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Mechanics on a Galileean manifold (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

In this paper we examine some geometric properties of the dynamics on a Galileean manifold. This consists [**1**] of the so called absolute time bundle $t: M \to R$ (M is the event space) together with a Riemannian metric g in the spacelike bundle $\sigma = \text{Ker. d}t$. Hence a Galileean manifold generalizes the well known Newtonian-Galileean manifold in which the bundle $t: M \to R$ is linear affine and g is Euclidean [**5**], [**6**].

The « arena » of dynamics is the kinematic space K which is defined as the submanifold of the tangent space TM such that $\mathrm{d}t|K = 1$. If $\pi: K \to M$ is the canonical projection, then the bundle $\pi: K \to M$ has a canonical affine structure over the vector bundle σ. All the basic concepts are related to the kinematic space K. For example, a motion c in M is a section of the bundle $t: M \to R$ and hence its velocity $\dot{c}$ is a section of the bundle $c^*\pi$ (the pull-back of the bundle $\pi: K \to M$ over c); a framing F is a section of the bundle $\pi: K \to M$; a force f is a section of the vertical bundle $V_K = \text{Ker. } T\pi \hookrightarrow \tau_K$ and a dynamic equation X (that is an equation of motion) is a section of an affine subbundle of the tangent bundle τ_K modeled on the vector bundle V_K. Moreover, the concept of acceleration and that of Galileean connection on M (cfr. (c), **3**) lead to consider vertical projections $\Gamma \in \text{Hom}(\tau_K, V_K)$ providing affine isomorphisms between forces and dynamic equations [**4**]. However, we

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are « forced » to work in the space K in order to use « absolute » concepts avoiding to choose a distinguished framing F. In this connexion, let us emphasize the role played through the paper by the vertical endomorphism v (see (2), (18), (26), (37), etc).

But physical interpretation require the use of framings. For this we give a detailed analysis of all the concepts related to a framing F. For example, to F there are associated a projection $p_F \in \operatorname{Hom}(\tau_M, \sigma)$, a vertical projection $\Gamma_F \in \operatorname{Hom}(\tau_K, V_K)$ used to define the acceleration relative to F, a dynamic equation ξ_F, a contact structure ω_F on K, a Galileean connection ∇^F on M which is used to define the vorticity 2-form Ω_F of the framing F, a kinetic energy T_F, the apparent force $-m\Delta_F$ which arises from the deformation of F, etc. Certain novelties should be found in the way in which some of these concepts are introduced (inertial forces, contact structures on the kinematic space K, etc.).

The work is motivated by applications, mainly to classical analytical mechanics. For this reason, we let M be an $n+1$ dimensional manifold, $n \geqslant 1$. However, in the § 4 on Dynamics, it may be supposed $n=3$, though this is not relevant in the context of the discussion. As a simple example, note that an « holonomic constraint » in the Newtonian-Galileean manifold (M, t, g) is merely a « Galileean submanifold » of (M, t, g). For example, if φ is a function on M, we get an holonomic constraint $N = \varphi^{-1}(0)$ (it is assumed that 0 is a regular value of φ and that $N \cap t^{-1}(\tau) \neq \phi$, $\tau \in R$) by requiring that $\operatorname{Ker.} d\varphi(p) \neq \operatorname{Ker.} dt(p)$ for all $p \in N$. Indeed, it is clear that $(N, t/N, j^*g)$ is a Galileean manifold (here j is the injection $N \hookrightarrow M$). Recall that the traditional approach assumes a Cartesian product structure of the event space M, that is a choice of a distinguished framing is made (usually inertial).

To finish, let us remark that definitions and formulae are always written in an intrinsic way. However, in the proofs local coordinates are used (proofs omitted are simple).

2. - Preliminaries

We start by establishing the notation. Then we consider some basic concepts.

(a) *Notation.* This is the one usually found in any text on modern differential geometry (we refer to [2]). However, about connections and their associated horizontal subbundles and sprays, we refer to [**3**].

All manifolds, bundles and tensor fields will be C^∞. If M is a manifold, TM denotes its tangent space (T_pM denotes the tangent space at $p \in M$) and $\tau_M\colon TM \to M$ its tangent bundle. We denote by $C^\infty(M)$ the ring of the functions on M and by $\mathscr{X}(M)$ the $C^\infty(M)$-module of the vector fields on M. If c is a curve in M, we denote by $\dot{c}$ and by $\ddot{c}$ its canonical liftings to TM and to TTM, respectively.

