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Some classes of solutions of the functional equation $f(L + x) = f(L) + f(x)$ (**)

1 - In this paper we consider the functional equation

$$(*)_m \qquad f(L + x) = f(L) + f(x) , \qquad x \in L ,$$

where L is an open convex cone in $\boldsymbol{R}^n$ (with the euclidean norm) with the vertex at the origin and $f: L \to \boldsymbol{R}^m$ (the symbol $(*)_m$ points out this fact) ($L + x$ has the usual algebraic meaning and $f(L + x)$, $f(L)$ denote the f-transforms of $L + x$ and L respectively).

The solution of $(*)_1$ were extensively studied in [**1**], $[\mathbf{3}]_{1,2}$ and the equation $(*)_m$ has been studied, looking for some special class of continuous solutions, in [**2**].

In this paper we characterize completely some classes of solutions of $(*)_m$, where L is a proper cone in $\boldsymbol{R}^2$ (i.e. L is not a half-plane).

From now on L is a proper cone in $\boldsymbol{R}^2$ and r, s are the generatrices of L. In $\boldsymbol{R}^2$ we introduce a system (x_1, x_2) of coordinates, for which r is the positive x_1-axis and s is the positive x_2-axis. The cone L, as usual, induces in $\boldsymbol{R}^2$ a partial order, that is $x \geqslant y$ if and only if $x = y$ or $x - y \in L$; we assume that $\boldsymbol{R}^2$ is ordered as above and we denote with the same symbol « $\geqslant$ » the natural order on r and s. For every $x \in L$, let

$$R(x) = \{y \in L:\ y < x\} = L \cap (x - L) ,$$

then $R(x)$ is open. Observe that a set $A \subset L$ is bounded in the sense of the norm if and only if $A \subset R(x)$ for some $x \in L$.

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We use the following notations: L_x instead of $L + x$; $\mathscr{A}^+$ is the set of the continuous solutions of $(*)_1$, bounded from below and without zeros (see [3]$_1$, theorem 1); $\mathscr{N}^+$ is the set of the solutions of $(*)_1$, bounded from below and with zeros (except $f \equiv 0$) (see [3]$_2$).

We recall that if $f = (f_1, \ldots, f_m)$ is a solution of $(*)_m$, then f_i $(i = 1, \ldots, m)$ is a solution of $(*)_1$ (see [2]).

2 - In this section we assume that $f = (f_1, f_2)$ is a solution of $(*)_2$, where $f_2 \in \mathscr{N}^+$ and f_1 is a solution, bounded from below, of $(*)_1$, $f_1 \not\equiv 0$. We denote by M_0 and N_0 the sets of the zeros of f_1 and f_2 respectively (M_0 may be empty).

Theorem 1. *Let $f = (f_1, f_2)$ be a solution of $(*)_2$ and $N_0 \setminus M_0 \neq \emptyset$; then N_0 is unbounded.*

Proof. Let $x_0 \in N_0 \setminus M_0$, then $(f_1(x_0), 0) \in f(L)$ and $f_1(x_0) \neq 0$; by $(*)_2$, $f(L) + f(x_0) \subset f(L)$; therefore $(nf_1(x_0), 0) \in f(L)$ for every positive integer n. Hence $\sup f_1(N_0) = \infty$ and N_0 is unbounded, otherwise there exists $\bar{x} \in L$ such that $N_0 \subset R(\bar{x})$, so $\sup f_1(N_0) \leqslant f_1(\bar{x})$.

Remark 2. If $N_0 \setminus M_0 \neq \emptyset$ and $M_0 \setminus N_0 \neq \emptyset$ then N_0 and M_0 are unbounded.

The following examples show that if $N_0 = M_0$ every situation can occur.

