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Mechanics on a Galileean manifold (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

In this paper we examine some geometric properties of the dynamics on a Galilean manifold. This consists [1] of the so called absolute time bundle \( t: M \rightarrow R \) (\( M \) is the event space) together with a Riemannian metric \( g \) in the spacelike bundle \( \sigma = \text{Ker. } dt \). Hence a Galilean manifold generalizes the well known Newtonian-Galilean manifold in which the bundle \( t: M \rightarrow R \) is linear affine and \( g \) is Euclidean [5], [6].

The arena of dynamics is the kinematic space \( K \) which is defined as the submanifold of the tangent space \( TM \) such that \( dt|_K = 1 \). If \( \pi: K \rightarrow M \) is the canonical projection, then the bundle \( \pi : K \rightarrow M \) has a canonical affine structure over the vector bundle \( \sigma \). All the basic concepts are related to the kinematic space \( K \). For example, a motion \( c \) in \( M \) is a section of the bundle \( t: M \rightarrow R \) and hence its velocity \( \dot{c} \) is a section of the bundle \( \sigma^*\pi \) (the pullback of the bundle \( \pi : K \rightarrow M \) over \( c \)); a framing \( F \) is a section of the bundle \( \pi : K \rightarrow M \); a force \( f \) is a section of the vertical bundle \( V_K = \text{Ker. } T\pi \simeq \tau_K \) and a dynamic equation \( X \) (that is an equation of motion) is a section of an affine subbundle of the tangent bundle \( \tau_K \) modeled on the vector bundle \( V_K \).

Moreover, the concept of acceleration and that of Galilean connection on \( M \) (cfr. (c), 3) lead to consider vertical projections \( \Gamma \in \text{Hom} (\tau_K, V_K) \) providing affine isomorphisms between forces and dynamic equations [4]. However, we

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are "forced" to work in the space $\mathcal{K}$ in order to use "absolute" concepts avoiding to choose a distinguished framing $\mathcal{F}$. In this connexion, let us emphasize the role played through the paper by the vertical endomorphism $v$ (see (2), (18), (26), (37), etc).

But physical interpretation require the use of framings. For this we give a detailed analysis of all the concepts related to a framing $\mathcal{F}$. For example, to $\mathcal{F}$ there are associated a projection $p_{\mathcal{F}} \in \text{Hom}(\tau_M, \sigma)$, a vertical projection $\Gamma_{\mathcal{F}} \in \text{Hom}(\tau_K, \nu_K)$ used to define the acceleration relative to $\mathcal{F}$, a dynamic equation $\xi_{\mathcal{F}}$, a contact structure $\omega_{\mathcal{F}}$ on $K$, a Galilean connection $\nabla^\mathcal{F}$ on $\mathcal{M}$ which is used to define the vorticity 2-form $\Omega_\mathcal{F}$ of the framing $\mathcal{F}$, a kinetic energy $T_{\mathcal{F}}$, the apparent force $m\dot{A}_{\mathcal{F}}$ which arises from the deformation of $\mathcal{F}$, etc. Certain novelties should be found in the way in which some of these concepts are introduced (inertial forces, contact structures on the kinematic space $K$, etc.).

The work is motivated by applications, mainly to classical analytical mechanics. For this reason, we let $\mathcal{M}$ be an $n + 1$ dimensional manifold, $n \geq 1$. However, in the $\S$ 4 on Dynamics, it may be supposed $n = 3$, though this is not relevant in the context of the discussion. As a simple example, note that an "holonomic constraint" in the Newtonian-Galilean manifold $(\mathcal{M}, t, g)$ is merely a "Galilean submanifold" of $(\mathcal{M}, t, g)$. For example, if $\varphi$ is a function on $\mathcal{M}$, we get an holonomic constraint $\mathcal{N} = \varphi^{-1}(0)$ (it is assumed that 0 is a regular value of $\varphi$ and that $\mathcal{N} \cap t^{-1}(\tau) \neq \emptyset$, $\tau \in \mathcal{R}$) by requiring that $\text{Ker.} \, d\varphi(p) \neq \text{Ker.} \, dt(p)$ for all $p \in \mathcal{N}$. Indeed, it is clear that $(\mathcal{N}, t/\mathcal{N}, j^* g)$ is a Galilean manifold (here $j$ is the injection $\mathcal{N} \hookrightarrow \mathcal{M}$). Recall that the traditional approach assumes a Cartesian product structure of the event space $\mathcal{M}$, that is a choice of a distinguished framing is made (usually inertial).

To finish, let us remark that definitions and formulae are always written in an intrinsic way. However, in the proofs local coordinates are used (proofs omitted are simple).

2. - Preliminaries

We start by establishing the notation. Then we consider some basic concepts.

(a) Notation. This is the one usually found in any text on modern differential geometry (we refer to [2]). However, about connections and their associated horizontal subbundles and sprays, we refer to [3].