If σ is a vector bundle, we denote by Sec $(\Lambda^p\sigma^*)$ the module of the p-forms on σ. Some tensor products and canonical isomorphisms of vector bundles are used: for example, $L(\tau_M; \sigma) \cong \tau_M^* \otimes \sigma$ (the bundle $L(\sigma; \sigma)$ is denoted by L_σ), etc.

To finish, the symbol d denotes the exterior derivative, L_X the Lie derivative along a vector field X, and i the interior product.

(b) *The absolute time bundle.* This is the bundle $t: M \to R$ where M is the *event space* and the projection t is the *absolute time*. We shall usually write simple M to denote this bundle. The n-dimensional submanifolds $\Sigma_\tau = t^{-1}(\tau) \subset M$ (the fiber of M at $\tau \in R$) are the *spaces of absolute symultaneity*.

A *motion in* M, usually denoted by c, is a section of M over an open interval $I \subset R$.

On M we shall always use *adapted* charts (U, x^α) where $x^0 = t\,|\,U$.

(c) *Observers and framings.* (i) An *instantaneous observer* is an ordered pair (p, u) where $p \in M$ and $u \in T_pM$ is such that $\langle \mathrm{d}t(p), u\rangle = 1$.

Let $K \subset TM$ be the $(2n+1)$-dimensional submanifold of the instantaneous observers and let $\pi: K \to M$ be the canonical projection. Then the bundle $\pi: K \to M$ has a canonical affine structure on the vector bundle $\sigma = \mathrm{Ker.d}t$. It is clear that σ is an involutive subbundle of τ_M and that $\tau_{\Sigma_s} \cong \sigma\,|\,\Sigma_s$, $s \in R$. Elements of Sec σ are called *spacelike vector fields on* M. Note that spacelike form on M cannot be defined.

Let c be a motion. Then $\dot{c} \in \mathrm{Sec}\,(c^*\pi)$ is its (*absolute*) *velocity* [$c^*\pi$ is the pull-back of π over c]. For this reason, K is also called the *kinematic space*.

Let (U, x^α) be a chart on M. We shall denote by $\{\partial/\partial x^i\}$ the induced basis of the module Sec $(\sigma|U)$ and by $(\pi^{-1}(U), q^\alpha, \dot{q}^i)$ the induced chart on K.

(ii) We define an *observer* to be a motion. However we need a « cooperation » of observers in order to interpret in physical terms the laws of dynamics written in the space K. Hence we define a *framing*, typically denoted by F, to be a section of K, that is $F \in \mathrm{Sec}\,\pi$. If c is an observer, we say that c is an *observer in* F iff c is an integral curve of F. The *velocity of* c *relative to* F is defined to be $\dot{c} - F \circ c \in \mathrm{Sec}\,(c^*\sigma)$.

Let F be a framing. Then to F there is associated the projection $p_F \in \mathrm{Hom}\,(\tau_M, \sigma)$ given by

$$p_F(u) = u - \langle \mathrm{d}t(p), u\rangle F(p)\,, \qquad u \in T_pM\,. \tag{1}$$

Clearly we have $\tau_M = \sigma \oplus \mathrm{Ker.}\,p_F$.

Let F be a framing and let $p \in M$. Then there exists a chart (U, x^α) at p such that $F|U = \partial/\partial x^0$. Charts with such a property are said to be *adapted to* F.

(d) *Dynamic equations.* (i) Let V_K be the vertical subbundle Ker. $T\pi \hookrightarrow \tau_K$. Note that we have $V_K \cong \pi^*\sigma$. If (U, x^α) is a chart on M, we denote by $\{\partial/\partial \dot{q}^i\}$ the induced basis of the module Sec $(V_K|\pi^{-1}(U))$.

We now introduce the vertical endomorphism v characterized locally by

$$v(\frac{\partial}{\partial q^0}) = -\dot{q}^i \frac{\partial}{\partial \dot{q}^i}, \qquad v(\frac{\partial}{\partial q^i}) = \frac{\partial}{\partial \dot{q}^i}, \qquad v(\frac{\partial}{\partial \dot{q}^i}) = 0 . \tag{2}$$

It follows at once that v^* is characterized by

$$v^*(\mathrm{d}q^\alpha) = 0 , \qquad v^*(\mathrm{d}\dot{q}^i) = -\dot{q}^i \mathrm{d}q^0 + \mathrm{d}q^i . \tag{3}$$

The vertical endomorphism v will be used later to get a 2-form on the space K from a Galileean connection on M.