Examples. (1) Let $L = \{(x_1, x_2): x_1 > 0 \text{ and } x_2 > 0\}$, $f_1(x_1, x_2) = [x_1]$, $f_2(x_1, x_2) = \alpha[x_1]$, where $\alpha > 0$ ($[x]$ is the greatest integer not exceeding x); $N_0 = M_0 = \{(x_1, x_2): 0 < x_1 < 1\}$. (2) Let $L = \{(x_1, x_2): x_1 > 0 \text{ and } x_2 > 0\}$ and let $S_{i,j} = \{(x_1, x_2): i \leqslant x_1 < i + 1 \text{ and } j \leqslant x_2 < j + 1\} \cap L$, where i, j are non-negative integers. Let $f_1(x_1, x_2) = i + j$ if $(x_1, x_2) \in S_{i,j}$ and $f_2(x_1, x_2) = 2i + j$ if $(x_1, x_2) \in S_{i,j}$.

$f(L) = \bigcup_{h=0}^{\infty} \bigcup_{k=h}^{2h} (h, k)$; if $x \in L$ and $f(x) = (\bar{h}, \bar{k})$ where $\bar{h} \leqslant \bar{k} \leqslant 2\bar{h}$, it is $f(L) + f(x) = \bigcup_{h=\bar{h}}^{\infty} \bigcup_{k=\bar{k}+h-\bar{h}}^{\bar{k}+2(h-\bar{h})} (h, k) = f(L + x)$, so f is a solution of $(*)_2$ and $N_0 = M_0 = S_{0,0}$.

Theorem 3. *Let $f = (f_1, f_2)$ be a solution of $(*)_2$ and $N_0 \setminus M_0 \neq \emptyset$; then for every $u \in f_2(L)$ and every $x \in f_2^{-1}(u)$, $L_x \cap f_2^{-1}(u)$ is unbounded.*

Proof. As we have seen in the proof of Theorem 1, $f_1(N_0)$ is unbounded. For every $x \in N_0$, it is $f_1(L_x \cap N_0) = \{t: (t, 0) \in f(L_x)\} = \{t: (t, 0) \in f(L) + (f_1(x), 0)\} = \{t: (t - f_1(x), 0) \in f(L)\} = f_1(N_0) + f_1(x)$, so f_1 is unbounded on

$L_x \cap N_0$, therefore $L_x \cap N_0$ is unbounded, otherwise $L_x \cap N_0 \subset R(\bar{x})$ for some $\bar{x}$ and $\sup f_1(L_x \cap N_0) \leqslant f_1(\bar{x})$.

Let $u \in f_2(L)$, $u \neq 0$, and $x \in f_2^{-1}(u)$; because $f_1(L_x \cap f_2^{-1}(u)) = \{t\colon (t, u) \in f(L_x)\} = \{t\colon (t, u) \in f(L) + (f_1(x), u)\} = \{t\colon (t - f_1(x), 0) \in f(L)\} = f_1(N_0) + f_1(x)$, f_1 is unbounded on $L_x \cap f_2^{-1}(u)$, thus $L_x \cap f_2^{-1}(u)$ is unbounded.

Remark 4. The Theorems 1 and 3 hold even if $L \subset \boldsymbol{R}^n$.

Theorem 5. *Let $f = (f_1, f_2)$ be a solution of $(*)_2$ and $N_0 \setminus M_0 \neq \emptyset$, then $N_0 = L \setminus \bar{L}_{n_1}$ $(n_1 \in \bar{L})$.*

Proof. For every $x \in N_0$, the set $L_x \cap N_0$ is unbounded (Th. 3); for a fixed $x \in N_0$ it is either

(a) $\sup \{x_2\colon (x_1, x_2) \in L_x \cap N_0\} = +\infty$ or

(b) $\sup \{x_1\colon (x_1, x_2) \in L_x \cap N_0\} = +\infty$.

If (a) holds then $s + x \subset N_0$, indeed, for every $y \in s + x$, $f_2(y) \geqslant f_2(x) = 0$, by (a) there exists $z \in L_x \cap N_0$ such that $z > y$, so $f_2(y) \leqslant f_2(z) = 0$ and $f_2(y) = 0$.

If (a) holds for every $x \in N_0$, then $N_0 = \bigcup_{x \in N_0} (s + x) = L \setminus \bar{L}_{n_1}$, where $n_1 = (n_{1,1}, 0)$ and $n_{1,1} = \sup \{x_1\colon (x_1, x_2) \in N_0\}$.

