $\overline{S}$ be a subsemigroup of $S$ with prescribed property.

Remark. $K(S)$ will denote a minimal ideal of $S$, $E(S)$—the set of all idempotent elements of $S$. Moreover, if $A$, $B$ are subsemigroups of $S$, then $A \subset B$ means that $A$ is a proper subset of $B$.

1. Let $S$ be a semigroup.

Definition 1.1. Let $a \in S$. We define a relation $\rho_a$ by

$$x \rho_a y \iff ax = ay \quad (x, y \in S).$$

$\rho_a$ is an equivalence relation.

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Let \( C(a, x) = \{ y \in S \mid ax = ay \} \) the equivalence class of \( x \). The equivalence \( \varrho_a \) defines a partition \( \pi_a(S) \) of \( S \) where the parts of \( \pi_a(S) \) are the elements of the quotient set \( S/\varrho_a = \{ C(a, x) \mid x \in S \} \).

**Theorem 1.2.**

(i) \( C(a, x) \subseteq C(sa, x), \forall s \in S. \)

(ii) If \( a \) is a left cancellable element of \( S \), then every class \( C(a, x) \) consists of a single element.

(iii) If \( S \) is left simple, then \( \pi_a(S) = \pi_b(S) \) for all \( a, b \in S \).

(iv) If \( ax = bx \) holds for all \( x \in S \), then \( \pi_a(S) = \pi_b(S) \).

**Proof.** (i) is evident.

(ii) \( ax = ay \) implies \( x = y \), thus \( C(a, x) = \{ x \}, \forall x \in S. \)

(iii) It holds \( Sa = S \) for all \( a \in S \). Let \( y \in C(a, x) \). Then for any element \( b \) of \( S \) there is an element \( s \in S \) such that \( b = sa \). Hence \( C(a, x) \subseteq C(b, x) \) by (i). The converse inclusion can be obtained similarly, and thus \( C(a, x) = C(b, x) \) for each \( x \in S \), that is \( \pi_a(S) = \pi_b(S) \).

(iv) Let \( y \in C(a, x) \), i.e. \( ay = ax \). But \( ax = bx \) and \( ay = by \), whence \( bx = by \), \( y \in C(b, x) \) and \( C(a, x) \subseteq C(b, x) \). Similarly, \( C(b, x) \subseteq C(a, x) \) and we get \( C(a, x) = C(b, x) \) for all \( x \in S \). Thus Theorem 1.2. is proved.

**Remarks.** (a) In general, \( \pi_a(S) = \pi_b(S) \) does not imply \( ax = bx \), \( \forall x \in S \).

(b) If \( S \) is a left zero semigroup, then \( ax = ay = a, \forall y \in S \) and thus \( C(a, x) = S, \forall a, x \in S \), that is \( \pi_a(S) \) has a single class (\( \forall a \in S \)).

(c) Let \( a \) be a left magnifying element of \( S \), i.e. \( aM = S \) holds for a proper subset \( M \) of \( S \). Then every class \( C(a, x) \) of \( \pi_a(S) \) contains at least one element of \( M \). Indeed, there is an element \( m \in M \) such that \( ax = am \), whence \( m \in C(a, x) \). Choosing an element \( \bar{m} \), in \( C(a, x_i) \) (\( i \in I \)), then \( \bar{M} = \{ \bar{m}_i; i \in I \} \) is a minimal subset of \( S \) having the property \( a\bar{M} = S \) (cfr. also [2]).

**Theorem 1.3.** Let \( S \) be a semigroup, \( e \in E(S) \) such that \( Se \) is a minimal left ideal of \( S \). If \( s \) is an element of \( S \) such that \( es = ese \), then \( \pi_s(S) = \pi_{es}(S) \).

**Proof.** We have to show that \( ess = esy \) (\( x, y \in S \)) implies \( ex = ey \) and conversely. Let \( ess = esy \). Since \( Se \) is a minimal left ideal of \( S \), \( eSe \) is the
maximal subgroup of \( S \) containing \( e \). Denote \((ese)^{-1}\) the inverse of \( ese \) in \( eSe \).
Then \( ex = (ese)^{-1}esse = (ese)^{-1}exy = (ese)^{-1}ese = ey \).
Conversely, let \( ex = ey \). Then \( esx = es(ex) = es(ey) = (ese)y = esy \).