Let F be a framing. Then to F there are associated a vertical field V_F and a vertical endomorphism v_F characterized locally by (using charts adapted to F)

$$V_F = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}, \tag{4}$$

and by

$$v_F(\frac{\partial}{\partial q^0}) = v_F(\frac{\partial}{\partial \dot{q}^i}) = 0 , \qquad v_F(\frac{\partial}{\partial q^i}) = \frac{\partial}{\partial \dot{q}^i}, \tag{5}$$

respectively.

(ii) A *dynamic equation*, typically denoted by X, is a vector field on K such that

$$\langle \mathrm{d}t, X\rangle = 1 , \qquad v(X) = 0 .$$

It follows at once that the local expression of a dynamic equation X is given by

$$X = \frac{\partial}{\partial q^0} + \dot{q}^i \frac{\partial}{\partial q^i} + b^i \frac{\partial}{\partial \dot{q}^i}, \qquad b^i \in C^\infty(\pi^{-1}(U)) . \tag{6}$$

Note that dynamic equations (which are some sort of second order differential equations on M) are sections of an affine subbundle of τ_K modeled on the vertical bundle V_K. Hence we define a *force*, typically denoted by f, to be an element of Sec (V_K).

(e) *Spacelike connections.* A (linear) connection ∇ on M is called *spacelike* iff $\nabla \mathrm{d}t = 0$. Let ∇ be a connection on M and let $\Gamma^{\gamma}_{\alpha\beta} \in C^{\infty}(U)$ be the connection parameters for ∇ determined by a chart (U, x^{α}). Then we have locally

$$\nabla \mathrm{d}t = -\Gamma^{0}_{\alpha\beta}\,\mathrm{d}x^{\alpha} \otimes \mathrm{d}x^{\beta}$$

and hence ∇ is spacelike iff $\Gamma^{0}_{\alpha\beta} = 0$. Note that from a spacelike connection ∇ we get a connection, also denoted by ∇, on the vector bundle σ.

Let ∇ be a spacelike connection and let F be a framing. Then $\nabla F \in \mathrm{Sec}\, L(\tau_M; \sigma)$ and from the canonical projection $\tau^{*}_{M} \to \sigma^{*}$ we get the *spatial part* $\widehat{\nabla F} \in \mathrm{Sec}\, L_{\sigma}$. If (U, x^{α}) is a chart adapted to F, the local expression of $\widehat{\nabla F}$ is given by

$$\widehat{\nabla F} = \Gamma^{k}_{i0}\,\mathrm{d}x^{i} \otimes \frac{\partial}{\partial x^{k}}\,. \tag{7}$$

We also need the following results about spacelike connections.

Proposition 1. Let ∇ be a spacelike connection on M and let $H_{TM} \hookrightarrow \tau_{TM}$ be the horizontal subbundle associated to it [**3**]. Then the restriction $H_{TM}|K$ gives an horizontal subbundle of τ_K, that is $V_K \oplus H_{TM}|K = \tau_K$. A local basis of this subbundle is given by

$$E_{\alpha} = \frac{\partial}{\partial q^{\alpha}} - (\Gamma^{k}_{\alpha 0} + \Gamma^{k}_{\alpha j}\dot{q}^{j})\,\frac{\partial}{\partial \dot{q}^{k}}\,. \tag{8}$$

Moreover, the spray ξ of ∇ restricts to a dynamic equation, also denoted by ξ, whose local expression is

$$\xi = E_0 + \dot{q}^{i} E_i\,. \tag{9}$$

If $\Gamma \in \mathrm{Hom}\,(\tau_K, V_K)$ is the vertical projection, then we have $\Gamma(\xi) = 0$ and Γ restricts to an affine isomorphism between dynamic equations and forces [**4**].

Remark. More generally, framings and connections, dynamic equations and forces must be considered over open sets $U \subset M$ and $W \subset K$, respectively. However, later we shall be interested to the case where $W = \pi^{-1}(U)$.

3. - The riemannian bundle (σ, g) and some consequences

Now we equip the bundle σ with a (positive definite) Riemannian metric g. We say that M becomes a *Galileean manifold*. The following are some consequences of this structure.

(a) *Lowering and raising indices*. The isomorphism $\sigma \to \sigma^*$ induced by the metric g will be denoted by $\flat$. Note that since $V_K \cong \pi^*\sigma$, V_K inherits the metric g.