All manifolds, bundles and tensor fields will be $C^\infty$. If $\mathcal{M}$ is a manifold, $T\mathcal{M}$ denotes its tangent space ($T_p\mathcal{M}$ denotes the tangent space at $p \in \mathcal{M}$) and $\tau_\mathcal{M}: T\mathcal{M} \to \mathcal{M}$ its tangent bundle. We denote by $C^\infty(\mathcal{M})$ the ring of the functions on $\mathcal{M}$ and by $\mathfrak{X}(\mathcal{M})$ the $C^\infty(\mathcal{M})$-module of the vector fields on $\mathcal{M}$. If $c$ is a
curve in $M$, we denote by $\dot{c}$ and by $\ddot{c}$ its canonical liftings to $TM$ and to $TTM$, respectively.

If $\sigma$ is a vector bundle, we denote by $\text{Sec}(A^p \sigma^q)$ the module of the $p$-forms on $\sigma$. Some tensor products and canonical isomorphisms of vector bundles are used: for example, $L(\tau_M; \sigma) \cong \tau^*_M \otimes \sigma$ (the bundle $L(\sigma; \sigma)$ is denoted by $L_\sigma$, etc.

To finish, the symbol $d$ denotes the exterior derivative, $L_X$ the Lie derivative along a vector field $X$, and $i$ the interior product.

(b) The absolute time bundle. This is the bundle $t: M \to R$ where $M$ is the event space and the projection $t$ is the absolute time. We shall usually write simple $M$ to denote this bundle. The $n$-dimensional submanifolds $\Sigma_t = t^{-1}(\tau) \subset M$ (the fiber of $M$ at $\tau \in R$) are the spaces of absolute simultaneity.

A motion in $M$, usually denoted by $\dot{c}$, is a section of $M$ over an open interval $I \subset R$.

On $M$ we shall always use adapted charts $(U, x^a)$ where $x^a = t \mid U$.

(c) Observers and framings. (i) An instantaneous observer is an ordered pair $(p, u)$ where $p \in M$ and $u \in T_p M$ is such that $\langle dt(p), u \rangle = 1$.

Let $K \subset TM$ be the $(2n + 1)$-dimensional submanifold of the instantaneous observers and let $\pi: K \to M$ be the canonical projection. Then the bundle $\pi: K \to M$ has a canonical affine structure on the vector bundle $\sigma = \text{Ker}. dt$. It is clear that $\sigma$ is an involutive subbundle of $\tau_M$ and that $\tau^*_M \cong \sigma \mid \Sigma_t$, $s \in R$. Elements of $\text{Sec} \sigma$ are called spacelike vector fields on $M$. Note that spacelike form on $M$ cannot be defined.

Let $c$ be a motion. Then $\dot{c} \in \text{Sec}(\sigma^a \pi)$ is its (absolute) velocity [$\sigma^a \pi$ is the pull-back of $\pi$ over $c$]. For this reason, $K$ is also called the kinematic space.

Let $(U, x^a)$ be a chart on $M$. We shall denote by $\{\partial/\partial x^a\}$ the induced basis of the module $\text{Sec}(\sigma \mid U)$ and by $(x^{-1}(U), q^a, \dot{q}^a)$ the induced chart on $K$.

(ii) We define an observer to be a motion. However we need a « cooperation » of observers in order to interpret in physical terms the laws of dynamics written in the space $K$. Hence we define a framing, typically denoted by $F$, to be a section of $K$, that is $F \in \text{Sec} \pi$. If $c$ is an observer, we say that $c$ is an observer in $F$ iff $c$ is an integral curve of $F$. The velocity of $c$ relative to $F$ is defined to be $\dot{c} = F \circ c \in \text{Sec}(\sigma^a \sigma)$.

Let $F$ be a framing. Then to $F$ there is associated the projection $p_F \in \text{Hom}(\tau_M, \sigma)$ given by

\begin{equation}
 p_F(u) = u - \langle dt(p), u \rangle F(p), \quad u \in T_p M.
\end{equation}

Clearly we have $\tau_M = \sigma \oplus \text{Ker}. p_F$.

Let $F$ be a framing and let $p \in M$. Then there exists a chart $(U, x^a)$ at $p$ such that $F \mid U = \partial/\partial x^a$. Charts with such a property are said to be adapted to $F$. 20
(d) **Dynamic equations.** (i) Let $\mathcal{V}_K$ be the vertical subbundle $\text{Ker. } T\pi \hookrightarrow \tau_K$. Note that we have $\mathcal{V}_K \cong \pi^*\sigma$. If $(\mathcal{U}, \sigma)$ is a chart on $M$, we denote by $\{\frac{\partial}{\partial q^i}\}$ the induced basis of the module $\text{Sec} (\mathcal{V}_K|\pi^{-1}(\mathcal{U}))$.

