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Notes on abstract almost-periodicity (**)

A Giorgio Sestini per il suo 70° compleanno

Introduction

In this work we prove some results concerning differential equations in Banach spaces in connection with Bochner's abstract almost-periodicity [5]$_1$. Several articles were written on this topic. The pioneering research to be mentioned is the paper [6] by Bochner and von Neumann. From the many papers in the area, we shall remember Amerio [1]$_{1,2}$, Biroli [4], Bochner [5]$_2$, Günzler-Zaidman [10], V. V. Jikov [12]$_{1,2}$, Jikov-Levitan [13], S. Zaidman [15]$_{1...10,14}$ Usually, the results in these papers came out from considerations connected with partial differential equations of the evolution type. Most of the time the equations are of the form $u'(t) = A(t)u(t) + f(t)$ and the operators in the family $A(t)$ are linear unbounded in a Banach or Hilbert space.

On the other hand, extending results from the classical theory of almost-periodic ordinary differential equations, various authors (see Massera-Schäffer [14], Daleckiĭ-Krein [7], Zaidman [15]$_{11,12}$) have investigated almost-periodic solutions of equations $u'(t) = A(t)u(t) + f(t)$, where $A(t)$ are bounded almost-periodic operators in infinite-dimensional spaces and $f(t)$ is an almost-periodic right-hand side.

Returning now to the specific matter of our present investigation

Our Theorem 1 is similar (in the abstract way) to lemma 1 in the paper [3].

The remaining two Theorems (2, and 3) are closely connected with theorem 5.7 in Fink's recent monograph [8].

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« If $A(t)$ is an almost-periodic matrix and x is an almost-periodic solution to $x' = A(t)x$, then $\inf_{t\in R} |x(t)| > 0$ or else $x(t) \equiv 0$ ».

This is quite a simple, but salient fact, which has important further applications (still in the classical framework, see th. 6.3 (corollary 6.4) in [**8**]). The comment in ([**8**], p. 95) says that « Theorem 5.7 also has an almost automorphic version. If x is almost automorphic then the conclusion is true with exactly same proof » (we gave a simple version of this remark in infinite dimensional Banach spaces in our forthcoming work $[\mathbf{15}]_{13}$, th. 6).

In the present paper we prove two more results on the same line; in the first one (Theorem 2) the equation is $x'(t) = Ax(t)$ in a general Banach space; A is an operator independent of t but unbounded (generator of a semi-group of class C_0); the second one (Theorem 3) exhibits an equation $x'(t) = A(t)x(t)$ where now A depends on t (in an almost-periodic way) but is a bounded operator (for any t) in a separable Banach space. As far as we know, these results are new; we hope that they will find useful applications.

1. – Our first result here is strongly connected with lemma 1 of the interesting paper [**3**].

We shall consider a Banach space X and a strongly almost-periodic group of linear operators: $G(t)$, $t \in R \to \mathscr{L}(X; X)$. Then let be A its infinitesimal generator and $F(t)$, $t \in R \to X$ be a strongly continuous function such that $\int_0^\infty \|F(t)\|\, dt < \infty$.

We now state

Theorem 1 *Let be $u(t)$, $t \in R \to \mathscr{D}(A)$—domain of A—a strong solution of the differential equation*

$$u'(t) = Au(t) + F(t)\,. \tag{1.1}$$

Then $u(t)$ can be expressed as the sum: $u(t) = V(t) + W(t)$, $\forall t \in R$, where $V(t)$ is an X-valued almost-periodic function and $W(t)$ is X-valued continuous function tending to 0 as $t \to +\infty$.

Proof. Using lemma 1 (th. 3.2) in our paper $[\mathbf{15}]_{14}$ we see that $u(t)$ is expressible under the formula

$$u(t) = G(t)u(0) + \int_0^t G(t-\tau)F(\tau)\, d\tau \qquad \forall t \in R\,. \tag{1.2}$$

Remark now that the group $G(t)$ is uniformly bounded on the real line, as follows from the uniform boundedness theorem ([**11**], th. 2.5.5) and from almost-periodicity of $G(t)x$, $\forall x \in X$.

Then, the continuous function, $R \to X$, $G(-\tau)F(\tau)$ (this continuity is easy to establish), is integrable on R^+ because of estimate

$$\|G(-\tau)F(\tau)\| \leqslant \sup_{s\in R} (\|G(s)\|)\, \|F(\tau)\|$$

and of the assumed integrability of $F(\tau)$ on R^+ (the improper integral $\int_0^\infty H(\tau)\,d\tau$ for X-valued continuous functions is defined as $\lim_{R\to\infty} \int_0^R H(\tau)\,d\tau$ in the well-known manner).