(b) *Framings, dynamic equations and contact structures*. (i) First of all note that the restriction $g|\Sigma_\tau$ is a Riemannian metric on $\Sigma_\tau \subset M$, $\tau \in R$. Now let g_{ij}, $g^{kh} \in C^\infty(U)$ be the components of the metric g determined by a chart (U, x^α) and put

$$\gamma^k_{ij} = \frac{1}{2}\, g^{kh}\left(\frac{\partial g_{hi}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h}\right). \tag{10}$$

It follows that $\gamma^k_{ij}\,|\,U_\tau$ are the Christoffel symbols of the Levi-Civita connection of $g\,|\,\Sigma_\tau$ determined by the chart $(U_\tau, x^i\,|\,U_\tau)$ on Σ_τ, $U_\tau = \Sigma_\tau \cap U$, $\tau \in R$.

To define the acceleration relative to F, we need the following proposition.

Proposition 2. Let F be a framing and let (U, x^α) be a chart adapted to F. Then there exists an horizontal subbundle $H_K \hookrightarrow \tau_K$ with local basis

$$E_0 = \frac{\partial}{\partial q^0}, \qquad E_i = \frac{\partial}{\partial q^i} - \gamma^k_{ij}\,\dot q^j\,\frac{\partial}{\partial \dot q^k}. \tag{11}$$

Proof. Indeed if (U', x'^α) is another chart adapted to F such that $U \cap U' \neq \phi$, we easily see that $E'_\varrho = (\partial x^\alpha/\partial x'^\varrho)E_\alpha$. Hence the result follows at once.

Let $\Gamma_F \in \mathrm{Hom}\,(\tau_K, V_K)$ be the vertical projection determined by this subbundle and let ξ_F be the dynamic equation uniquely determined by $\Gamma_F(\xi_F) = 0$. Let (U, x^α) be a chart adapted to F. Then the local expression of ξ_F is given by

$$\xi_F = \frac{\partial}{\partial q^0} + \dot q^i\,\frac{\partial}{\partial q^i} - \gamma^k_{ij}\,\dot q^i\,\dot q^j\,\frac{\partial}{\partial \dot q^k}, \tag{12}$$

as we see from (6) and (11). If c is a motion, we define $\Gamma_F \circ \ddot c$ to be the *acceleration of c relative to F*. Note that we can write $\Gamma_F \circ \ddot c \in \mathrm{Sec}\,(c^*\sigma)$ since $V_K \cong \pi^*\sigma$.

Locally we have (using charts adapted to F)

$$\Gamma_F \circ \ddot{c} = [\ddot{c}^k + (\gamma^k_{ij} \circ c)\, \dot{c}^i \dot{c}^j] \left(\frac{\partial}{\partial x^k} \circ c\right), \qquad c^i = x^i \circ c. \tag{13}$$

It is clear that an observer c has no acceleration relative to F iff is a base integral curve of ξ_F (note that observers in F are not accelerated relative to F).

(ii) Let F be a framing. Then we get a quadratic function $G_F \in C^\infty(K)$ by putting (cfr. (1))

$$G_F(u) = \tfrac{1}{2} g(p_F(u), p_F(u)), \qquad u \in K. \tag{14}$$

Its local expression is given by (henceforth charts adapted to F will be used)

$$G_F = \tfrac{1}{2} g_{ij}\, \dot{q}^i \dot{q}^j. \tag{15}$$

From the vertical differentiation d_{v_F} associated to the framing F (cfr. [**2**]), we get the 1-form on K

$$\mathrm{d}_{v_F} G_F = g_{ij} \dot{q}^i \mathrm{d}q^j. \tag{16}$$

This is an example of a *spacelike form relative to* F (on K). Such forms are easily characterized. Let us denote by d_F the exterior derivative acting on these forms (d_F is induced by F in an obvious way). Then we get the spacelike 2-form on K

$$\mathrm{d}_F \mathrm{d}_{v_F} G_F = \frac{\partial g_{jh}}{\partial q^i} \dot{q}^h \mathrm{d}q^i \wedge \mathrm{d}q^j - g_{ij} \mathrm{d}q^i \wedge \mathrm{d}\dot{q}^j. \tag{17}$$

In the following proposition we introduce the contact structure associated to a framing F.

