Riv. Mat. Univ. Parma (4) 6 (1980), 1-5

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On fixed point mappings and constant mappings (**)

In a recent paper, see [1], M. S. Khan considers two mappings S and T on a metric space X satisfying the inequality

$$d(Sx, Ty) \leqslant c \left\{ \frac{d(x, Sx)\, d(x, Ty) + d(y, Ty)\, d(y, Sx)}{d(x, Ty) + d(y, Sx)} \right\}, \quad 0 \leqslant c < 1 .$$

In the following theorem we consider two mappings S and T on a metric space X, satisfying a similar type of inequality.

Theorem 1. *Let S and T be mappings of the complete metric space X into itself satisfying the inequality*

$$d(Sx, Ty) \leqslant \frac{b\, d(x, Sx)\, d(x, Ty) + c\, d(y, Ty)\, d(y, Sx)}{d(x, Sx) + d(y, Ty)}$$

if $d(x, Sx) + d(y, Ty) \neq 0$ and the equality

$$d(Sx, Ty) = 0$$

if $d(x, Sx) + d(y, Ty) = 0$, for all x, y in X, where $b, c \geqslant 0$ and $bc < 1$. Then S and T have a unique common fixed point z.

Proof. Let x be an arbitrary point in X and put

$$u_{2n} = d\big((ST)^n x, T(ST)^n x\big) , \qquad u_{2n+1} = d\big(T(ST)^n x, (ST)^{n+1} x\big) \qquad (n = 0, 1, 2, \dots).$$

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Then

$$u_{2n}(u_{2n-1} + u_{2n}) \leqslant bu_{2n-1}d\big(T(ST)^{n-1}x,\ T(ST)^n x\big) \leqslant bu_{2n-1}(u_{2n-1} + u_{2n})$$

whether $u_{2n-1} + u_{2n} = 0$ or not, and so $u_{2n}^2 + (1-b)u_{2n}u_{2n-1} - bu_{2n-1}^2 \leqslant 0$. This implies that $-u_{2n-1} \leqslant u_{2n} \leqslant bu_{2n-1}$; but since $u_{2n} \geqslant 0$, we must have $u_{2n} \leqslant bu_{2n-1}$, or equivalently

$$d\big((ST)^n x,\ T(ST)^n x\big) \leqslant bd\big(T(ST)^{n-1}x,\ (ST)^n x\big) \qquad (n = 1, 2, \ldots)\,.$$

We can prove similarly that

$$d\big(T(ST)^{n-1}x,\ (ST)^n x\big) \leqslant cd\big((ST)^{n-1}x,\ T(ST)^{n-1}x\big) \qquad (n = 1, 2, \ldots)\,,$$

and so for $n = 1, 2, \ldots$

$$d\big((ST)^n x,\ T(ST)^n x\big) \leqslant bd\big(T(ST)^{n-1}x,\ (ST)^n x\big) \leqslant (bc)^n d(x,\ Tx)\,.$$

Since $bc < 1$, it follows that the sequence $\{x, Tx, STx, \ldots, (ST)^n x, T(ST)^n x, \ldots\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X.

If now $Tz \neq z$, we have

$$d\big((ST)^n x,\ Tz\big) \leqslant \frac{bu_{2n-1}d\big(T(ST)^{n-1}x,\ Tz\big) + c\,d(z,\ Tz)\,d\big(z,\ ST)^n x\big)}{u_{2n-1} + d(z,\ Tz)}$$

and on letting n tend to infinity we see that $d(z, Tz) \leqslant 0$, which implies that we must in fact have $Tz = z$.

Similarly, by considering $d\big(Sz, T(ST)^n x\big)$, it follows that $Sz = z$ and so z is a common fixed point of S and T.

Now suppose that T has a second fixed point z'. Then we will have $d(z, Sz) + d(z', Tz') = 0$ and so $d(Sz, Tz') = 0$.

It follows that $z = z'$ and so the common fixed point z of S and T is unique.

On putting $S = T$ in the theorem we have the following

Corollary. *Let T be a mapping of the complete metric space X into itself satisfying the inequality*

$$d(Tx,\ Ty) \leqslant \frac{b\,d(x,\ Tx)\,d(x,\ Ty) + c\,d(y,\ Ty)\,d(y,\ Tx)}{d(x,\ Tx) + d(y,\ Ty)}$$

if $d(x, Tx) + d(y, Ty) \neq 0$ *and the equality*

$$d(Tx, Ty) = 0$$

if $d(x, Tx) + d(y, Ty) = 0$, *for all* x, y *in* X, *where* $b, c \geqslant 0$ *and* $bc < 1$. *Then* T *has a unique fixed point* z.

When $b = 0$ in Theorem 1, we have the following stronger result

Theorem 2. *Let* S *and* T *be mappings of the complete metric space* X *into itself satisfying the inequality*

$$d(Sx, Ty) \leqslant \frac{cd(y, Ty)\, d(y, Sx)}{d(x, Sx) + d(y, Ty)}$$

if $d(x, Sx) + d(y, Ty) \neq 0$ *and the equality*

$$d(Sx, Ty) = 0$$

if $d(x, Sx) + d(y, Ty) = 0$, *for all* x, y *in* X, *where* $c \geqslant 0$.

Then S *and* T *have a unique common fixed point* z *and* S *is a constant mapping with* $Sx = z$ *for all* x *in* X.

