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Almost complex conditions and weakly Kähler manifolds (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

Let V be an almost Hermite manifold.

In a previous paper ($[7]_4$, 1977) I investigated about the effects induced on V by some formal conditions, depending only on the almost complex structure J of V, on the tensor field DJ, obtained from J by covariant differentiation.

An analogous investigation is performed in the present paper for the Nijenhuis field N, for the Kähler field K and for the fields you can derive from DJ, N, K, by using some convenient isomorphisms, depending on the Riemannian structure or on the almost complex structure of V (Sec. **2**, **3**).

The five theorems of Sec. **5** give a complete answer to the question. They may be regarded also as characterization theorems for some known classes of « weakly » Kähler manifolds. It is worth remarking that all the classes of manifolds, that occur in the results, fit nicely into a recent classification due to A. Gray and L. M. Hervella.

Some complementary results are also obtained; as an instance, a new form of definition of underkähler manifold (Sec. **6**) and a complete study about the properties of symmetry and of skew-symmetry of the tensor fields, which occur in the paper (Sec. **7**).

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2. - Isomorphisms α, γ, W, λ

Let V be an *almost Hermite manifold* of dimension $2n$ and of class C^{2n+1} (1).

Let $\mathscr{T}^r_s$ be the linear space of tensor fields of type (r, s) on V. In particular, let g be the symmetric field of $\mathscr{T}^0_2$ of class C^1, defining the Riemannian metric on V and let J be the field of $\mathscr{T}^1_1$ of class C^{2n}, defining the almost complex structure on V.

Some *isomorphisms* of $\mathscr{T}^1_2$ occur in the following: namely α, γ, W, λ.

Let σ, ε be the homomorphisms of symmetry, of skew-symmetry of $\mathscr{T}^1_2$; then $\alpha = \sigma - \varepsilon$.

Denote now by G the symmetric tensor field of $\mathscr{T}^2_0$, satisfying $c^1_1(g \otimes \mathrm{G}) = \delta$ (2). Then the isomorphism γ is defined for any field L of $\mathscr{T}^1_2$ by

$$\gamma \mathrm{L} = c^2_2(c^1_1(g \otimes \mathrm{L}) \otimes \mathrm{G}) \,.$$

It is worth remarking that γ *depends only on the Riemannian structure of* V (3).

On the contrary, *the isomorphisms* W, λ *defined by* (4), (3), of [7]$_4$, *depend only on the almost complex structure of* V (4).

Some formal properties of the isomorphism α, γ, W, λ, are useful in the following (5).

$$\alpha\alpha = \gamma\gamma = \mathrm{WW} = 1 \,, \qquad \lambda\lambda = -1 \,, \tag{1}$$

$$\alpha\lambda = \lambda\alpha \,, \qquad \alpha\gamma\alpha = \gamma\alpha\gamma \,, \qquad \mathrm{W}\lambda = \lambda\mathrm{W} \,, \tag{2}$$

$$\alpha\mathrm{W}\alpha\mathrm{W} = \mathrm{W}\alpha\mathrm{W}\alpha = -\gamma\mathrm{W}\gamma \,, \tag{3}$$

$$\lambda\gamma\lambda\gamma = \alpha\mathrm{W}\alpha \,. \tag{4}$$

We conclude the Section with some remarks.

Ler $\mathscr{L}$ be a tensor field of $\mathscr{T}^0_3$ and put $\mathrm{L} = c^1_3(\mathrm{G} \otimes \mathscr{L})$; then

P_1 - $\mathscr{L}$ *is symmetric, skew-symmetric in the second and the third index, if and only if*

$$(1 - \alpha\gamma\alpha)\,\mathrm{L} = 0 \,, \qquad (1 + \alpha\gamma\alpha)\,\mathrm{L} = 0 \,, \tag{5}$$

respectively.

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(1) For the basic facts about almost Hermite manifolds see K. Yano [**8**], ch. 9; S. Kobayashi - K. Nomizu [**5**], II, Ch. 9.

(2) The symbol c^r_s denotes contraction ([**1**], p. 45) and δ is the classical Kronecker field of $\mathscr{T}^1_1$.