Theorem 1.3. is completely proved.

The converse of Theorem 1.3. holds if \( S \) is right reductive, i.e. \( ax = bx \)
(\( \forall x \in S \)) implies \( a = b \) (\( a, b \in S \)).

**Theorem 1.4.** Let \( S \) be a right reductive semigroup, \( e \in E(S) \). If \( s \) is
an element of \( S \) such that \( q_e \subseteq q_{es} \), then \( es = ese \).

**Proof.** By hypothesis, \( ex = ey \) implies \( esx = esy \) (\( x, y \in S \)). Hence for
each \( x \in S \) we have \( ex = e(ex) \), i.e. \( (es)x = (es)ex = (ese)x \). Since \( S \) is right
reductive, we get \( es = ese \).

Theorem 1.3. and Theorem 1.4. imply the following

**Theorem 1.5.** If \( S \) is a right reductive semigroup and \( e \in E(S) \) such
that \( S_e \) is a minimal left ideal of \( S \), then the following conditions are equivalent

(i) \( es = ese \);
(ii) \( \pi_e(S) = \pi_{es}(S) \) (\( s \in S \)).

The next result is known (see [1], theorem 1.17), we prove it for the sake
of completeness.

**Theorem 1.6.** Let \( K(S) \) be a completely simple minimal ideal of \( S \). If
\( e \in E(K(S)) \), the following are equivalent

(i) \( es \in S e \),
(ii) \( es = ese \),
(iii) \( L e \subseteq L \), where \( L = S e \) is a minimal left ideal,
(iv) \( fs \in S f \) for all \( f \in E(L) = E(K(S)) \cap L \).

**Proof.** (i) \( \Rightarrow \) (ii). (i) implies that there is an element \( v \in S \) such that
\( esv = ve \). Thus \( ese = (ve)e = ve = es \).

(ii) \( \Rightarrow \) (iii). Since \( es = ese \), we get \( Le = Ses = Serie \subseteq Se = L \).

(iii) \( \Rightarrow \) (iv). If \( f \in E(L) \), then \( L = Sf \) and \( fe \in Le \subseteq L = Sf \).
Finally, (iv) implies (i) evidently.
By Theorem 1.3., any of conditions (i)-(iv) of Theorem 1.6. implies \( \pi_e(S) = \pi_{es}(S) \). If \( S \) is right reductive, then \( C(e, x) \subseteq C(es, x) \), \( \forall x \in S \) implies (i)-(iv) of Theorem 1.6. by Theorem 1.4.

**Theorem 1.7.** Let \( S \) be a right reductive semigroup containing a completely simple minimal ideal \( K(S) \). If \( e \in E(K(S)) \) and \( s \in S \) the following are equivalent:

(i) \( es \in Se \),
(ii) \( es = ese \),
(iii) \( Ls \subseteq L \), where \( L = Se \) is a minimal left ideal,
(iv) \( fs \in Sf \) for all \( f \in E(L) \),
(v) \( \pi_e(S) = \pi_{es}(S) \).

**Proof.** By Theorems 1.5. and 1.6.

**Theorem 1.8.** Let \( K(S) \) be a completely simple minimal ideal of a semigroup \( S \). Let \( e \in E(K(S)) \) and thus \( L = Se \) is a minimal left ideal. Then \( L = K(S) \) implies \( \pi_e(S) = \pi_{es}(S) \), \( \forall s \in S \). Conversely, if \( S \) is right reductive and \( g_e \subseteq g_{es} \), \( \forall s \in S \), then \( L = Se = K(S) \).