Proof. By Theorem 1, S and T have a unique common fixed point z. If now $Sx \neq x$, we have

$$d(Sx, z) = d(Sx, Tz) \leqslant \frac{c\, d(z, Tz)\, d(z, Sx)}{d(x, Sx) + d(z, Tz)} = 0$$

and so $Sx = z$. If for some x, $Sx = x \neq z$, we have $d(x, Sx) + d(z, Tz) = 0$ which implies that $d(Sx, Tz) = 0 = d(x, z)$.

It follows that $x = z$, giving a contradiction. Thus S is a constant mapping with $Sx = z$ for all x in X.

On putting $S = T$ in the theorem we have the following

Corollary. *Let* T *be a mapping of the complete metric space* X *into itself satisfying the inequality*

$$d(Tx, Ty) \leqslant \frac{c\, d(y, Ty)\, d(y, Tx)}{d(x, Tx) + d(y, Ty)}$$

if $d(x, Tx) + d(y, Ty) \neq 0$ *and the equality*

$$d(Tx, Ty) = 0$$

if $d(x, Tx) + d(y, Ty) = 0$, *for all* x, y *in* X, *where* $c \geqslant 0$.

Then T *has a unique fixed point* z *and* $Tx = z$ *for all* x *in* X.

We finally prove an analogous result to Theorem 1 for compact metric spaces.

Theorem 3. *Let* S *and* T *be continuous mappings of the compact metric space* X *into itself satisfying the inequality*

$$d(Sx, Ty) < \frac{c^{-1} d(x, Sx) d(x, Ty) + c\, d(y, Ty) d(y, Sx)}{d(x, Sx) + d(y, Ty)}$$

if $d(x, Sx) + d(y, Ty) \neq 0$ *and the equality*

$$d(Sx, Ty) = 0$$

if $d(x, Sx) + d(y, Ty) = 0$, *for all* x, y *in* X, *where* $c > 0$.

Then S *and* T *have a unique common fixed point* z.

Proof. Suppose first of all that there exists $b < c^{-1}$ such that

$$d(Sx, Ty) \leqslant \frac{b\, d(x, Sx) d(x, Ty) + c\, d(y, Ty) d(y, Sx)}{d(x, Sx) + d(y, Ty)}$$

for all x, y with $d(x, Sx) + d(y, Ty) \neq 0$. The result then follows from Theorem 1. If no such b exists, let $\{b_n\}$ be a monotonically increasing sequence of positive real numbers converging to c^{-1}. We can then find sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$d(Sx_n, Ty_n) > \frac{b_n d(x_n, Sx_n) d(x_n, Ty_n) + c\, d(y_n, Ty_n) d(y_n, Sx_n)}{d(x_n, Sx_n) + d(y_n, Ty_n)}$$

for $n = 1, 2, \ldots$. Since X is compact, we can find convergent subsequences $\{x_{n(r)}\} = \{x_r'\}$ and $\{y_{n(r)}\} = \{y_r'\}$ of $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively. Putting $\{b_{n(r)}\} = \{b_r'\}$, we have

$$d(Sx_r', Ty_r') > \frac{b_r' d(x_r', Sx_r') d(x_r', Ty_r') + c\, d(y_r', Ty_r') d(y_r', Sx_r')}{d(x_r', Sx_r') + d(y_r', Ty_r')}$$

for $r = 1, 2, \ldots$. If $d(x, Sx) + d(y, Ty) \neq 0$, we have on letting r tend to infinity

$$d(Sx, Ty) \geqslant \frac{c^{-1} d(x, Sx)\, d(x, Ty) + c\, d(y, Ty)\, d(y, Sx)}{d(x, Sx) + d(y, Ty)},$$

since S and T are continuous, which gives a contradiction. It follows that $x = Sx$, $y = Ty$, $Sx = Ty$ and so $x = y = z$ is a common fixed point of S and T. The uniqueness of z follows immediately.

Corollary. *Let T be a continuous mapping of the compact metric space X into itself satisfying the inequality*

$$d(Tx, Ty) < \frac{c^{-1} d(x, Tx)\, d(x, Ty) + c\, d(y, Ty)\, d(y, Tx)}{d(x, Tx) + d(y, Ty)}$$

if $d(x, Tx) + d(y, Ty) \neq 0$ and the equality

$$d(Tx, Ty) = 0$$

if $d(x, Tx) + d(y, Ty) = 0$, for all x, y in X, where $c > 0$.
Then T has a unique fixed point z.

Reference

[1] M. S. Khan, *A fixed point theorem in bi-metric spaces*, Riv. Mat. Univ. Parma, (4) **4** (1978), 41-44.

Sommario

Si dimostra che, date due applicazione T ed S di uno spazio metrico completo $\langle X, d\rangle$ in sè, se

$$d(Sx, Ty) \leqslant \frac{b d(x, Sx)\, d(x, Ty) + c d(y, Ty)\, d(y, Sx)}{d(x, Sx) + d(y, Ty)}$$

per $d(x, Sx) + d(y, Ty) \neq 0$, $(b, c \geqslant 0;\ bc < 1)$,

$d(Sx, Ty) = 0$ *per* $d(x, Sx) + d(y, Ty) = 0$,

allora S e T hanno un unico punto unito, il medesimo per entrambe.

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