Proposition 3. Let F be a framing and let $\omega_F \in \mathrm{Sec}\,(\Lambda^2 \tau^*_K)$ be given by

$$\omega_F(X, Y) = g(vY, \Gamma_F X) - g(vX, \Gamma_F Y), \qquad X, Y \in \mathscr{X}(K). \tag{18}$$

Then ω_F is closed (relative to d_F) and of maximal rank on K, and we have

$$\omega_F = \mathrm{d}_F\, \mathrm{d}_{v_F} G_F - \mathrm{d}G_F \wedge \mathrm{d}t. \tag{19}$$

Moreover, ξ_F is the unique vector field on K that is a characteristic vector field of ω_F and satisfies $i_{\xi_F} \mathrm{d}t = 1$.

(c) *Admissible and Galileean connections.* (i) A connection ∇ in the (involutive) subbundle $\sigma \hookrightarrow \tau_M$ is called *spacelike symmetric* iff we have $\nabla_\phi \psi - \nabla_\psi \phi = [\phi, \psi]$ for all $\phi, \psi \in \operatorname{Sec} \sigma$. Moreover, a spacelike symmetric Riemannian connection in (σ, g) is called *admissible.* We have the following proposition.

Proposition 4. Let ∇ be a Riemannian connection in (σ, g). Then ∇ is spacelike symmetric (and hence admissible) iff its restriction to $\tau_{\Sigma_s} \cong \sigma | \Sigma_s$ coincides with the Levi-Civita connection of the Riemannian metric $j_s^* g$ on Σ_s (j_s is the injection $\Sigma_s \hookrightarrow M$) for all $s \in R$. Moreover, let ∇ be a connection in the vector bundle σ and let $\Gamma^k_{\alpha j}$ be the connection parameters for ∇ determined by a chart (U, x^α). Then ∇ is admissible iff we have

$$\Gamma^k_{ij} = \gamma^k_{ij}\,, \qquad \frac{\partial g_{ij}}{\partial x^0} = g_{ik}\Gamma^k_{0j} + g_{jk}\Gamma^k_{0i}\,. \tag{20}$$

The following proposition, in a somewhat different formulation, is to be found in [**1**].

Proposition 5. Let ∇ be an admissible connection in the Riemannian bundle (σ, g) and let $h \in \operatorname{Sec}(\tau^*_M \otimes L_\sigma)$. Then also the connection $\nabla + h$ is admissible iff we have

$$g(h_X \phi, \psi) + g(\phi, h_X \psi) = 0\,, \qquad h_\phi \psi = 0\,, \tag{21}$$

for any vector field X on M and any $\phi, \psi \in \operatorname{Sec} \sigma$. Moreover, there is a bijection between the tensor fields h satisfying (21) and the 2-forms $\eta \in \operatorname{Sec}(\Lambda^2 \sigma^*)$ given by

$$g(\phi, h_X \psi) = \langle \mathrm{d}t, X \rangle \eta(\phi, \psi)\,. \tag{22}$$

The local expression of (21) and (22) are given by

$$g_{ik} h^k_{0j} + g_{jk} h^k_{0i} = 0\,, \qquad h^k_{ij} = 0\,, \tag*{(21)$'$}$$

$$h = h^k_{0j}\, \mathrm{d}x^0 \otimes \mathrm{d}x^j \otimes \frac{\partial}{\partial x^k} \Leftrightarrow \eta = \frac{1}{2}\, g_{ik}\, h^k_{0j}\, \mathrm{d}x^i \wedge \mathrm{d}x^j\,. \tag*{(22)$'$}$$

respectively.

(ii) An admissible connection in the Riemannian bundle can be obtained by a *Galileean* connection in M, that is a spacelike, symmetric connection ∇

on M inducing a Riemannian connection ∇ (we use the same symbol ∇) in the Riemannian bundle (σ, g). Here we are interested to the fact that to a framing F we can associate a Galileean connection ∇^F and hence an admissible connection in (σ, g). Indeed we have the following proposition (cfr. [**1**]).

Proposition 6. Let F be a framing and let g_F be the Riemannian metric on M given by $g_F = p_F^* g + \mathrm{d}t \otimes \mathrm{d}t$. Let h be the element of $\mathrm{Sec}\,(\tau_M^* \otimes L_{\tau_M})$ so defined

$$g_F(h_X Y, Z) = \tfrac{1}{2}\langle \mathrm{d}t, Z\rangle L_F g(p_F X, p_F Y)\,,$$

for any $X, Y, Z \in \mathscr{X}(M)$ and let ∇^F be the connection on M obtained by adding h to the Levi-Civita connection of g_F. Then ∇^F is a Galileean connection on M. If (U, x^α) is a chart adapted to F, the connection parameters for ∇^F determined by this chart are

$$(\Gamma^F)^k_{ij} = \gamma^k_{ij}\,, \qquad (\Gamma^F)^k_{00} = 0\,, \qquad (\Gamma^F)^k_{0j} = \frac{1}{2}\, g^{kh}\, \frac{\partial g_{hj}}{\partial x^0}\,. \tag{24}$$