We now introduce the vertical endomorphism $v$ characterized locally by

$$v\left(\frac{\partial}{\partial q^0}\right) = -\dot{q}^i \frac{\partial}{\partial q^i}, \quad v\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial q^i}, \quad v\left(\frac{\partial}{\partial q^i}\right) = 0.$$ \hfill (2)

It follows at once that $v^*$ is characterized by

$$v^*(dq^0) = 0, \quad v^*(dq^i) = -\dot{q}^i dq^0 + dq^i.$$ \hfill (3)

The vertical endomorphism $v$ will be used later to get a 2-form on the space $K$ from a Galilean connection on $M$.

Let $F$ be a framing. Then to $F$ there are associated a vertical field $V_F$ and a vertical endomorphism $v_F$ characterized locally by (using charts adapted to $F$)

$$V_F = \dot{q}^i \frac{\partial}{\partial q^i},$$ \hfill (4)

and by

$$v_F\left(\frac{\partial}{\partial q^0}\right) = v_F\left(\frac{\partial}{\partial q^i}\right) = 0, \quad v_F\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial q^i},$$ \hfill (5)

respectively.

(ii) A **dynamic equation**, typically denoted by $X$, is a vector field on $K$ such that

$$\langle dt, X \rangle = 1, \quad v(X) = 0.$$ \hfill (ii)

It follows at once that the local expression of a dynamic equation $X$ is given by

$$X = \frac{\partial}{\partial q^0} + \dot{q}^i \frac{\partial}{\partial q^i} + b^i \frac{\partial}{\partial q^i}, \quad b^i \in C^\infty(\pi^{-1}(\mathcal{U})).$$ \hfill (6)

Note that dynamic equations (which are some sort of second order differential equations on $M$) are sections of an affine subbundle of $\tau_K$ modeled on the vertical bundle $V_K$. Hence we define a force, typically denoted by $f$, to be an element of $\text{Sec} (V_K)$. 


(e) **Spacelike connections.** A (linear) connection $\nabla$ on $M$ is called spacelike iff $\nabla dt = 0$. Let $\nabla$ be a connection on $M$ and let $\Gamma^\sigma_{\alpha\beta} \in C^0(U)$ be the connection parameters for $\nabla$ determined by a chart $(U, x^\alpha)$. Then we have locally

$$\nabla dt = - \Gamma^\alpha_{\beta\delta} dx^\alpha \otimes dx^\delta$$

and hence $\nabla$ is spacelike iff $\Gamma^0_{\alpha\beta} = 0$. Note that from a spacelike connection $\nabla$ we get a connection, also denoted by $\nabla$, on the vector bundle $\sigma$.

Let $\nabla$ be a spacelike connection and let $F$ be a framing. Then $\nabla F \in \text{Sec} L(\tau_M; \sigma)$ and from the canonical projection $\tau^*_M \rightarrow \sigma^*$ we get the spatial part $\nabla F \in \text{Sec} L_\sigma$. If $(U, x^\alpha)$ is a chart adapted to $F$, the local expression of $\nabla F$ is given by

$$\nabla F = \Gamma^i_{i0} dx^i \otimes \frac{\partial}{\partial x^i}.$$  

(7)

We also need the following results about spacelike connections.

**Proposition 1.** Let $\nabla$ be a spacelike connection on $M$ and let $H_{TM} \hookrightarrow \tau_{TM}$ be the horizontal subbundle associated to it [3]. Then the restriction $H_{TM}|K$ gives an horizontal subbundle of $\tau_K$, that is $V_K \oplus H_{TM}|K = \tau_K$. A local basis of this subbundle is given by

$$E_\alpha = \frac{\partial}{\partial q_\alpha} - (\Gamma^\nu_{\alpha0} + \Gamma^\nu_{\alpha i} q^i) \frac{\partial}{\partial q^\nu}.$$  

(8)

Moreover, the spray $\xi$ of $\nabla$ restricts to a dynamic equation, also denoted by $\xi$, whose local expression is

$$\xi = E_0 + q^i E_i.$$  

(9)

If $\Gamma \in \text{Hom} (\tau_K, V_K)$ is the vertical projection, then we have $\Gamma(\xi) = 0$ and $\Gamma$ restricts to an affine isomorphism between dynamic equations and forces [4].

**Remark.** More generally, framings and connections, dynamic equations and forces must be considered over open sets $U \subset M$ and $W \subset K$, respectively. However, later we shall be interested to the case where $W = \pi^{-1}(U)$. 


3. - The riemannian bundle \((\sigma, g)\) and some consequences

Now we equip the bundle \(\sigma\) with a (positive definite) Riemannian metric \(g\). We say that \(M\) becomes a Galilean manifold. The following are some consequences of this structure.