We have the obvious equality

$$\int_0^t G(t-\tau)F(\tau)\,d\tau = \int_0^\infty G(t)G(-\tau)F(\tau)\,d\tau - \int_t^\infty G(t)G(-\tau)F(\tau)\,d\tau\,, \qquad t \in R\,. \tag{1.3}$$

Hence, from (1.2) and (1.3) we deduce $\forall t \in R$

$$\begin{aligned} u(t) &= G(t)u(0) + G(t)\int_0^\infty G(-\tau)F(\tau)\,d\tau - \int_t^\infty G(t)G(-\tau)F(\tau)\,d\tau \\ &= G(t)[u(0) + \int_0^\infty G(-\tau)\,F(\tau)\,d\tau] - \int_t^\infty G(t)G(-\tau)F(\tau)\,d\tau\,. \end{aligned} \tag{1.4}$$

The first term in the right-hand side is almost-periodic; the second term is estimated by

$$\Big\|\int_t^\infty G(t-\tau)F(\tau)\,d\tau\Big\| \leqslant \sup_{s\in R} \|G(s)\| \int_t^\infty \|F(\tau)\|\,d\tau \to 0 \qquad \text{as } t \to +\infty\,.$$

This proves Th. 1.

2. – In this section we consider two results inspired by th. 5.7 in Fink's monograph [**8**]; the second is in fact a more general one.

We start with consideration of a strongly continuous semi-group (of class C_0) T_t; $t \geqslant 0 \to \mathscr{L}(X; X)$, in the Banach space X. Let us remark that, if A is the infinitesimal generator of this semi-group, and if $x(t)$, $t \in R \to \mathscr{D}(A)$ is a solution on the whole real line of the equation $x'(t) = Ax(t)$, then $x(t)$ admits, for any real $s \in R$ and for $t \geqslant s$, the representation $x(t) = T_{t-s}x(s)$ (in fact, if we put, for fixed s and $t > s$, $V(t) = T_{t-s}x(s)$, then we see that $V'(t) = AV(t)$, $t > s$; $V(s) = x(s)$ so that from unicity of Cauchy problem (see [**9**]) we get $x(t) = V(t) = T_{t-s}x(s)$, $\forall t \geqslant s$).

Now, if we substitute $\tau \geqslant 0$ for $t - s$ we obtain for $x(t)$ the new representation formula: $x(s+\tau) = T_\tau x(s)$, for any real s and for any $\tau \geqslant 0$. We can now state the following

Theorem 2. *Let be $x(t)$ an almost-periodic solution of the differential equation $x'(t) = Ax(t)$ on the real line. Then*

$$x(t) = \theta \quad \forall t \in R \qquad \text{or} \qquad \inf_{t\in R} \|x(t)\| > 0 .$$

Proof. If we assume that $\inf_{t\in R} \|x(t)\| = 0$, we can find a sequence of real numbers, $\{\alpha_n'\}_1^\infty$, such that $x(\alpha_n') \to \theta$ as $n \to \infty$. Using Bochner's criterion for almost-periodicity, we can extract a subsequence $\{\alpha_n\}_1^\infty \subset \{\alpha_n'\}_1^\infty$, with the property that $\lim_{n\to\infty} x(t+\alpha_n) = y(t)$ exists uniformly on $t \in R$ and $\lim_{n\to\infty} y(t-\alpha_n) = x(t)$, again uniformly on $t \in R$. Let us use now the equality

$$x(\alpha_n + t) = T_t x(\alpha_n) , \qquad \text{which is valid for } n = 1, 2, \ldots \text{ and } t \geqslant 0 .$$

Because of the relation: $\lim_{n\to\infty} x(\alpha_n) = \theta$, we find, for any $t \geqslant 0$, that $\lim_{n\to\infty} x(t + \alpha_n) = \theta$. Hence $y(t) = \theta$, $\forall t \geqslant 0$. On the other hand, $y(t)$ is almost-periodic, as uniform limit of $x(t+\alpha_n)$ on $t \in R$. Take an arbitrary fixed $t < 0$, and any $\varepsilon > 0$. From Bohr's definition of almost-periodicity (see [**2**]) we can find an ε-almost-period, τ_ε, such that $t + \tau_\varepsilon > 0$ (take τ_ε in $[|t|, |t| + L_\varepsilon]$). Then $\|y(t)\| \leqslant \|y(t) - y(t+\tau_\varepsilon)\| + \|y(t+\tau_\varepsilon)\| \leqslant \varepsilon$, $\forall \varepsilon > 0$. This $\Rightarrow y(t) = \theta$, $\forall$ real t; hence $x(t) = \theta$, $\forall t \in R$.