If (b) holds for every $x \in N_0$, then $N_0 = L \setminus \bar{L}_{n_1}$ where $n_1 = (0, n_{1,2})$ and $n_{1,2} = \sup \{x_2\colon (x_1, x_2) \in N_0\}$.

If there exist $v, w \in N_0$ such that (a) and (b) hold respectively, then for every $x \in R(v) \cap R(w)$ we have $r + x \subset N_0$ and $s + x \subset N_0$; therefore $N_0 = L \setminus \bar{L}_{n_1}$ where $n_1 = (n_{1,1}, n_{1,2})$ and $n_{1,1} = \sup \{t_1\colon (t_1, t_2) \in N_0$ and $\sup \{x_2\colon (x_1, x_2) \in L_{(t_1,t_2)} \cap N_0\} = +\infty\}$, $n_{1,2} = \sup \{t_2\colon (t_1, t_2) \in N_0$ and $\sup \{x_1\colon (x_1, x_2) \in L_{(t_1,t_2)} \cap N_0\} = +\infty\}$.

Theorem 6. *Let $f = (f_1, f_2)$ be a solution of $(*)_2$ and $N_0 \setminus M_0 \neq \emptyset$; then f_2 is constant on ∂N_0 and $f_2(L)$ is the semigroup generated by $f_2(\partial N_0)$. (∂N_0 is the boundary of N_0 in L).*

Proof. If f_2 is not constant on ∂N_0, then there esist $z, w \in \partial N_0$ and $w \in \partial L_z$, such that $f_2(w) > f_2(z)$. (If $N_0 = L \setminus \bar{L}_{n_1}$, where $n_1 \in L$, we can choose as z the point n_1). For every $x \in N_0$ it is, by theorem 6 in [3]$_2$,

$$\inf \{f_2(\partial N_0)\} = \inf \{f_2(L_x) \setminus \{0\}\} = \inf \{f_2(\partial N_0 \cap L_x)\}.$$

The shape of N_0 implies the existence of $x \in N_0$ such that $w \notin L_x \cap \partial N_0$ and

$\partial L_z \supset L_x \cap \partial N_0$, so

$$f_2(w) \leqslant \inf\{f_2(L_x \cap \partial N_0)\} = \inf\{f_2(L_x)\setminus\{0\}\} = \inf\{f_2(\partial N_0)\} \leqslant f_2(z)\ ,$$

i.e. $f_2(w) = f_2(z)$; a contradiction.

If $f_2(\partial N_0) = \{a\}$, by theorem 8 in [**3**]$_2$, it is $f_2(L) = a\boldsymbol{N}$.

We denote by N_k the set $f_2^{-1}(ka)$, $k \in \boldsymbol{N}$, whenever $f_2(L) = a\boldsymbol{N}$. Analogously, if $f_1(L) = b\boldsymbol{N}$, we shall denote by M_k the set $f_1^{-1}(kb)$.

Theorem 7. *Let* $f = (f_1, f_2)$ *be a solution of* $(*)_2$ *and* $N_0\setminus M_0 \neq \emptyset$. *Then* $N_k = (\bar{L}_{nk}\setminus\bar{L}_{n_{k+1}}) \cap L$, *where* $\{n_k\}$ *is an increasing sequence in* r (*or in* s) *such that* $n_k \to +\infty$ *for* $k \to +\infty$.

Proof. We prove the theorem by induction.

Let $k = 1$; it is, by Th. 5, $N_0 = L\setminus\bar{L}_{n_1}$. Assume that $n_1 \in r$. For every $x \in N_1$ either

(a) $\sup\{x_2 \colon (x_1, x_2) \in L_x \cap N_1\} = +\infty$;

(b) $\sup\{x_1 \colon (x_1, x_2) \in L_x \cap N_1\} = +\infty$ hold.