**Proof.** If \( L = K(S) \), then \( L \) is a right ideal and \( Ls \subseteq L \). By Theorems 1.6. and 1.3. we obtain \( \pi_e(S) = \pi_{es}(S) \), \( \forall s \in S \). Conversely, if \( C(e, x) \subseteq C(es, x) \), \( \forall x, s \in S \) and \( S \) is right reductive, then Theorems 1.4. and 1.6. imply \( Ls \subseteq L \), \( \forall s \in S \), that is, \( L (= Se) \) is a right ideal of \( S \). But \( L \) is minimal, and hence it follows that \( L = K(S) \).

2. – The equivalence relation \( g_e \) defined in \( I \) will be a congruence relation under certain conditions.

Suppose that a semigroup \( S \) has an element \( x_0 \) such that \( x_0 S = \overline{S} \subseteq S \), and

(a) \( \overline{S} x_0 = \overline{S} \); (b) \( ss' = ss'' \) implies \( s' = s'' \) for all \( s, s', s'' \in \overline{S} \). Let us consider the classes \( C(x_0, y) \) of the relation \( g_e \). Let us fix an element \( y_i \) (\( i \in I \)) in every class. Then \( \overline{S} = \bigcup \{ C(x_0, y_i) \} \), where \( C(x_0, y_i) \cap C(x_0, y_j) = \phi \) (\( i \neq j \)).

**Theorem 2.1.** Every class \( C(x_0, y_i) \) contains at most one element of \( \overline{S} \).

**Proof.** If \( s_1, s_2 \in \overline{S} \) and \( x_0 s_1 = x_0 s_2 \), then \( x_0^2 s_1 = x_0^2 s_2 \), and in view of (b), \( s_1 = s_2 \) follows \( (x_0^2 \in \overline{S}) \).

**Theorem 2.2.** If \( \overline{S} \) is a finite or a right simple semigroup, then every class \( C(x_0, y_i) \) contains exactly one element of \( \overline{S} \).
Proof. If $\overline{S}$ is finite, then $x_0\overline{S} = \overline{S}$. For if $s_1$, $s_2$ are different elements of $\overline{S}$, then $x_0s_1 \neq x_0s_2$ by Theorem 2.1, whence $x_0\overline{S} = \overline{S}$ because of $|\overline{S}| = |x_0\overline{S}|$. Thus every class $C(x_0, y_i)$ contains exactly one element of $\overline{S}$. If $\overline{S}$ is right simple then $x_0\overline{S} = \overline{S}$. For a class $C(x_0, y_i)$ we have $x_0s_i \in \overline{S}$. Hence $x_0\overline{S} = \overline{S}$ and there is an element $s \in \overline{S}$ such that $x_0s = x_0y_i$, that is $x_0(x_0s) = x_0y_i$, whence $x_0s \in C(x_0, y_i)$. But $x_0s \in \overline{S}$.

Theorem 2.3. $C(x_0, y_i) = C(s, y_i)$ for all $s \in \overline{S}$ $(i \in I)$.

Proof. Let $x_0y_i = s_1 (s_i \in \overline{S})$; $x \in C(x_0, y_i)$ if and only if $x_0x = s_1$. Let $s_2 \in \overline{S}$ for any element $x$ of $C(s_2x_0, y_i)$ it holds $s_2x_0x = s_2s_1$. Hence it follows that $x_0x = s_1$, i.e., $x \in C(x_0, y_i)$. Thus $C(s_2x_0, y_i) = C(x_0, y_i)$, where $s_2 \in \overline{S}$. But $\overline{S} = S$ by condition (a) and $C(s, y_i) = C(x_0, y_i)$.

Evidently, if $y_i \neq y_j$ (that is, $y_j \in C(x_0, y_i)$, $i, j \in I$) then $C(s, y_i) \neq C(s', y_i)$ $(s, s' \in \overline{S})$. For if $C(s, y_i) = C(s', y_j)$ then it follows that $C(x_0, y_i) = C(x_0, y_j)$ which is a contradiction. Thus the classes $C(s, y_i)$, $s \in \overline{S}$ are different when $y_i$ runs over different $\overline{S}$ equivalence classes.

By Theorem 2.3, $C(s, y_i)$ is a function of $y_i$ but it is independent from $s$, we can write $C(y_i)$ instead of $C(s, y_i)$.

