Riv. Mat. Univ. Parma (4) 6 (1980), 15-23

K. B. LAL and SHAFIULLAH (*)

On wave solutions of Kilmister and Newmann's weakened field equations in general relativity (**)

1 - Introduction

Einstein's field equation of general relativity in empty space is given by

$$R_{ij} = 0 . \tag{1.1}$$

Kilmister and Newmann [1], [2] have proposed an alternative set of field equations in vacuo which are given by

$$J_{ijk} \equiv R^{a}_{ijk;a} = 0 , \tag{1.2}$$

$$\varepsilon^{rs} \equiv (-g)^{\frac{1}{2}}[(g^{rs}g^{tu} - \tfrac{1}{2}g^{rt}g^{su} - \tfrac{1}{2}g^{ru}g^{st})R_{;ut} + R(R^{sr} - \tfrac{1}{4}g^{sr}R)] = 0 , \tag{1.3}$$

with the properties (a) $\varepsilon^{rs} = \varepsilon^{sr}$, (b) $\varepsilon^{sr}{}_{;r} = 0$ and

$$H^{ij}_{k} \equiv R^{ij}{}_{;k} = 0 , \tag{1.4}$$

where a semi-colon (;) denotes covariant differentiation with respect to Christoffel symbols $\left\{ {k \atop ij} \right\}$. These field equations are weaker than (1.1) in the sense that they admit a class of solutions for which (1.1) holds. They have called such field equations « Weakened field equations ».

(*) Indirizzi: K. B. LAL, Department of Mathematics and Statistics, University, Gorakhpur 273001, U.P. India; SHAFIULLAH, Department of Mathematics, Shibli National PG College, Azamgarh, U.P. India.
(**) Ricevuto: 1-III-1978.

Thompson [7] has investigated these field equations and found different static spherically symmetric solutions. Further D. Lovelock $[\mathbf{5}]_{1,2}$ has solved these field equations and obtained a static spherically symmetric solution. Lal and Singh [3] have obtained cylindrical wave solutions of the field equation (1.2) which include as a particular case, the solutions of the empty space field equation (1.1) of general relativity. Recently Lal and Srivastava [4] have found plane wave solutions of the weakned field equations.

In this paper the authors propose to consider a spacetime whose metric is given by

$$ (1.5) \qquad ds^2 = -(1-A)\,dx^2 - (1+A)\,dy^2 - C\,dz^2 + C\,dt^2 + 2B\,dx\,dy\,, $$

where A, B are functions of $\xi = z - t$, $C \equiv C(z, t)$, $(\det.(g_{ij}) = -mC^2$, $m = 1 - A^2 - B^2)$, and to obtain the solutions of the weakned field equations. The metric (1.5) reduces to Rao and Pandey [6] plane wave metric, if $C = 1$ and plane wave is taken along negative x-axis direction.

2 - Curvature properties of the metric (1.5)

The non-vanishing components of contravariant tensor g^{ij} in the metric (1.5) are

$$ (2.1) \qquad g^{11} = -\frac{1+A}{m}, \quad g^{12} = g^{21} = -\frac{B}{m}, \quad g^{22} = -\frac{1-A}{m}, \quad g^{33} = -g^{44} = -\frac{1}{C} $$

and the non-vanishing Christoffel symbols of second kind are

$$ (2.2) \qquad \begin{gathered} \left\{ {1 \atop 13} \right\} = -\left\{ {1 \atop 14} \right\} = -\frac{(1+A)\bar{A} + B\bar{B}}{2m}, \left\{ {1 \atop 23} \right\} = -\left\{ {1 \atop 24} \right\} = -\frac{(1+A)\bar{B} - B\bar{A}}{2m}, \\ \left\{ {2 \atop 13} \right\} = -\left\{ {2 \atop 14} \right\} = -\frac{(1-A)\bar{B} + B\bar{A}}{2m}, \left\{ {2 \atop 23} \right\} = -\left\{ {2 \atop 24} \right\} = -\frac{(1-A)\bar{A} - B\bar{B}}{2m}, \\ \left\{ {3 \atop 11} \right\} = -\left\{ {3 \atop 22} \right\} = \left\{ {4 \atop 11} \right\} = -\left\{ {4 \atop 22} \right\} = \frac{\bar{A}}{2C}, \quad \left\{ {3 \atop 12} \right\} = \left\{ {4 \atop 12} \right\} = \frac{\bar{B}}{2C}, \\ \left\{ {3 \atop 33} \right\} = \left\{ {3 \atop 44} \right\} = \left\{ {4 \atop 34} \right\} = \frac{C_3}{2C}, \quad \left\{ {3 \atop 34} \right\} = \left\{ {4 \atop 33} \right\} = \left\{ {4 \atop 44} \right\} = \frac{C_4}{2C}, \end{gathered} $$