For a fixed $x \in N_1$, assume that (a) holds, then, as in the proof of Th. 5, we have $s + x \subset N_1$ and every y, $y < x$, $y \in L$, $y \notin N_0$, is in N_1, because $0 < f_2(y) \leqslant a$, so $f_2(y) = a$. If for every $x \in N_1$ (a) holds, then $N_1 = (\bar{L}_{n_1}\setminus\bar{L}_{n_2}) \cap L$, where $n_2 \in r$, $n_{2,1} = \sup\{x_1 \colon (x_1, x_2) \in N_1\}$ and $n_{2,1} > n_{1,1}$.

If there exist v, $w \in N_1$ for which (a) and (b) hold respectively, then every $x \in \{R(v) \cap R(w)\}\setminus N_0$ satisfies both (a) and (b), so there exists $n_2 \in L$, $n_{2,1} > n_{1,1}$, such that $N_1 = (\bar{L}_{n_1}\setminus\bar{L}_{n_2}) \cap L$.

Theorem 6 and theorem 3 in [**3**]$_2$ imply that the set of the second coordinate of points $x \in N_1\setminus\partial N_0$ is unbounded; then for some of such points (a) holds, otherwise $N_1 = L\setminus N_0$ contradicting the fact that $f_2(L) = a\boldsymbol{N}$. We have an analogous result if $n_1 \in s$. Now we assume that $n_1 \in L$; then there exists $v \in N_1$ for which (a) holds and $w \in N_1$ for which (b) holds, so $N_1 = (\bar{L}_{n_1}\setminus\bar{L}_{n_2}) \cap L$, where $n_2 > n_1$. In all cases dealing as in the proof of Th. 6, we obtain $f_2(\partial L_{n_2} \cap L) = \{2a\}$. Let now $\bar{k} > 1$; if for every $k < \bar{k}$, $n_k \in r$, then, as for $k = 1$, it is $n_{\bar{k}+1} \in r$ or $n_{\bar{k}+1} \in L$ and $n_{\bar{k}+1,1} > n_{\bar{k},1}$. If there is $k < \bar{k}$ such that $n_k \in r$ and $n_{k+1} \in L$, $n_{k,1} < n_{k+1,1}$, then $n_{\bar{k}+1} \in L$ and $n_{\bar{k}} < n_{\bar{k}+1}$.

It is $n_{k,1} \to +\infty$ or $n_{k,2} \to +\infty$, otherwise if $n_{k,1} < H_1$ and $n_{k,2} < H_2$, every $x = (x_1, x_2) \in L$ with $x_1 > H_1$ and $x_2 > H_2$ is such that $x \notin N_k$ for every $k \in \boldsymbol{N}$. We now prove that actually for every k, $n_k \notin L$. We assume that $n_{k,1} \to +\infty$. If $n_{k+1} \in L$, then, for every $j > 0$, $n_{k+j} \in L$; let z, $y \in N_k$ with

$z_2 < n_{k+1,2}$ and $y_2 < n_{k+1,2}$, then for every j large enaugh it is $L_z \cap N_{k+j} = L_y \cap N_{k+j}$.

$$\begin{aligned} f(L_z \cap N_{k+j}) &= \{(u, (k+j)a) : (u, (k+j)a) \in f(L_z)\} \\ &= \{(u, (k+j)a) : (u - f_1(z), ja) \in f(L)\} \\ &= \{(u, (k+j)a) : (u - f_1(z), ja) \in f(N_j)\} \\ &= \{(u, (k+j)a) : (u, ja) \in f(N_j) + (f_1(z), 0)\} = f(L_y \cap N_{k+j}) \\ &= \{(u, (k+j)a) : (u, ja) \in f(N_j) + (f_1(y), 0)\} ; \end{aligned}$$

hence, being $f_1(N_j)$ bounded from below, it is $f_1(z) = f_1(y) = \alpha$. Now let $w \in N_k$, $w_2 \geqslant n_{k+1,2}$ and $f_1(w) = \beta$; if we take $z \in N_k$, with $z < w$ and $z_2 < n_{k+1,2}$ and $y \in N_k$, with $y_2 < n_{k+1,2}$, $y_1 > n_{k+1,1}$, it is $(\beta, ka) = f(w) \in f(L_z) = f(L_y)$, (being $f(z) = f(y)$ from above), but $f(L_y \cap N_k) = \{(\alpha, ka)\}$, so $\beta = \alpha$. We have proved that if $n_{k+1} \in L$ then f_1 is constant on N_k; so if $n_1 \in L$, f_1 is constant on N_0, but $\inf f_1(L) = \inf f_1(N_0) = 0$, then $M_0 \supset N_0$; so $n_1 \notin L$.