(a) Lowering and raising indices. The isomorphism \(\sigma \to \sigma^{*}\) induced by the metric \(g\) will be denoted by \(\tilde{\sigma}\). Note that since \(V_{\sigma} \cong \pi^{*}\sigma\), \(V_{\sigma}\) inherits the metric \(g\).

(b) Framings, dynamic equations and contact structures. (i) First of all note that the restriction \(g|_{\Sigma_{\tau}}\) is a Riemannian metric on \(\Sigma_{\tau} \subset M\), \(\tau \in R\). Now let \(g_{ij}, \ g^{ik} \in C_{\infty}(U)\) be the components of the metric \(g\) determined by a chart \((U, \sigma^{\alpha})\) and put

\[
\gamma_{ij}^{k} = \frac{1}{2} \ g^{kh} \left( \frac{\partial g_{ki}}{\partial x^{h}} + \frac{\partial g_{hi}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{h}} \right).
\]

It follows that \(\gamma_{ij}^{k}|_{U_{\tau}}\) are the Christoffel symbols of the Levi-Civita connection of \(g|_{\Sigma_{\tau}}\) determined by the chart \((U_{\tau}, \sigma^{\alpha}|_{U_{\tau}})\) on \(\Sigma_{\tau}, \ U_{\tau} = \Sigma_{\tau} \cap U, \ \tau \in R\).

To define the acceleration relative to \(F\), we need the following proposition.

Proposition 2. Let \(F\) be a framing and let \((U, \sigma^{\alpha})\) be a chart adapted to \(F\). Then there exists an horizontal subbundle \(H_{F} \hookrightarrow \tau_{\sigma}\) with local basis

\[
E_{\alpha} = \frac{\partial}{\partial q^{\alpha}}, \quad E_{i} = \frac{\partial}{\partial q^{i}} - \gamma_{ij}^{k} \frac{\partial}{\partial \dot{q}^{k}}.
\]

Proof. Indeed if \((U', \sigma'^{\alpha})\) is another chart adapted to \(F\) such that \(U \cap U' \neq \emptyset\), we easily see that \(E_{\alpha}' = (\partial x^{\alpha}/\partial x^{\alpha}) E_{\alpha}\). Hence the result follows at once.

Let \(\Gamma_{F} \in \text{Hom}(\tau_{\sigma}, V_{\sigma})\) be the vertical projection determined by this subbundle and let \(\xi_{F}\) be the dynamic equation uniquely determined by \(\Gamma_{F}(\xi_{F}) = 0\). Let \((U, \sigma^{\alpha})\) be a chart adapted to \(F\). Then the local expression of \(\xi_{F}\) is given by

\[
\xi_{F} = \frac{\partial}{\partial q^{\alpha}} + \dot{q}^{i} \frac{\partial}{\partial \dot{q}^{i}} - \gamma_{ij}^{k} \dot{q}^{i} \dot{q}^{j} \frac{\partial}{\partial \dot{q}^{k}},
\]

as we see from (6) and (11). If \(c\) is a motion, we define \(\Gamma_{\sigma} c\dot{c}\) to be the acceleration of \(c\) relative to \(F\). Note that we can write \(\Gamma_{\sigma} c\dot{c} \in \text{Sec}(\sigma^{*}\sigma)\) since \(V_{\sigma} \cong \pi^{*}\sigma\).
Locally we have (using charts adapted to $F$)

\[ I_P \circ \dot{c} = \left( \dot{c}^k + (\gamma^k_{ij} \circ c) \; \dot{c}^i \; \dot{c}^j \right) \left( \frac{\partial}{\partial \dot{c}^k} \circ c \right) , \quad c^i = x^i \circ c . \]

It is clear that an observer $c$ has no acceleration relative to $F$ iff is a base integral curve of $\xi_P$ (note that observers in $F$ are not accelerated relative to $F$).

(ii) Let $F$ be a framing. Then we get a quadratic function $G_P \in C^\infty(K)$ by putting (cfr. (1))

\[ G_P(u) = \frac{1}{2} g_P(p_P(u), p_P(u)) , \quad u \in K . \]

Its local expression is given by (henceforth charts adapted to $F$ will be used)

\[ G_P = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j . \]

From the vertical differentiation $d_{s_P}$ associated to the framing $F$ (cfr. [2]), we get the 1-form on $K$

\[ d_{s_P} G_P = g_{ij} \dot{q}^i \stackrel{\cdot}{} \dot{q}^j . \]

This is an example of a spacelike form relative to $F$ (on $K$). Such forms are easily characterized. Let us denote by $d_P$ the exterior derivative acting on these forms ($d_P$ is induced by $F$ in an obvious way). Then we get the spacelike 2-form on $K$

\[ d_P d_{s_P} G_P = \frac{\partial q}{\partial q^j} \dot{q}^i \dot{q}^j \stackrel{\cdot}{} \dot{q}^i \stackrel{\cdot}{} \dot{q}^j - g_{ij} \dot{q}^i \stackrel{\cdot}{} \dot{q}^j - \dot{q}^i \dot{q}^j . \]

In the following proposition we introduce the contact structure associated to a framing $F$.