Proof. Since we have

$$L_F g = \frac{\partial g_{ij}}{\partial x^0}\, \mathrm{d}x^i \otimes \mathrm{d}x^j\,, \tag{25}$$

from the Christoffel symbols of the Levi-Civita connection of g_F and from (23), we easily see that ∇^F is spacelike and that (24) holds. On the other hand, (20) is satisfyed and hence ∇^F is admissible.

(iii) Let ∇ be a Galileean connection on M and let ξ and $\Gamma \in \mathrm{Hom}\,(\tau_K, V_K)$ be respectively the dynamic equation and the vertical projection associated to it [see Prop. 1]. Then we get a 2-form on K as follows [cfr. Prop. 3].

Proposition 7. Let ∇ be a Galileean connection on M and let $\omega \in \mathrm{Sec}\,(\Lambda^2 \tau_K^*)$ be given by

$$\omega(X, Y) = g(vY, \Gamma X) - g(vX, \Gamma Y)\,, \qquad X, Y \in \mathscr{X}(K)\,. \tag{26}$$

Then ω is of maximal rank on K and ξ is the unique vector field on K that is a characteristic vector field of ω and satisfies $i_\xi \mathrm{d}t = 1$.

Proof. Let (U, x^α) be a chart on M. Then from (26) and from (2) we get

$$\omega\Big(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial \dot{q}^j}\Big) = -g_{ij}, \qquad \omega\Big(\frac{\partial}{\partial \dot{q}^i}, \frac{\partial}{\partial \dot{q}^j}\Big) = 0, \tag{27}$$

from which we see that ω is of maximal rank on K. Note also that $i_\xi \omega = 0$ since $v(\xi) = \Gamma(\xi) = 0$.

Remark. In the case of the Newtonian-Galileean manifold, if ∇ is the canonical connection on M, ω coincides with the 2-form used in [**5**]. Then ω is closed. We shall consider again this point.

(d) *Accelerations of transport field of a framing F.* Relative to a Galileean connection ∇, this is defined to be $\nabla_F F \in \operatorname{Sec}\sigma$. Using charts adapted to F, the local expression of $\nabla_F F$ (as a vertical field on K) is given by

$$\nabla_F F = \Gamma^k_{00} \frac{\partial}{\partial \dot{q}^k}. \tag{28}$$

If c is a motion, then $\Gamma \circ \ddot{c}$ is its (*absolute*) *acceleration* and we have $\Gamma \circ \ddot{c} = 0$ iff c is a geodesic of ξ. Note that we have $\nabla_F F = 0$ iff any observer in F is a geodesic of ξ (in this case we say that F is *not accelerated* or *geodesic*).

(e) *Vorticity 2-form of a framing F.* (i) Relative to a Galileean connection ∇, this is defined to be the 2-form $\Omega_F \in \operatorname{Sec}(\Lambda^2\sigma^*)$ that we get from the tensor $\nabla^F - \nabla \in \operatorname{Sec}(\tau^*_M \otimes L_\sigma)$ via the bijection (22). Using charts adapted to F, the local expression of Ω_F, as follows at once from (22′) and (24), is given by

$$2\Omega_F = g_{kj}[\Gamma^k_{0i} - (\Gamma^F)^k_{0i}]\, dx^i \wedge dx^j. \tag{29}$$

The meaning of the vorticity 2-form becomes the usual one in the following situation.

Proposition 8. Let F and F' be framings and let $\Omega^F_{F'}$ be the vorticity 2-form of the framing F' relative to the Galileean connection ∇^F. Then we have

$$2\Omega^F_{F'} = d\,\phi^\flat, \qquad \phi = F' - F \in \operatorname{Sec}\sigma. \tag{30}$$

Proof. Let (U, x^α) be a chart adapted to F and put $\phi^j = \langle dx^j, \phi\rangle$, $\phi_i = g_{ij}\phi^j$. Then a simple calculation shows that

$$(\Gamma^{F'})^k_{0j} = \frac{1}{2} g^{kh} \frac{\partial g_{hj}}{\partial x^0} + \frac{1}{2} g^{kh} \Big(\frac{\partial \phi_j}{\partial x^h} - \frac{\partial \phi_h}{\partial x^j}\Big), \tag{31}$$

and hence the result follows from (24) and (29).