3. – Our last theorem here contains a generalization (to separable Banach spaces) of th. 5.7 in [**8**].

We shall consider an operator-valued function $A(t)$, from $t \in R$ to $\mathscr{L}(X; X)$ (where X is now a separable Banach space), which is strongly almost-periodic, that is $A(t)x$, $t \in R \to X$ is almost-periodic $\forall x \in X$. We shall prove

Theorem 3. *Let be $x(t)$, $t \in R \to X$ be an almost-periodic solution of the differential equation: $x'(t) = A(t)x(t)$. Then $x(t) \equiv \theta$ or $\inf_{t\in R} \|x(t)\| > 0$.*

Proof. Let us assume: $\inf_{t\in R} \|x(t)\| = 0$. Then, $\exists$ a real sequence $\{\alpha_n'\}_1^\infty$ such that: $\lim_{n\to\infty} x(\alpha_n') = \theta$.

Using a result of us in [**15**]$_{15}$, we see that $A(t)x(t)$ is an almost-periodic function, $\mathbf{R} \to X$; we shall extract a subsequence $\{\alpha_n\}_1^\infty \subset \{\alpha_n'\}_1^\infty$ in order to have

$$\text{(i)} \quad \lim_{n\to\infty} A(t+\alpha_n)x(t+\alpha_n) = y(t) \text{ uniformly on } R\,,$$

$$\text{(ii)} \quad \lim_{n\to\infty} y(t-\alpha_n) = A(t)x(t) \text{ uniformly on } R\,,$$

$$\text{(iii)} \quad \lim_{n\to\infty} x(t+\alpha_n) = \tilde{x}(t) \text{ uniformly on } R.$$

On the other hand, from the equation: $x'(t) = A(t)x(t)$ we derive

$$x'(t+\alpha_n) = A(t+\alpha_n)x(t+\alpha_n)\,,$$

so that

$$x(t+\alpha_n) = \int_0^t A(\tau+\alpha_n)x(\tau+\alpha_n)\,\mathrm{d}\tau + x(\alpha_n)\,.$$

When $n\to\infty$ we get obviously the relation $\tilde{x}(t) = \int_0^t y(\tau)\,\mathrm{d}\tau$, (these considerations are valid without assuming separability of X); unfortunately, we are unable to deduce from our last relation that $\tilde{x}(t) \equiv \theta$ (which would imply $x(t) = \lim_{n\to\infty} \tilde{x}(t-\alpha_n) = \theta$, $\forall t \in R$). Consequently, we shall first prove the following

Lemma. *Let be X a separable Banach space, and $A(t)$, $R \to \mathscr{L}(X, X)$ be a strongly almost-periodic operator-valued function. Then, given any real sequence $\{\alpha_n'\}_1^\infty$, there exists a subsequence $\{\alpha_n\}_1^\infty \subset \{\alpha_n'\}_1^\infty$, such that*

$$\lim_{n\to\infty} A(t+\alpha_n)x \quad \textit{exists } \forall x \in X, \textit{ uniformly on } t \in R\,.$$

Let us consider in fact a countable dense set in X, $(x_n)_1^\infty$. From almost-periodicity of $A(t)x_1$, we can find a subsequence $\{\alpha_n^1\} \subset \{\alpha_n'\}$ such that

$$\lim_{n\to\infty} A(t+\alpha_n^1)x_1 \quad \text{exists, uniformly on } t \in R\,.$$

From almost-periodicity of $A(t)x_2$, we find then a subsequence $\{\alpha_n^2\} \subset \{\alpha_n^1\}$ such that

$$\lim_{n\to\infty} A(t+\alpha_n^2)x_2 \quad \text{exists, uniformly on } t \in R\,,$$

(and because $\{\alpha_n^2\} \subset \{\alpha_n^1\}$), we have also

$$\lim_{n\to\infty} A(t+\alpha_n^2)x_1 \quad \text{exists, uniformly on } t \in R\,.$$

Continuing this way we find sequences $\{\alpha_n^p\} \subset \{\alpha_n^{p-1}\} \subset \ldots \subset \{\alpha_n^1\} \subset \{\alpha_n'\}$ such that

$$\lim_{n\to\infty} A(t+\alpha_n^p)x_j \qquad \text{exists, uniformly on } t\in R, \text{ for } j=1,2,\ldots,p\,.$$

Let us consider now the diagonal sequence $\{\alpha_p^p\}$. We have

$$\lim_{p\to\infty} A(t+\alpha_p^p)x_k \qquad \text{exists, uniformly on } t\in R, \ \forall k=1,2,\ldots\,.$$

In fact, fixing x_k, we see that $\lim_{n\to\infty} A(t+\alpha_n^k)x_k$ exists, uniformly on $t\in R$.