**Proposition 3.** Let $F$ be a framing and let $\omega_P \in \text{Sec} \left( A^2 \tau^*_P \right)$ be given by

\[ \omega_P(X, Y) = g(\nu Y, I_P X) - g(\nu X, I_P Y), \quad X, Y \in \mathcal{X}(K) . \]

Then $\omega_P$ is closed (relative to $d_P$) and of maximal rank on $K$, and we have

\[ \omega_P = d_P d_{s_P} G_P - \dot{d} G_P \wedge \stackrel{\cdot}{} \dot{d} . \]

Moreover, $\xi_P$ is the unique vector field on $K$ that is a characteristic vector field of $\omega_P$ and satisfies $i_{\xi_P \dot{d}} = 1$. 
(c) Admissible and Galilean connections. (i) A connection $\nabla$ in the (involutive) subbundle $\sigma \hookrightarrow \tau_M$ is called spacelike symmetric iff we have $\nabla_\phi \psi = -\nabla_\psi \phi = [\phi, \psi]$ for all $\phi, \psi \in \text{Sec} \sigma$. Moreover, a spacelike symmetric Riemannian connection in $(\sigma, g)$ is called admissible. We have the following proposition.

Proposition 4. Let $\nabla$ be a Riemannian connection in $(\sigma, g)$. Then $\nabla$ is spacelike symmetric (and hence admissible) iff its restriction to $\tau_{\Sigma^s} \cong \sigma | \Sigma^s$, coincides with the Levi-Civita connection of the Riemannian metric $J^s_i g$ on $\Sigma$ ($j_s$ is the injection $\Sigma_s \hookrightarrow \mathcal{M}$) for all $s \in R$. Moreover, let $\nabla$ be a connection in the vector bundle $\sigma$ and let $\Gamma^s_{ij}$ be the connection parameters for $\nabla$ determined by a chart $(U, x^s)$. Then $\nabla$ is admissible iff we have

$$\Gamma^s_{ij} = \gamma^s_{ij}, \quad \frac{\partial g_{ij}}{\partial x^0} = g_{ik} \Gamma^s_{0j} + g_{jk} \Gamma^s_{0i}. \quad (20)$$

The following proposition, in a somewhat different formulation, is to be found in [1].

Proposition 5. Let $\nabla$ be an admissible connection in the Riemannian bundle $(\sigma, g)$ and let $h \in \text{Sec} (\tau^*_s \otimes L_\sigma)$. Then also the connection $\nabla + h$ is admissible iff we have

$$g(h_x \phi, \psi) + g(\phi, h_x \psi) = 0, \quad h_x \psi = 0, \quad (21)$$

for any vector field $X$ on $\mathcal{M}$ and any $\phi, \psi \in \text{Sec} \sigma$. Moreover, there is a bijection between the tensor fields $h$ satisfying (21) and the 2-forms $\eta \in \text{Sec} (A^* \sigma^*)$ given by

$$g(\phi, h_x \psi) = \langle dt, X \rangle \eta(\phi, \psi). \quad (22)$$

The local expression of (21) and (22) are given by

$$g_{ik} h_{ij}^x + g_{ij} h_{ki}^x = 0, \quad h_{ij}^x = 0, \quad (21)'$$

$$h = h_{0i} dx^0 \otimes dx^i \otimes \frac{\partial}{\partial x^i} \Leftrightarrow \eta = \frac{1}{2} g_{ik} h_{0j} dx^i \wedge dx^j, \quad (22)'$$

respectively.

(ii) An admissible connection in the Riemannian bundle can be obtained by a Galilean connection in $\mathcal{M}$, that is a spacelike, symmetric connection $\nabla$.
on $\mathcal{M}$ inducing a Riemannian connection $\nabla$ (we use the same symbol $\nabla$) in the Riemannian bundle $(\sigma, g)$. Here we are interested to the fact that to a framing $\mathcal{F}$ we can associate a Galilean connection $\nabla^\phi$ and hence an admissible connection in $(\sigma, g)$. Indeed we have the following proposition (cfr. [1]).