(ii) Let F be a framing. Later, in connexion with the apparent forces relative to F, we need the vertical field $P_F \in \mathrm{Sec}\,(V_K)$ such that (cfr. (4))

$$(P_F)^\flat = i_{V_F} \Omega_F\,. \tag{32}$$

Here Ω_F is viewed as an element of $\mathrm{Sec}\,(\Lambda^2 V_K^*)$ [recall that $V_K \cong \pi^*\sigma$]. Using charts adapted to F, the local expression of P_F is given by

$$P_F = [\Gamma^k_{0j} - (\Gamma^F)^k_{0j}]\dot{q}^j \frac{\partial}{\partial \dot{q}^k}\,. \tag{33}$$

Note that $P_F = 0$ iff $\Omega_F = 0$ [in this case we say that F is *not rotating*] and that we have $P_F \circ \dot{c} = 0$ if c is an observer in F.

(f) *Rate of deformation tensor of a framing F.* This is defined to be the symmetric tensor field $L_F g \in \mathrm{Sec}\,(V^2\sigma^*)$. The framing F is said to be *rigid* iff $L_F g = 0$. Later, in connexion with apparent forces, we need the vertical field $\Delta_F \in \mathrm{Sec}\,(V_K)$ such that

$$v_F^*\big((\Delta_F)^\flat\big) = \mathrm{d}_{v_F} L_F G_F\,. \tag{34}$$

Using charts adapted to F, the local expression of Δ_F is given by

$$\Delta_F = 2(\Gamma^F)^k_{0j}\, \dot{q}^j \frac{\partial}{\partial \dot{q}^k}\,, \tag{35}$$

where we have used (15) and (24). Note that $\Delta_F = 0$ iff F is rigid and that we have $\Delta_F \circ \dot{c} = 0$ if c is an observer in F.

(g) *Three important theorems.* The first theorem give, in our scheme, a well known result in the case of the Newtonian-Galileean manifold.

Theorem 9. *Let ∇ be a Galileean connection and let F be a framing. Then if $\hat{\nabla} F \in \mathrm{Sec}\, L(\sigma, \sigma^*)$ is the spatial part of ∇F, we have*

$$\hat{\nabla} F = \tfrac{1}{2} L_F g + \Omega_F\,.$$

Proof. Using charts adapted to F, the result follows from (7), (24), (25) and (29).

The next theorem will be used in connexion with apparent forces.

Theorem 10. *Let ∇ be a Galileean connection and let F be a framing. Then we have*

$$\xi_F - \xi = \nabla_F F + 2P_F + \Delta_F . \tag{36}$$

Proof. Using charts adapted to F, the local expression of $\xi_F - \xi$ is given by

$$\xi_F - \xi = (\Gamma^k_{00} + 2\Gamma^k_{0j}\,\dot{q}^j)\,\frac{\partial}{\partial \dot{q}^k}\,.$$

Hence the result follows from (29), (33) and (35).

The proof of the last theorem is left as a simple exercise.

Theorem 11. *Let ∇ be a Galileean connection and let F be a framing. Then the following conditions are equivalent*

(i) $\xi = \xi_F$,

(ii) $\omega = \omega_F$,

(iii) *F is geodesic, not rotating and rigid,*

(iv) $\nabla F = 0$ *(that is, F is parallel with respect to ∇),*

(v) *F is rigid and $\nabla^F = \nabla$.*

Corollary 12. *Let F and F' be framings and suppose that F satisfies one of the preceding equivalent conditions. Then the same holds for F' iff F' is rigid, non rotating with respect to F (that is $\Omega^F_{F'} = 0$) and geodesic.*

4. - Dynamics

A formulation of dynamics on the kinematic space K requires a Galileean connection ∇ on the event space M. We proceed as follows.

(a) *Inertial structure on M.* This is a Galileean connection ∇ satisfying the condition that for any $p \in M$ there is a chart (U, x^α) at p such that $\Gamma^k_{00} = \Gamma^k_{0j} = 0$.

(b) *Inertial observers and inertial framings.* The first are geodesics of ξ (*inertial dynamic equation*). The second are framings satisfying one of the equivalent conditions of Th. 11 (locally on open sets $U \subset M$ and $W = \pi^{-1}(U) \subset K$). Note that if F is inertial, then also all the observers in F are inertial (recall that this is equivalent to F of being geodesic: cfr. (d), **3**).