where the indices 3, 4 attached to C denote its partial derivatives with respect to z and t respectively. A overhead bar denotes partial differentiation with respect to ξ.

The non-vanishing components of the curvature tensor R_{ijkl} obtained with the help of (2.2) are as follows

$$(2.3)\quad \begin{aligned} &R_{1313} = -R_{1314} = R_{1414} = \frac{P}{2}, \qquad R_{1323} = -R_{1324} = R_{1424} = \frac{Q}{2}, \\ &R_{2324} = -R_{2323} = -R_{2424} = \frac{L}{2}, \quad R_{3434} = -\frac{C_{33} - C_{44}}{2} + \frac{C_3^2 - C_4^2}{2C} = N, \end{aligned}$$

where

$$P = -\bar{\bar{A}} - \frac{2B\bar{A}\bar{B} + \bar{A}^2\bar{B}^2 + A(\bar{A}^2 - \bar{B}^2)}{2m} + \frac{\bar{A}(C_3 - C_4)}{2C},$$

$$Q = -\bar{\bar{B}} - \frac{2A\bar{A}\bar{B} - B(\bar{A}^2 - \bar{B}^2)}{2m} + \frac{\bar{B}(C_3 - C_4)}{2C},$$

$$L = -\bar{\bar{A}} - \frac{2B\bar{A}\bar{B} - \bar{A}^2 - \bar{B}^2 + A(\bar{A}^2 - \bar{B}^2)}{2m} + \frac{\bar{A}(C_3 - C_4)}{2C}.$$

The non-vanishing components of the Ricci tensor R_{ij} obtained either from its expression or from (2.3) on contraction with the help of (2.1) are given by

$$(2.4)\qquad R_{33} = M - \frac{N}{C}, \qquad R_{34} = -M, \qquad R_{44} = M + \frac{N}{C},$$

where

$$M = \left[A(P + L) + 2BQ - \frac{\bar{A}^2 + \bar{B}^2}{m}\right]/2m.$$

It can easily be shown that the non-vanishing components of R^i_j and R^{ij} are given by

$$(2.5)\quad (a)\ R^3_3 = -\frac{M}{C} + \frac{N}{C^2}, \qquad R^3_4 = -R^4_3 = \frac{M}{C}, \qquad R^4_4 = \frac{M}{C} + \frac{N}{C^2},$$

$$(2.5)\quad (b)\ R^{33} = \frac{M}{C^2} - \frac{N}{C^3} = X,\ R^{34} = R^{43} = \frac{M}{C^2} = Y,\ R^{44} = \frac{M}{C^2} + \frac{N}{C^3} = Z,$$

and the scalar curvature R is given by

$$R = \frac{2N}{C^2}. \tag{2.6}$$

3 - Solutions of the weakened field equations

(a) *Solution of the field equation* (1.2). The curvature tensor R^a_{ijk} satisfies the Bianchi identity

$$R^a_{ijk;l} + R^a_{ikl;j} + R^a_{ilj;k} = 0. \tag{3.1}$$

Let us put $a = 1$ and sum with respect to a and remembering that R^a_{ijk} is skew-symmetric in j, k we obtain

$$R^a_{ijk;a} + R_{ik;j} - R_{ij;k} = 0 \qquad \text{or} \qquad R^a_{ijk;a} = R_{ij;k} - R_{ik;j}.$$

Therefore the weakened field equation (1.2) reduces to

$$R_{ij;k} - R_{ik;j} = 0. \tag{3.2}$$

Using (2.4) in (3.2), we have two equations

$$R_{33;4} - R_{34;3} = M_3 + M_4 - \frac{N_4}{C} + \frac{2NC_4}{C^2} = 0, \tag{3.3}$$

$$R_{44;3} - R_{43;4} = M_3 + M_4 + \frac{N_3}{C} - \frac{2NC_3}{C^2} = 0. \tag{3.4}$$