**Proposition 6.** Let $\mathcal{F}$ be a framing and let $g_\mathcal{F}$ be the Riemannian metric on $\mathcal{M}$ given by $g_\mathcal{F} = p_\mathcal{F}^*g + dt \otimes dt$. Let $h$ be the element of $\text{Sec}(\tau_\mathcal{K}^\times \otimes L_{\tau_\mathcal{K}})$ so defined

$$g_\mathcal{F}(h_X Y, Z) = \frac{1}{2} \langle dt, Z \rangle L_{\tau_\mathcal{F}}g(p_\mathcal{F}X, p_\mathcal{F}Y),$$

for any $X, Y, Z \in \mathcal{X}(\mathcal{M})$ and let $\nabla^\phi$ be the connection on $\mathcal{M}$ obtained by adding $h$ to the Levi-Civita connection of $g_\mathcal{F}$. Then $\nabla^\phi$ is a Galilean connection on $\mathcal{M}$. If $(U, x^\alpha)$ is a chart adapted to $\mathcal{F}$, the connection parameters for $\nabla^\phi$ determined by this chart are

\begin{equation}
(I^\phi)^k_\alpha = \gamma^k_{ij}, \quad (I^\phi)^{\alpha_0} = 0, \quad (I^\phi)^{\alpha_0}_{\beta \alpha} = \frac{1}{2} g^{\beta \alpha_0} \frac{\partial g_{\beta \alpha}}{\partial x^\alpha}.
\end{equation}

**Proof.** Since we have

\begin{equation}
L_{\tau_\mathcal{F}}g = \frac{\partial g_{ij}}{\partial x^k} dx^i \otimes dx^j,
\end{equation}

from the Christoffel symbols of the Levi-Civita connection of $g_\mathcal{F}$ and from (23), we easily see that $\nabla^\phi$ is spacelike and that (24) holds. On the other hand, (20) is satisfied and hence $\nabla^\phi$ is admissible.

(iii) Let $\nabla$ be a Galilean connection on $\mathcal{M}$ and let $\xi$ and $\Gamma \in \text{Hom}(\tau_\mathcal{K}, V_\mathcal{K})$ be respectively the dynamic equation and the vertical projection associated to it [see Prop. 1]. Then we get a 2-form on $K$ as follows [cfr. Prop. 3].

**Proposition 7.** Let $\nabla$ be a Galilean connection on $\mathcal{M}$ and let $\omega \in \text{Sec}(\Lambda^2\tau_\mathcal{K}^\times)$ be given by

\begin{equation}
\omega(X, Y) = g(\nu Y, \Gamma X) - g(\nu X, \Gamma Y), \quad X, Y \in \mathcal{X}(K).
\end{equation}

Then $\omega$ is of maximal rank on $K$ and $\xi$ is the unique vector field on $K$ that is a characteristic vector field of $\omega$ and satisfies $i_\xi dt = 1$. 


Proof. Let \((U, \omega^a)\) be a chart on \(M\). Then from (26) and from (2) we get

\[
\omega \left( \frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \right) = -g_{ij}, \quad \omega \left( \frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \right) = 0,
\]

from which we see that \(\omega\) is of maximal rank on \(K\). Note also that \(i_{q^i} \omega = 0\) since \(v(\xi) = I(\xi) = 0\).

Remark. In the case of the Newtonian-Galilean manifold, if \(\nabla\) is the canonical connection on \(M\), \(\omega\) coincides with the 2-form used in [5]. Then \(\omega\) is closed. We shall consider again this point.

(d) Accelerations of transport field of a framing \(F\). Relative to a Galilean connection \(\nabla\), this is defined to be \(\nabla_p F \in \text{Sec } \sigma\). Using charts adapted to \(F\), the local expression of \(\nabla_p F\) (as a vertical field on \(K\)) is given by

\[
\nabla_p F = I^{a_i} \frac{\partial}{\partial q^i}.
\]

If \(c\) is a motion, then \(I^{a_i} \ddot{c}\) is its (absolute) acceleration and we have \(I^{a_i} \ddot{c} = 0\) iff \(c\) is a geodesic of \(\xi\). Note that we have \(\nabla_p F = 0\) iff any observer in \(F\) is a geodesic of \(\xi\) (in this case we say that \(F\) is not accelerated or geodesic).

(e) Vorticity 2-form of a framing \(F\). (i) Relative to a Galilean connection \(\nabla\), this is defined to be the 2-form \(\Omega_p \in \text{Sec } (\Lambda^2 \sigma^a)\) that we get from the tensor \(\nabla - \nabla \in \text{Sec } (\tau^a_{\nu} \otimes L_\sigma)\) via the bijection (22). Using charts adapted to \(F\), the local expression of \(\Omega_p\), as follows at once from (22') and (24), is given by

\[
2\Omega_p = g_{ij} [\Gamma^k_{vi} - (\Gamma^k_{vi})_a] dx^i \wedge dx^j.
\]

The meaning of the vorticity 2-form becomes the usual one in the following situation.