Theorem 2.4. There exists $y_k \in S$ $(k \in I)$ such that $\forall x \in C(y_i)$ and $\forall y \in C(y_j)$ $(i, j \in I)$ it holds $xy \in C(y_k)$.

Proof. We have $x_0x = x_0y_i = s_i \in \overline{S}$ and $y \in C(y_i)$ implies $y \in C(s_i, y_i)$, that is $x_0y = s_iy_i$. In this case $x_0(xy) = s_0y = s_0y_i = x_0(y_0y_i)$, i.e. $xy \in C(x_0, y_0y_i) = C(x_0, y_k) = C(s_i, y_k) = C(y_k) (k \in I)$, that is $y_k \in C(y_k)$.

Corollary 2.5. $\overline{S}$ is a congruence relation on $S$, i.e. $S/\overline{S} = \{C(y_i)\}_{i \in I}$ is a quotient semigroup $\overline{C}$ with property $C(y_i)C(y_j) = C(y_k)$ $(k \in I)$, where $C(y_k) = C(y_k)$ $(i, j, k \in I)$.

Theorem 2.6. Let $C^*$ be a subset of $\overline{C}$ consisting of classes $C(y_i)$ which have an element of $\overline{S}$. Then $C^* \cong \overline{S}$.

Proof. By Theorem 2.1, the class $C(y_i)$ $(i \in I)$ has at most one element of $\overline{S}$. If $s_i \in C(y_i)$ and $s_i \in \overline{S}$, then $C(y_i) = C(s_i)$. The mapping $q: C^* \rightarrow \overline{S}$, $q(C(s_i)) = s_i$ is an isomorphism, because of $C(s_i)C(s_i) = C(s_i, s_i)$ by Theorem 2.5., and $C^*$ is a subsemigroup of $\overline{C}$.

Theorem 2.7. $C^* = \overline{C}$ if and only if $x_0\overline{S} = \overline{S}$ (i.e. $x_0\overline{S} \subset \overline{S}$ implies $C^* \subset \overline{C}$).
Proof. \( C^* = \overline{C} \) if and only if every class \( C(y_i) \) has an element \( s_i \in \overline{S} \). When \( y_i \) runs over the different classes \( C(y_i) \), \( x_0 y_i \) describes \( x_0 S = \overline{S} \). Thus \( x_0 \overline{S} = x_0 S = \overline{S} \). Hence it follows that \( C^* \subset \overline{C} \) if \( x_0 \overline{S} \subset \overline{S} \). Conversely, if \( x_0 \overline{S} = x_0 S = \overline{S} \), then by Theorem 2.2, \( C^* = \overline{C} \).

Remark. We can obtain analogous theorems if \( S x_0 = \overline{S} \subset S \) and (a') \( x_0 \overline{S} = \overline{S} \), (b') \( s' s = s'' s \Rightarrow s' = s'' \), \( \forall s, s', s'' \in \overline{S} \) hold instead of (a), (b).

3. We shall give necessary and sufficient conditions for the existence of subsemigroups \( \overline{S} \subset S \) satisfying conditions (a) and (b) of 2. We start from the following decomposition[3]

\[
S = \bigcup_{i=0}^{5} S_i, \tag{1}
\]

where

(2)_0 \( S_0 = \{ a \in S; a S \subset S \text{ and } \exists x \in S - \{0\} \text{ so that } ax = 0 \} \),

(2)_1 \( S_1 = \{ a \in S; a S = S \text{ and } \exists y \in S - \{0\} \text{ so that } ay = 0 \} \),

(2)_2 \( S_2 = \{ a \in S - (S_0 \cup S_1); a S \subset S \text{ and } \exists x_1, x_2 \in S \),

so that \( x_1 \neq x_2, \ ax_1 = ax_2 \},

(2)_3 \( S_3 = \{ a \in S - (S_0 \cup S_1); a S = S \text{ and } \exists y_1, y_2 \in S \),

so that \( y_1 \neq y_2, \ ay_1 = ay_2 \},

(2)_4 \( S_4 = \{ a \in S - \bigcup_{i=0}^{3} S_i; a S \subset S \} \),

(2)_5 \( S_5 = \{ a \in S - \bigcup_{i=0}^{3} S_i; a S = S \} \).