(3) The isomorphism γ was first introduced in [**7**]$_3$.

(4) The isomorphisms W, λ were first introduced in [**7**]$_2$. See also [**7**]$_4$, Sec. **3**.

(5) See [**7**]$_2$, n. 3, 5 and [**7**]$_3$, n. 3.

P_2 – *Conditions* (5) *are equivalent to conditions*

$$\varepsilon\gamma L = 0 \,, \qquad \sigma\gamma L = 0 \,, \tag{6}$$

respectively.

P_1 follows directly from the definitions of α and of γ; P_2 derives immediately from (6) of $[\mathbf{7}]_3$ and from a similar equation.

From P_1, P_2 we immediately derive the corollaries.

C_1 – $\mathscr{L}$ *is symmetric, skew-symmetric in all its indices, if and only if*

$$\varepsilon L = \varepsilon\gamma L = 0 \,, \qquad \sigma L = \sigma\gamma L = 0 \,,$$

respectively.

C_2 – *If* L *is symmetric, skew-symmetric and satisfies* $(6)_2$, $(6)_1$, *respectively, then* $L = 0$.

3. - Tensor fields DJ, N, K

It is well known that some tensor fields of $\mathscr{T}^1_2$, namely DJ, N, K play an essential role for the almost Hermite manifold V.

The first one is obtained from J by covariant differentiation with respect to the Levi-Civita connection (Riemannian connection) (6). The others, that may be written in the form

$$N = \tfrac{1}{2}\lambda\varepsilon W\varepsilon DJ \,, \qquad K = (1-\alpha-\gamma)DJ \,, \tag{7}$$

are known as the *Nijenhuis field* and the *Kähler field*, respectively (7).

Of course, starting from DJ, N, K and making use of the isomorphisms α, γ, W, λ we are able to derive other remarkable fields, all strictly related with the almost Hermitian structure of V.

As known, an interesting *relation* links the fields DJ, N, K; namely

$$4N = -\gamma\lambda DJ + \varepsilon\lambda WK \ (8) \,. \tag{8}$$

(6) The index of covariant differentiation is assumed to be the first lower index.

(7) Consider the skew-symmetric field of $\mathscr{T}^0_2$ $\mathscr{J} = c^1_1(g\otimes J)$ (Kähler form) and denote by $\mathscr{K}$ the differential of $\mathscr{J}$; then $K = c^1_3(G\otimes\mathscr{K})$.

(8) Compare with [**8**], (4.10), p. 141. Remark only that the definition of the Nijenhuis field in [**8**] differs by a factor from our definition $(7)_1$.

Relations

$$(9)\qquad WDJ = -DJ\,, \qquad WN = -N\,, \qquad \gamma K = -K\,,$$

$$(10)\qquad (W\gamma\lambda + \lambda\gamma)DJ = 0\,, \qquad (W\lambda\gamma + \gamma\lambda)DJ = 0$$

are also useful in the following (9).

4. - Almost complex conditions in $\mathcal{T}^1_2$

We make use in the following of the *conditions* A_1, A_2, B, C, D and of the symmetric *conditions* $\overline{A}_1$, $\overline{A}_2$, $\overline{B}$, $\overline{C}$, $\overline{D}$, listed in $[7]_4$ (10). The above conditions depend only on the almost complex structure of V (11).

Some remarks are essential.

Let L be a tensor field of $\mathcal{T}^1_2$; then

P_1 – *Conditions* A_1, A_2, B, C, D, $\overline{A}_1$, $\overline{A}_2$, $\overline{B}$, $\overline{C}$, $\overline{D}$ *for* αL *reduce to conditions* A_2, A_1, B, C, D, $\overline{A}_2$, $\overline{A}_1$, $\overline{B}$, $\overline{C}$, $\overline{D}$ *for* L, *respectively.*

P_2 – *Conditions* A_1, A_2, C, $\overline{A}_1$, $\overline{A}_2$, $\overline{C}$ *for* γL *reduce to conditions* $\overline{C}$, A_2, $\overline{A}_1$, C, $\overline{A}_2$, A_1 *for* L, *respectively.*