Proposition 8. Let \(F\) and \(F'\) be framings and let \(\Omega^a_{p'}\) be the vorticity 2-form of the framing \(F'\) relative to the Galilean connection \(\nabla')\). Then we have

\[
2\Omega^a_{p'} = d \phi^b, \quad \phi = F' - F \in \text{Sec } \sigma.
\]

Proof. Let \((U, \omega^a)\) be a chart adapted to \(F\) and put \(\phi^i = \langle dx^i, \phi\rangle\), \(\phi_i = g_{ij} \phi^j\). Then a simple calculation shows that

\[
(\Gamma^p)_{ij} = \frac{1}{2} g^{sh} \frac{\partial g_{ij}}{\partial x^h} + \frac{1}{2} g^{sh} \left( \frac{\partial \phi_j}{\partial x^h} - \frac{\partial \phi_i}{\partial x^h} \right),
\]

and hence the result follows from (24) and (29).
(ii) Let $F$ be a framing. Later, in connexion with the apparent forces relative to $F$, we need the vertical field $P_F \in \text{Sec}(V_K)$ such that (cfr. (4))

\[(P_F)^b = i_{V_F} \Omega_F.\]

Here $\Omega_F$ is viewed as an element of $\text{Sec}(A^2 V^*_K)$ [recall that $V_K \cong \mathcal{F}^* \sigma^*$. Using charts adapted to $F$, the local expression of $P_F$ is given by

\[P_F = [(T^i)_{kj} - (T^i)_{kj}] \frac{\delta}{\delta q^i}.\]

Note that $P_F = 0$ iff $\Omega_F = 0$ [in this case we say that $F$ is not rotating] and that we have $P_F c = 0$ if $c$ is an observer in $F$.

(f) *Rate of deformation tensor of a framing $F$. This is defined to be the symmetric tensor field $L_F g \in \text{Sec}(V^* \sigma^*)$. The framing $F$ is said to be rigid iff $L_F g = 0$. Later, in connexion with apparent forces, we need the vertical field $\Lambda_F \in \text{Sec}(V_K)$ such that

\[v^*_F((\Lambda_F)^b) = d_{\nu_F} L_F G_F.\]

Using charts adapted to $F$, the local expression of $\Lambda_F$ is given by

\[\Lambda_F = 2(T^i)_{kj} \frac{\delta}{\delta q^i},\]

where we have used (15) and (24). Note that $\Lambda_F = 0$ iff $F$ is rigid and that we have $\Lambda_F c = 0$ if $c$ is an observer in $F$.

(g) *Three important theorems. The first theorem give, in our scheme, a well known result in the case of the Newtonian-Galilean manifold.

**Theorem 9.** Let $\nabla$ be a Galilean connection and let $F$ be a framing. Then if $\hat{\nabla}F \in \text{Sec} L(\sigma, \sigma^*)$ is the spatial part of $\nabla F$, we have

\[\hat{\nabla}F = \frac{1}{2} L_F g + \Omega_F.\]

**Proof.** Using charts adapted to $F$, the result follows from (7), (24), (25) and (29).

The next theorem will be used in connexion with apparent forces.
Theorem 10. Let $\nabla$ be a Galilean connection and let $F$ be a framing. Then we have

\begin{equation}
\xi_F - \xi = \nabla_F \xi + 2 P_F + A_F.
\end{equation}

Proof. Using charts adapted to $F$, the local expression of $\xi_F - \xi$ is given by

\begin{equation}
\xi_F - \xi = (\Gamma^k_{00} + 2 \Gamma^k_{0j} \dot{q}^j) \frac{\partial}{\partial \dot{q}^k}.
\end{equation}

Hence the result follows from (29), (33) and (35).

The proof of the last theorem is left as a simple exercise.

Theorem 11. Let $\nabla$ be a Galilean connection and let $F$ be a framing. Then the following conditions are equivalent

(i) $\xi = \xi_F$,
(ii) $\omega = \omega_F$,
(iii) $F$ is geodesic, not rotating and rigid,
(iv) $\nabla F = 0$ (that is, $F$ is parallel with respect to $\nabla$),
(v) $F$ is rigid and $\nabla^F = \nabla$.

Corollary 12. Let $F$ and $F'$ be framings and suppose that $F$ satisfies one of the preceding equivalent conditions. Then the same holds for $F'$ iff $F'$ is rigid, non rotating with respect to $F$ (that is $\Omega^F_{\xi_F} = 0$) and geodesic.

4. - Dynamics

A formulation of dynamics on the kinematic space $K$ requires a Galilean connection $\nabla$ on the event space $M$. We proceed as follows.

(a) Inertial structure on $M$. This is a Galilean connection $\nabla$ satisfying the condition that for any $p \in M$ there is a chart $(U, x^a)$ at $p$ such that $\Gamma^k_{00} = \Gamma^k_{0j} = 0$.