Substracting (3.4) from (3.3), we get ($k = 3, 4$)

$$N_3 + N_4 = \frac{2N}{C}(C_3 + C_4) \qquad \text{or} \qquad \frac{\partial N}{\partial x^k} = \frac{2N}{C}\frac{\partial C}{\partial x^k},$$

which on integration yields

$$N = \lambda C^2 \qquad (\lambda \text{ constant of integration}). \tag{3.5}$$

Using (3.5) in (3.3) or (3.4), we obtain $M_3 + M_4 = 0$, which on integration gives

$$M = f(z - t)\,, \tag{3.6}$$

which gives a definite relation among all the components A, B, C of the fundamental tensor.

From (2.3) and (3.5), we have

$$-\frac{C_{33} - C_{44}}{2} + \frac{C_3^2 - C_4^2}{2C} = \lambda C^2\,, \qquad \text{i.e.}$$

$$-C\frac{\partial^2 C}{\partial z^2} + C\frac{\partial^2 C}{\partial t^2} + \left(\frac{\partial C}{\partial z}\right)^2 - \left(\frac{\partial C}{\partial t}\right)^2 = \lambda_1 C^3\,, \tag{3.7}$$

where $\lambda_1 = 2\lambda$. Equation (3.7) is a linear partial differential equation of second degree, which on integration gives

$$C = \tfrac{1}{2}[\varphi_2(z + t) - \varphi_1(z - t)]\,. \tag{3.8}$$

Hence the metric (1.5) represents the plane wave solution of the field equation (1.2) under the conditions (3.6) and (3.8).

(b) *Solution of the field equation* (1.3). The non-vanishing components of

$$\varepsilon^{rs} \equiv (-g)^{\frac{1}{2}}[(g^{rs}g^{tu} - \tfrac{1}{2}g^{rt}g^{su} - \tfrac{1}{2}g^{ru}g^{st})R_{;ut} + R(R^{sr} - \tfrac{1}{4}g^{sr}R)]$$

are as follows

$$\begin{aligned}
\varepsilon^{11} &\equiv \frac{1}{\sqrt{m}}\Big[\frac{\bar{A}}{2}(R_3 + R_4) - (1 + A)\,(R_{;44} - R_{;33} - \frac{R^2 C}{4})\Big],\\
\varepsilon^{12} = \varepsilon^{21} &\equiv \frac{1}{\sqrt{m}}\Big[\frac{\bar{B}}{2}(R_3 + R_4) - B\,(R_{;44} - R_{;33} - \frac{R^2 C}{4})\Big],\\
\varepsilon^{22} &\equiv \frac{1}{\sqrt{m}}\Big[-\frac{\bar{A}}{2}(R_3 + R_4) - (1 - A)\,(R_{;44} - R_{;33} - \frac{R^2 C}{4})\Big],\\
\varepsilon^{33} &\equiv \sqrt{m}\,C\,\Big[-\frac{(A\bar{A} + B\bar{B})(R_3 + R_4)}{mC^2} - \frac{1}{C^2}R_{;44} + R\,(R^{33} + \frac{R}{4C})\Big],\\
\varepsilon^{34} = \varepsilon^{43} &\equiv \sqrt{m}\,C\,\Big[\frac{1}{C^2}R_{;34} + RR^{43}\Big],\\
\varepsilon^{44} &\equiv \sqrt{m}\,C\,\Big[\frac{(A\bar{A} + B\bar{B})(R_3 + R_4)}{mC^2} - \frac{1}{C^2}R_{;33} + R\,(R^{44} - \frac{R}{4C})\Big].
\end{aligned} \tag{3.9}$$