The sets \( S_i \) \((i = 0, 1, 2, 3, 4, 5)\) are disjoint subsemigroups of \( S \) and the following relations hold

(3)_1 \( S_5 S_i \subset S_i, S_i S_5 \subset S_i \) \((0 \leq i \leq 5)\),

(3)_2 \( S_4 S_3 \subset S_2, S_4 S_2 \subset S_2, S_4 S_1 \subset S_0 \),

(3)_3 \( S_4 S_0 \subset S_0, S_2 S_3 \subset S_2, S_0 S_1 \subset S_0 \).
We obtain a similar decomposition

\( S = \bigcup_{i=0}^{5} D_i \),

if in (2) \((i = 0, \ldots, 5)\) the multiplication by \( a \) is on the right.

The result of this § 3 is the first step in this field of research.

Any semigroup with at most one \( O \) annihilator has a unique decomposition (1) as well as one of type (4). Let \( x_0 S = \overline{S} \subset S \). Let us consider the decompositions (1) and (4) of \( \overline{S} \)

\( \overline{S} = \bigcup_{i=0}^{5} \overline{S}_i = \bigcup_{i=0}^{5} \overline{D}_i \).

It is easy to see that property (a) holds if and only if \( x_0^2 \in \overline{D}_1 \cup \overline{D}_3 \cup \overline{D}_5 \).

For \( x_0^2 \in \overline{S} \) and \( \overline{S} x_0 = x_0 \overline{S} x_0 \subset x_0 S = \overline{S} \).

On the other hand, if \( x_0^2 \in \overline{D}_1 \cup \overline{D}_3 \cup \overline{D}_5 \), \( \overline{S} = \overline{S} x_0 \subset \overline{S} x_0 \) whence \( \overline{S} x_0 = \overline{S} \) and (a) holds. Conversely, if (a) holds, then \( \overline{S} x_0 = \overline{S} \) and \( x_0^2 \in \overline{D}_1 \cup \overline{D}_3 \cup \overline{D}_5 \).

If \( \overline{S} = \overline{S}_1 \cup \overline{S}_2 \) then \( \overline{S}_1 = \overline{S}_2 \) implies \( \overline{s}_1 = \overline{s}_2 \) for all \( \overline{s} \in \overline{S} \), \( \overline{s}_1, \overline{s}_2 \in \overline{S} \) (property (b)). Conversely, if \( \overline{S} \) has the property (b), then for every element \( \overline{s} \in \overline{S} \) we have \( \overline{s} \in \overline{S}_1 \cup \overline{S}_2 \), that is \( \overline{S} = \overline{S}_1 \cup \overline{S}_2 \). Therefore we obtain the following

**Theorem 3.1.** The semigroup \( x_0 S = \overline{S} \subset S \) has properties (a), (b) if and only if \( x_0^2 \in D_1 \cup D_3 \cup D_5 \) and \( \overline{S} = \overline{S}_1 \cup \overline{S}_2 \).

**Remark.** The decomposition (5) of \( \overline{S} \) isn't independent on the decomposition (1) and (4) of \( S \). This problem will be discussed later on.

**Bibliography**


Sunto

Si studiano (1) certe proprietà generali, in un semigruppo $S$, della relazione di equivalenza $\equiv_a$ ($a \in S$) definita da $x \equiv_a y \iff ax = ay$ ($x, y \in S$). Se in $S$ esiste un sottosemigruppo proprio $\bar{S}$ con certe proprietà, $\equiv_a$ risulta una congruenza; si studia il semigruppo quoziente $S/\equiv_a$ (2). Infine in 3 si determina una condizione necessaria e sufficiente affinché in $S$ esista un sottosemigruppo $\bar{S}$ con le proprietà richieste, ricorrendo alla decomposizione di Szép di un semigruppo.

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