(b) Inertial observers and inertial framings. The first are geodesics of $\xi$ (inertial dynamic equation). The second are framings satisfying one of the equivalent conditions of Th. 11 (locally on open sets $U \subset M$ and $W = \pi^{-1}(U) \subset K$). Note that if $F$ is inertial, then also all the observers in $F$ are inertial (recall that this is equivalent to $F$ of being geodesic: cfr. (d), 3).
Remark. However, in the case of the Newtonian-Galileean manifold, a
gedesic framing $F$ over (all!) $M$, which is linear affine, is inertial.

(c) The inertial contact structure on $K$. This is the 2-form $\omega$ associated
to the inertial connection $\nabla$, cfr. (26). We leave to prove that we have $d\omega = 0$.
Recall that the inertial dynamic equation $\xi$ is characterized by $i_\xi \omega = 0$
and $i_\xi dt = 1$.

(d) The 2-form associated to a force. Relative to a force $f \in \text{Sec} (V_K)$ and
to a body of mass $m$, $m > 0$, this is the 2-form of maximal rank on $K$ so defined

\[ \varrho = m\omega + \psi^b (f^b) \wedge dt. \]

The following proposition is of basic importance in dynamics.

Proposition 13. There is a unique dynamic equation $X$ on $K$ such
that $\Gamma(X) = f/m$, that is

\[ X = \xi + f/m. \]

Moreover, $X$ is the unique vector field on $K$ that is a characteristic vector field
of $\varrho$ and satisfies $i_X dt = 1$.

(e) Absolute formulation of dynamics.

Definition 14. We say that a motion $c$ is dynamically admissible for a
body of mass $m$ ($m > 0$) under the action of a force $f$ (called effective), iff we have

\[ m(\Gamma o c) = f o c. \]

Theorem 15. A motion $c$ is dynamically admissible for a body of mass $m$
under the action of an effective force $f$ iff is a solution of the dynamic equation (38),
that is $X o c = \dot{c}$.

(f) Framings and apparent forces. Only by choosing a framing $F$, the
absolute formulation given in (e) can be interpreted from a physical point
of view.

Definition 16. Let $F$ be a framing. Then we say that $m(\xi - \xi_F) \in \in \text{Sec} (V_K)$ is the apparent force relative to $F$ of a body of mass $m$, $m > 0$.

From Th. 10 we see that the apparent force $m(\xi - \xi_F)$ is made up of the
three terms.
(i) \(-m\nabla_F F\) which arises solely from the acceleration of observers in 
\(F\) (cfr. (d), 3.);

(ii) \(-2mP_F\) which is the Coriolis force;

(iii) \(-mA_F\) which arises solely from the deformation of \(F\).

**Theorem 17.** Let \(F\) be a framing. Then a motion \(c\) is dynamically admissible for a body of mass \(m (m > 0)\) under the action of an effective force \(f\) iff we have

\[
m(\Gamma_F \omega \ddot{c}) = (f + m(\xi - \xi_F)) \omega \dot{c}.
\]

**Proof.** In fact, we have \(\Gamma_F \omega \ddot{c} = \Gamma \omega \ddot{c} + (\xi - \xi_F) \omega \dot{c}\), as is easily seen, for example, by using a chart \((U, \omega)\) adapted to \(F\) (see also (13)).

This is the second law of dynamics written in an arbitrary framing.

(g) **Kinetic energy and Lagrange's equations.** Let \(F\) be a framing. Then it is clear that \(X\) (cfr. (38)) is the unique dynamic equation such that

\[
\Gamma_F(X) = f + m \xi - \xi_F.
\]

However, by considering the kinetic energy relative to \(F\), that is \(T_F = mG_F\), there is another way of characterizing \(X\) that leads to the Lagrange's equations [but then \(F\) must be rigid]. For this we need the 2-form of maximal rank on \(K\) so defined (cfr. (37))

\[
\sigma_F = m\omega_F + [v_F^b(\pi)] \wedge dt,
\]

where \(\pi = (f + m(\xi - \xi_F))^b \in \text{Sec} (V_F^*)\).

In the traditional approach the vector field \(X\) (and hence Lagrange's equations) is deduced from the 2-form \(\sigma_F\) (or from some other equivalent way) [2]. However, this is not the present case since we have given an "absolute" characterization of \(X\). The following theorem can be proved.

**Proposition 18.** Let \(F\) be a framing. Then \(X\) is the unique characteristic vector field of \(\sigma_F\) such that \(i_X dt = 1\). Moreover, if \(F\) is rigid and \((U, x)\) is a chart on \(M\), a motion \(c\) is a base integral curve of \(X\) iff locally satisfies Lagrange's equations

\[
\frac{d}{dt} \left( \frac{\partial T_F}{\partial q^i} \right) - \frac{\partial T_F}{\partial q^i} = \pi_i, \quad \pi_i = \langle \pi, \frac{\partial}{\partial q^i} \rangle.
\]
References


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