Using (3.9) in (1.3) we have the following equations

(3.10)

$$\text{(a)}\quad \frac{\bar{A}}{2}(R_3+R_4)-(1+A)(R_{;44}-R_{;33}-\frac{R^2C}{4})=0\,,$$

$$\text{(b)}\quad \frac{\bar{B}}{2}(R_3+R_4)-B\,(R_{;44}-R_{;33}-\frac{R^2C}{4})=0\,,$$

$$\text{(c)}\quad \frac{\bar{A}}{2}(R_3+R_4)+(1-A)\,(R_{;44}-R_{;33}-\frac{R^2C}{4})=0\,,$$

$$\text{(d)}\quad -\frac{(A\bar{A}+B\bar{B})(R_3+R_4)}{mC^2}-\frac{1}{C^2}R_{;44}+R\,(R^{33}+\frac{R}{4C})=0\,,$$

$$\text{(e)}\quad \frac{1}{C^2}R_{;34}+RR^{43}=0\,,$$

$$\text{(f)}\quad \frac{(A\bar{A}+B\bar{B})(R_3+R_4)}{mC^2}-\frac{1}{C^2}R_{;33}+R\,(R^{44}-\frac{R}{4C})=0\,.$$

Multiplying (3.10) (a), (b), (c) by $(1-A)/2$, $-B$, $-(1+A)/2$ respectively and adding, we get

$$\frac{(A\bar{A}+B\bar{B})(R_3+R_4)}{2}+m\,[(R_{;44}-R_{;33})-\frac{R^2C}{4}]=0\,,\qquad \text{i.e.}$$

$$\text{(3.11)}\qquad \frac{(A\bar{A}+B\bar{B})(R_3+R_4)}{m}=-2\,[(R_{;44}-R_{;33})-\frac{R^2C}{4}]\,.$$

Subtracting (3.10) (d) from (3.10) (f), we obtain

$$\text{(3.12)}\qquad \frac{2(A\bar{A}+B\bar{B})(R_3+R_4)}{mC^2}+\frac{1}{C^2}(R_{;44}-R_{;33})+R(R^{44}-R^{33}-\frac{R}{2C})=0\,.$$

From (3.11) and (3.12), we have

$$\text{(3.13)}\qquad -\frac{3}{C^2}(R_{;44}-R_{;33})+R(R^{44}-R^{33})+\frac{R^2}{2C}=0\,.$$

Adding (3.10) (a) and (3.10 (c), we obtain

$$\text{(3.14)}\qquad R_{;44}-R_{;33}=\frac{R^2C}{4}\,.$$

Using (3.14) into (3.13), we obtain $NC^3 = 0$. But $C \neq 0$ and therefore

$$N = 0\,, \qquad \text{i.e.} \qquad -\frac{C_{33} - C_{44}}{2} + \frac{C_3^2 - C_4^2}{2C} = 0\,, \tag{3.15}$$

which on integration yields $\log C = \{\varphi_1(z+t) + \varphi_2(z-t)\}$, i.e.

$$C = \exp\cdot\{\varphi_1(z+t) + \varphi_2(z-t)\}\,. \tag{3.16}$$

From (3.10) (e), we have

$$R_3 C_4 + R_4 C_3 = \frac{4MN}{C}\,, \tag{3.17}$$

which establishes a definite relation among A, B, C.

Hence g_{ij} given by (1.5) represents the plane wave solution of the field equation (1.3) under the conditions (3.16) and (3.17).

(c) *Solution of the field equation* (1.4). The non-vanishing components of $H_k^{ij} \equiv R^{ij}{}_{;k}$ are given by

$$\begin{aligned}
&H_3^{33} \equiv X_3 + \frac{XC_3}{C} + \frac{YC_4}{C}\,, \quad H_4^{33} \equiv X_4 + \frac{XC_4}{C} + \frac{YC_3}{C}\,,\\
&H_3^{34} \equiv Y_3 + \frac{YC_3}{C} + \frac{XC_4}{2C} + \frac{ZC_4}{2C}\,, \quad H_4^{34} \equiv Y_4 + \frac{YC_4}{C} + \frac{XC_3}{2C} + \frac{ZC_3}{2C}\,,\\
&H_3^{44} \equiv Z_3 + \frac{ZC_3}{C} + \frac{YC_4}{C}\,, \quad H_4^{44} \equiv Z_4 + \frac{ZC_4}{C} + \frac{YC_3}{C}\,.
\end{aligned} \tag{3.18}$$