P_3 – *Conditions* A_1, A_2, B, C, $\overline{A}_1$, $\overline{A}_2$, $\overline{B}$, $\overline{C}$ *for* WL *reduce to conditions* A_1, A_2, B, C, $\overline{A}_1$, $\overline{A}_2$, $\overline{B}$, $\overline{C}$ *for* L, *respectively. Conditions* D, $\overline{D}$ *for* WL *reduce to symmetry, skew-symmetry for* L, *respectively.*

P_4 – *Conditions* A_1, A_2, B, C, D, $\overline{A}_1$, $\overline{A}_2$, $\overline{B}$, $\overline{C}$, $\overline{D}$ *for* λL *reduce to conditions* A_1, A_2, B, C, D, $\overline{A}_1$, $\overline{A}_2$, $\overline{B}$, $\overline{C}$, $\overline{D}$ *for* L, *respectively.*

Taking account of (1), (2), (3) of Sec. **2**, we can easily prove P_1, P_3, P_4. Remark now that from (3) we have

$$(11)\qquad \gamma W\alpha W\alpha\gamma L = -WL\,, \qquad \gamma W\gamma L = -W\alpha W\alpha L\,.$$

Since $\gamma\gamma = WW = 1$, the comparison shows that condition A_2 for γL is equivalent to condition A_2 for L. In a similar way we can prove the other statements of P_2.

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(9) For relation (9) see $[7]_4$, P_1; $[7]_1$, O_9; $[7]_3$ (29). To prove $(10)_1$, just apply γ, taking account of (3), (4) of Sec. **2** and then of $(2)_1$, $(2)_3$, $(9)_1$. Since $WW = 1$, $(10)_2$ results to be equivalent to $(10)_1$.

(10) A Vezzani recently remarked that condition B_1 and condition B_2 are equivalent. The same happens for $\overline{B}_1$ and $\overline{B}_2$.

(11) Only α and W occur in their expressions. The first five conditions were introduced in $[7]_1$.

5. - Characterization theorems

We begin now an investigation about the consequences of the action of the conditions of Sec. **4** on the tensor fields DJ, N, K of Sec. **3** and on the tensor fields we derive from them, by applying one of the isomorphisms α, γ, W, λ of Sec. **2**.

Remark first that by P_1, P_4 the cases of the isomorphisms α, λ may be omitted. Therefore, by virtue of relation (9) of Sec. **3** the only fields we have to study are

$$\mathrm{DJ},\ \mathrm{N},\ \mathrm{K},\ \gamma\mathrm{DJ},\ \ \gamma\mathrm{N},\ \mathrm{WK}\,.$$

A complete examination of the first case has been performed in $[\mathbf{7}]_4$.

Remark also that N, K, γDJ are skew-symmetric fields ([12]); so only conditions A, B, C, D have to be considered for them ([13]). Finally, by P_3 only conditions D, $\overline{\mathrm{D}}$ result to be interesting in the case of the field WK.

In conclusion, taking account of the previous remarks, we are now able to state that the following theorems give a complete answer to our investigation.

Th. 1. *The Nijenhuis field* N *always satisfies conditions* B, C. *If* N *satisfies one of the conditions* A, D, *then* V *is a Hermite manifold; and conversely.*

Th. 2 *If the Kähler field* K *satisfies one of the conditions* A, D, *then* V *is an almost Kähler manifold; and conversely. If* K *satisfies one of the conditions* B, C, *then* V *is an almost Kōto manifold; and conversely.*

Th. 3. *The field* γDJ *always satisfies condition* C. *If* γDJ *satisfies condition* A, B, D, *then* V *is a Hermite manifold, an almost Kōto manifold, a Kähler manifold, respectively; and conversely.*

Th. 4. *If* γN *satisfies one of the conditions* B, $\overline{\mathrm{D}}$, *then* V *is an underkähler manifold and conversely. If* γN *satisfies one of the conditions* $\overline{\mathrm{A}}$, D, *then* V *is a Hermite manifold; and conversely.*

Th. 5. *The field* WK *always satisfies condition* $\overline{\mathrm{D}}$. *If* WK *satisfies condition* D, *then* V *is an almost Kähler manifold.*

([12]) This property is well known for N, K. The property for γDJ follows easily from $(7)_2$.