On the other hand, the diagonal sequence $\{\alpha_p^p\}$, after deleting a finite number of terms, is included in $(\alpha_n^k)_{n=1}^\infty$ Hence $\lim_{p\to\infty} A(t+\alpha_p^p)x_k$ exists, uniformly on R.

Remark now that $\sup_{t\in R}\|A(t)\|_{\mathscr{L}(X;X)} = L < \infty$ as follows from strong almost-periodicity.

Take an arbitrary $x\in X$ and a sequence $(U_j)_1^\infty \subset (x_n)_1^\infty$, such that $\lim_{j\to\infty} U_j = x$. Let us prove now that

$$\lim_{p\to\infty} A(t+\alpha_p^p)x \text{ exists, uniformly on } t\in R\,.$$

We shall use Cauchy's criterion; for $p, q\in N$, we have

$$\begin{aligned} &A(t+\alpha_p^p)x - A(t+\alpha_q^q)x \\ &= [A(t+\alpha_p^p) - A(t+\alpha_q^q)][x-U_j] + [A(t+\alpha_p^p) - A(t+\alpha_q^q)]\,U_j \end{aligned}$$

and henceforth the estimate

$$\|A(t+\alpha_p^p)x - A(t+\alpha_q^q)x\| \leqslant 2L\|x-U_j\| + \|A(t+\alpha_p^p)\,U_j - A(t+\alpha_q^q)\,U_j\|\,.$$

Consider now $\varepsilon>0$; fix an U_j such that $\|x-U_j\|\leqslant\varepsilon/4L$. Next, there exists $N(\varepsilon)$ such that for $p, q\geqslant N(\varepsilon)$

$$\|A(t+\alpha_p^p)\,U_j - A(t+\alpha_q^q)\,U_j\| < \frac{\varepsilon}{2}\,, \qquad \forall t\in R\,.$$

(Actually N depends on j also, but j was already fixed in dependence of ε). So, we found that for $p, q > N(\varepsilon)$ and $\forall t\in R$,

$$\|A(t+\alpha_p^p)x - A(t+\alpha_q^q)\,x\| < \varepsilon\,.$$

This proves Lemma 1.

Let us consider the operators $B(t)$, defined $\forall x \in X$ by $B(t)x = \lim_{p\to\infty} A(t + \alpha_p^p)x$ (which is uniform on $t \in R$). From th. 2.11.4 in [**11**] we deduce that $B(t) \in \mathscr{L}(X; X)$, $\forall t \in R$. ($B(t)$ is also a strongly almost-periodic function because of the uniform convergence above.)

Let us end now the proof of the Theorem.

We can assume to have already done all successive extractions of subsequences so that to (i), (ii), (iii) we can now add

(iv) $\lim_{n\to\infty} A(t + \alpha_n)x = B(t)x$ exists $\forall x \in X$, uniformly on $t \in R$.

Remark now that

(v) $\lim_{n\to\infty} A(t + \alpha_n)x(t + \alpha_n) = B(t)\tilde{x}(t)$, $\forall t \in R$.

In fact

$$A(t + \alpha_n)x(t + \alpha_n) - B(t)\tilde{x}(t)$$

$$= A(t + \alpha_n)[x(t + \alpha_n) - \tilde{x}(t)] + [A(t + \alpha_n) - B(t)]\tilde{x}(t) ,$$

hence

$$\|A(t + \alpha_n)x(t + \alpha_n) - B(t)\tilde{x}(t)\| \leqslant L\|x(t + \alpha_n) - \tilde{x}(t)\| + \|[A(t + \alpha_n) - B(t)]\tilde{x}(t)\| ,$$

which implies (v) (with pointwise convergence).

Now we see that

$$\lim_{n\to\infty} \int_0^t A(\tau + \alpha_n)x(\tau + \alpha_n)\, d\tau = \int_0^t B(\tau)\tilde{x}(\tau)\, d\tau ,$$

from the dominated convergence theorem (we can estimate $\|A(\tau + \alpha_n)x \cdot (\tau + \alpha_n)\|$ by a constant, due to almost-periodicity).

Hence, we obtain, when $n \to \infty$, $\tilde{x}(t) = \int_0^t B(\tau)\tilde{x}(\tau)\, d\tau$, $\forall t \in R$ where $B(\tau) \in \mathscr{L}(X; X)$ is strongly continuous.

In the well-known way (see [7]), we deduce $\tilde{x}(t) \equiv \theta$. Consequently, $x(t) = \theta$, $\forall t \in R$. This proves Th. 3.

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