Using (3.18) in (1.4), we have six equations as follows

$$\begin{aligned}
&\text{(a)} \quad \frac{M_3}{C^2} - \frac{MC_3}{C^3} - \frac{N_3}{C^3} + \frac{2NC_3}{C^4} + \frac{MC_4}{C^3} = 0\,,\\
&\text{(b)} \quad \frac{M_4}{C^2} - \frac{MC_4}{C^3} - \frac{N_4}{C^3} + \frac{2NC_4}{C^4} + \frac{MC_3}{C^3} = 0\,,\\
&\text{(c)} \quad \frac{M_3}{C^2} - \frac{MC_3}{C^3} + \frac{MC_4}{C^3} = 0\,,\\
&\text{(d)} \quad \frac{M_4}{C^2} - \frac{MC_4}{C^3} + \frac{MC_3}{C^3} = 0\,,\\
&\text{(e)} \quad \frac{M_3}{C^2} - \frac{MC_3}{C^3} + \frac{N_3}{C^3} - \frac{2NC_3}{C^4} + \frac{MC_4}{C^3} = 0\,,\\
&\text{(f)} \quad \frac{M_4}{C^2} - \frac{MC_4}{C^3} + \frac{N_4}{C^3} - \frac{2NC_4}{C^4} + \frac{MC_3}{C^3} = 0\,.
\end{aligned} \tag{3.19}$$

Subtracting (3.19) (e) from (3.19) (a) and (3.19) (f) from (3.19) (b) we get respectively

$$-2N_3 + \frac{4NC_3}{C} = 0\,, \qquad -2N_4 + \frac{4NC_4}{C} = 0\,.$$

Adding these two equations, we get

$$N_3 + N_4 = \frac{2N}{C}\,(C_3 + C_4) \qquad \text{or} \qquad \frac{\partial N}{\partial x^k} = \frac{2N}{C}\,\frac{\partial C}{\partial x^k} \qquad (k = 3, 4)\,,$$

which on integration gives

(3.20) $$N = \lambda C^2 \qquad (\lambda \text{ constant of integration})\,.$$

Adding (3.19) (c) and (3.19) (d), we get $M_3 + M_4 = 0$, which on integration gives

(3.21) $$M = f(z - t)\,.$$

The equations (3.20) and (3.21) are same as the equations (3.5) and (3.6) respectively.

Hence the metric (1.5) represents the plane wave solution of the field equation (1.4) under the conditions (3.6) and (3.8).

It is interesting to note that the above plane wave solutions of the weakened field equations include, as a particular case, the solutions of the empty space field equation (1.1) of general relativity.

References

[1] C. W. Kilmister, *Alternative field equations in general relativity*, Perspectives in Geometry and Relativity, Indiana Univ. Press. (1966), 201-211.

[2] C. W. Kilmister and D. J. Newmann, *The use of algebraic structure in physics*, Proc. Cambridge Phil. Soc. **57** (1961), 851-864.

[3] K. B. Lal and T. Singh, *On cylindrical wave solutions of Kilmister and Newmann's weakened field equations in general relativity*, Tensor **27** (1973), 287-290.

[4] K. B. Lal and V. K. Srivastava, *Weakened field equations in general relativity admitting a generalised Takeno's space-time*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **58** (1975), 394-397.

[5] D. LOVELOCK: [•]$_1$ *Weakened field equations in general relativity admitting an « unphysical » metric*, Comm. Math. Phys. **5** (1967), 205-214; [•]$_2$ *A spherically symmetric solution of Maxwell-Einstein equations*, Comm. Math. Phys. **5** (1967), 257-261.

[6] S. N. PANDEY and J. KRISHNA RAO, *Some relativistic fields of electromagnetic and gravitational radiation*, Proc. Nat. Sci. India **28** (1962), 425-436.

[7] A. H. THOMPSON, *The investigations of a set of weakened field equations for general relativity*, contract AF-61(052)-457 TN 10, Aerospace Research Laboratories U.S.A.F. (21 Aug. 1963).

Summary

Kilmister and Newmann have proposed an alternative set of field equations of general relativity in empty space which are called weakened field equations.

Thompson, D. Lovelock have obtained static spherically symmetric solutions of these equations. Lal and Singh, Lal and Srivastava have obtained wave solutions of these field equations in varius space-time.

In this paper plane wave solutions of weakened field equations have been investigated in a space-time which is more general than Pandey and Rao plane wave metric. It has been shown that the plane wave solutions exist.

* * *