([13]) See [**6**], O_2, O_3, n. 4,

6. - Remarks

The proofs of the five theorems will take place in Sec. **7**. In the present Section I add only some remarks.

As well known, *Hermite manifolds*, *almost Kähler manifolds*, *almost Kōto manifolds* (O^*-*spaces*) generalize Kähler manifolds (14).

G_1-*manifolds*, also known as *underkähler manifolds*, were first introduced by L. Hervella and E. Vidal in [**4**]; these manifolds also generalize Kähler manifolds. It is worth remarking that V is a G_1-*manifold, if and only if the field* γN *is skew-symmetric* (15).

An interesting *classification* of the manifolds generalizing Kähler manifolds can be found in the recent paper [**3**] of A. Gray and L. Hervella. All the classes of manifolds, that occur in our theorems, fit nicely into the scheme proposed by these Authors.

Finally, in paper [**2**] S. Donnini introduced the class of the almost Hermite manifolds, such that their Kähler fields K (Sec. **3**) satisfy condition D. Now Theorem 2 shows that the class considered by S. Donnini reduces to the class of almost Kähler manifolds.

7. - Proofs

The first statement of Th. 1 follows directly from O_9 of $[\mathbf{7}]_1$, O_1 of [**6**]. Since N is a skew-symmetric field, then by virtue of O_3 of [**6**] N satisfies also conditions $\overline{A}$, $\overline{D}$. Now, to prove the second statement, just remark that conditions A, $\overline{A}$, as well as conditions D, $\overline{D}$, imply $N = 0$.

The first step to prove Th. 2 is to remark that *for the Kähler field* K *conditions* B *and* C *are equivalent.*

Taking account of O_1 of [**6**], we have only to prove that condition C implies condition B. A Sawaki's lemma assures that condition C for K is equivalent to condition C for DJ (16). Therefore, using (3), we have

$$\alpha W \alpha W DJ = - \gamma W \gamma DJ = DJ \, . \tag{12}$$

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(14) See for example $[\mathbf{7}]_4$, where you can find the definitions we use in the following and some characterization theorems.

(15) See [**3**]. By C_1 of Sec. **3** the skew-symmetry of γN is equivalent to the skew-symmetry of the field $\mathcal{N} = c_4^1(N \otimes g)$ in all its indices.

(16) This form of Sawaki's Lemma is due to S. Donnini ([**2**] lemma L, p. 489).

By virtue of (12) and of $(9)_1$ we can write

$$\begin{aligned}W\alpha K &= -WK = -WDJ + W\alpha DJ + W\gamma DJ \\ &= -WDJ + \alpha WDJ - \gamma DJ = DJ - \alpha DJ - \gamma DJ = K\,.\end{aligned}$$

This proves the assumption.

Now, the part of Th. 2 concerning conditions B and C follows easily from Sawaki's lemma and from Theorem 5 of $[7]_4$. Consider then condition A for the tensor field K. Since A implies C, B is equivalent to $\bar{A}$ ([**6**], O_1, O_3), and, as we remarked, C and B are equivalent, so K satisfies conditions A, $\bar{A}$. Thus we have $K = 0$ and V is an almost Kähler manifold. The converse is trivial.

More difficult is to prove the statement concerning condition D. If K satisfies condition D, we have $\varepsilon WK = 0$. Hence, by relation (8) of Sec. **3**, we find $4N = -\gamma\lambda DJ$. Taking account of $(9)_2$, $(10)_1$, we can write $4N = -\lambda\gamma DJ$. Using $(9)_2$ we get equation $W\gamma DJ = -\gamma DJ$ which by (3) reduces to $\alpha W\alpha WDJ = DJ$. By virtue of Sawaki's lemma we conclude that the skew-symmetric field K satisfies condition C, equivalent to condition $\bar{D}$ ([**6**], O_3). Finally, from D, $\bar{D}$ we derive $K = 0$; so V is an almost Kähler manifold. The converse is trivial.