Remark. However, in the case of the Newtonian-Galileean manifold, a geodesic framing F over (all!) M, which is linear affine, is inertial.

(c) *The inertial contact structure on* K. This is the 2-form ω associated to the inertial connection ∇, cfr. (26). We leave to prove that we have $d\omega = 0$. Recall that the inertial dynamic equation ξ is characterized by $i_\xi \omega = 0$ and $i_\xi dt = 1$.

(d) *The 2-form associated to a force.* Relative to a force $f \in \operatorname{Sec}(V_K)$ and to a body of mass m, $m > 0$, this is the 2-form of maximal rank on K so defined

$$\varrho = m\omega + v^*(f^\flat) \wedge dt . \tag{37}$$

The following proposition is of basic importance in dynamics.

Proposition 13. There is a unique dynamic equation X on K such that $\Gamma(X) = f/m$, that is

$$X = \xi + f/m . \tag{38}$$

Moreover, X is the unique vector field on K that is a characteristic vector field of ϱ and satisfies $i_X dt = 1$.

(e) *Absolute formulation of dynamics.*

Definition 14. We say that a motion c is dynamically admissible for a body of mass m ($m > 0$) under the action of a force f (called *effective*), iff we have

$$m(\Gamma \circ \ddot{c}) = f \circ \dot{c} .$$

Theorem 15. *A motion c is dynamically admissible for a body of mass m under the action of an effective force f iff is a solution of the dynamic equation* (38), *that is $X \circ \dot{c} = \ddot{c}$.*

(f) *Framings and apparent forces.* Only by choosing a framing F, the absolute formulation given in (e) can be interpreted from a physical point of view.

Definition 16. Let F be a framing. Then we say that $m(\xi - \xi_F) \in \operatorname{Sec}(V_K)$ is the apparent force relative to F of a body of mass m, $m > 0$.

From Th. 10 we see that the apparent force $m(\xi - \xi_F)$ is made up of the three terms.

(i) $-m\nabla_F F$ which arises solely from the acceleration of observers in F(cfr. (d), **3.**);

(ii) $-2mP_F$ which is the *Coriolis force*;

(iii) $-m\Delta_F$ which arises solely from the deformation of F.

Theorem 17. *Let F be a framing. Then a motion c is dynamically admissible for a body of mass m $(m > 0)$ under the action of an effective force f iff we have*

$$m(\Gamma_F \circ \ddot{c}) = (f + m(\xi - \xi_F)) \circ \dot{c} \,. \tag{40}$$

Proof. In fact, we have $\Gamma_F \circ \ddot{c} = \Gamma \circ \ddot{c} + (\xi - \xi_F) \circ \dot{c}$, as is easily seen, for esample, by using a chart (U, x^α) adapted to F (see also (13)).

This is the second law of dynamics written in an arbitrary framing.

(g) *Kinetic energy and Lagrange's equations.* Let F be a framing. Then it s clear that X (cfr. (38)) is the unique dynamic equation such that

$$\Gamma_F(X) = f/m + \xi - \xi_F \,. \tag{41}$$

However, by considering the *kinetic energy relative to F*, that is $T_F = mG_F$, there is another way of characterizing X that leads to the Lagrange's equations [but then F must be rigid]. For this we need the 2-form of maximal rank on K so defined (cfr. (37))

$$\varrho_F = m\omega_F + [v_F^*(\pi)] \wedge \mathrm{d}t \,, \tag{42}$$

where $\pi = (f + m(\xi - \xi_F))^\flat \in \mathrm{Sec}\,(V_K^*)$.

In the traditional approach the vector field X (and hence Lagrange's equations) is deduced from the 2-form ϱ_F (or from some other equivalent way) [**2**]. However, this is not the present case since we have given an « absolute » characterization of X. The following theorem can be proved.

Proposition 18. Let F be a framing. Then X is the unique characteristic vector field of ϱ_F such that $i_X \mathrm{d}t = 1$. Moreover, if F is rigid and (U, x^α) is a chart on M, a motion c is a base integral curve of X iff locally satisfies Lagrange's equations

$$\frac{d}{\mathrm{d}t}\Big(\frac{\partial T_F}{\partial \dot{q}^i}\Big) - \frac{\partial T_F}{\partial q^i} = \pi_i \,, \qquad \pi_i = \Big\langle \pi, \frac{\partial}{\partial \dot{q}^i} \Big\rangle \,. \tag{43}$$

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