The proof of Th. 2 is now complete.

Remark now that by virtue of (11), $(9)_1$ we have $\gamma W\alpha W\alpha\gamma DJ = DJ$ thus γDJ always satisfies condition C. Since γDJ is a skew-symmetric field ($(7)_2$, Sec. **3**), then γDJ satisfies also condition $\bar{D}$ ([**6**], O_3). Therefore, if γDJ is assumed to satisfy condition D, it follows $\gamma DJ = 0$. Thus $DJ = 0$ and V is a Kähler manifold. The converse is trivial. Finally, by using (3) of Sec. **2** and, respectively, Theorem 6 and Theorem 5 of $[7]_4$, we prove easily the statements of Th. 3 concerning conditions A and B.

To prove Th. 4, remark first that by (1), (3) of Sec. **2** and by Th. 1 we can write

$$\alpha W\gamma N = \alpha\gamma\gamma W\gamma N = -\alpha\gamma\alpha W\alpha WN = -\alpha\gamma N\,. \tag{13}$$

Therefore, by virtue of $(2)_2$ conditions $\bar{B}$, B for γN are equivalent to conditions $(6)_1$, $(6)_2$ for N, respectively. Taking account then of the corollary C_2 of Sec. **2**, we conclude in the first case that $N = 0$; thus V is a Hermite manifold. In the second case the tensor field γN is skew-symmetric; so V is a G_1-manifold (Sec. **6**). To complete the proof of Th. 4 remark that, by applying the isomorphism W to relation (13), we see that conditions $\bar{B}$, D, as well as conditions B, $\bar{D}$, are equivalent.

Finally, since K is a skew-symmetric field (Sec. **5**), Th. 5 is an obvious consequence of proposition P_3 of Sec. **4**.

8. - Symmetry. Skew-symmetry

We want to end with some remarks about properties of symmetry and skew-symmetry for the fields DJ, N, K of Sec. **3** and the fields we can derive from them by applying the isomorphisms α, γ, W, λ of Sec. **2**.

Since α λ commute with $\sigma = \frac{1}{2}(1+\alpha)$, $\varepsilon = \frac{1}{2}(1-\alpha)$ $\big((2)_1$, Sec. **2**$\big)$, only the fields,

$$\mathrm{DJ},\ \mathrm{N},\ \mathrm{K},\ \gamma\mathrm{DJ},\ \gamma\mathrm{N},\ \mathrm{WK}$$

have to be considered (Sec. **3**).

Now N, K, γDJ are skew-symmetric fields (Sec. **5**). So an assumption of symmetry is equivalent to their vanishing and V reduces to a Hermite manifold, to an almost Kähler manifold, to a Kähler manifold, respectively.

For the other tensor fields we have

P_5 – *If* DJ *is symmetric, skew-symmetric, then* V *is, respectively, a Kähler manifold, an almost Tachibana manifold; and conversely.*

P_6 – *If* γN *is symmetric, skew-symmetric, then* V *is, respectively, a Hermite manifold, an underkähler manifold; and conversely.*

P_7 – *If* WK *is symmetric, skew-symmetric, then* V *is, respectively, an almost Kähler manifold, an almost Kōto manifold; and conversely.*

The first part of P_5 is known $\big($Theorem 2 [**7**]$\big)_4$. The second part is just the definition of almost Tachibana manifolds (nearly Kähler manifolds). The corollary C_2 of Sec. **2** and the characterization of the underkähler manifolds, given in Sec. **6**, lead to P_6. Finally, P_7 is an immediate consequence of Th. 2 of Sec. **5**.

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Sommario

Sia V *una varietà quasi Hermitiana e* J *la struttura quasi complessa di* V. *Si considerano il campo* DJ, *il campo di Nijenhuis* N, *il campo di Kähler* K *e i campi, che derivano dai precedenti, operando con opportuni isomorfismi, strettamente dipendenti dalla struttura di* V. *Se a questi campi tensoriali si impongono alcune condizioni formali, legate alla struttura quasi complessa* J, *si ottengono diverse caratterizzazioni di note classi di varietà, generalizzanti le varietà Kähleriane.*

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