By the hypothesis there is $x \in N_0$ such that $f_1(x) \neq 0$, so $(f_1(x), 0) \in f(L)$ and $(nf_1(x), 0) \in f(L)$, i.e. f_1 is unbounded on N_0.

If $n_{k+1} \in L$, then for every $x \in N_0$ there is $y \in N_k$ such that $x < y$, so $f_1(x) \leqslant f_1(y)$, but f_1 is constant on N_k, then we have a contradiction. Thus for every k, $n_k \notin L$.

3 - In this section we use the previous results on the structure of the sets N_k and M_k to characterize completely the solutions of $(*)_2$ when $f_2 \in \mathscr{N}^+$ and $f_1 \in \mathscr{A}^+$ or $f_1 \in \mathscr{N}^+$ and $N_0 \neq M_0$.

Theorem 8. *Let $f = (f_1, f_2)$, where $f_1 \in \mathscr{A}^+$ and $f_2 \in \mathscr{N}^+$. Then f is a solution of $(*)_2$ if and only if f_1 and f_2 depend only on x_2 and x_1 (or x_1 and x_2) respectively.*

Proof. The « if » part follows from the fact that if f_1 and f_2 depend on different variables then for every set $A \subset L$ it is $f(A) = f_1(A) \times f_2(A)$. Now assume that f is a solution of $(*)_2$; by Th. 7, for every $k \in \mathbf{N}$, $n_k \in r$ or $n_k \in s$; we suppose that $\{n_k\} \subset r$. For a fixed $\bar{k}$ and every $y = (y_1, y_2) \in N_{\bar{k}}$, it is $f(L_y) = f(L) + f(y) = f(L) + (f_1(y), \bar{k}a)$.

Now let $y' = (y_1', y_2') \in N_{\bar{k}}$ with $y_2' = y_2$, then for every $k > \bar{k}$ we have $L_y \cap N_k = L_{y'} \cap N_k$, so, as in the proof of Th. 7, we have $f_1(y) = f_1(y')$, i.e. f_1 does not depend on x_1 in $N_{\bar{k}}$. Since $\bar{k}$ was arbitrary, then $f_{1|x_2=c}$ is a step function and, by the continuity, $f_{1|x_2=c}$ is constant, i.e. f_1 depends only on x_2 in L.

If for every $k \in \boldsymbol{N}$, $n_k \in s$, we have the same result with x_1 in place of x_2.

Lemma 9. *Let $f = (f_1, f_2)$ be a solution of $(*)_2$, where $f_i \in \mathcal{N}^+$ $(i = 1, 2)$. If $N_0 \setminus M_0 \neq \emptyset$ and $M_0 \setminus N_0 \neq \emptyset$, then for every $k \in \boldsymbol{N}$, $N_0 \cap M_k \neq \emptyset$ (so $m_k \in N_0 \cup \partial L$) and $M_0 \cap N_k \neq \emptyset$ (so $n_k \in M_0 \cup \partial L$).*

Proof. $M_0 \setminus N_0 \neq \emptyset$ implies, by Th. 6, that $f_1(L) = b\boldsymbol{N}$. Let $y \in N_0 \setminus M_0$ and $f_1(y) = b$, then $(b, 0) \in f(L)$ and, by $(*)_2$, $(kb, 0) \in f(L)$ for every $k \in \boldsymbol{N}$, i.e. $N_0 \cap M_k \neq \emptyset$. Similarily we have $M_0 \cap N_k \neq \emptyset$ for every $k \in \boldsymbol{N}$.

Theorem 10. *Let $f = (f_1, f_2)$, where $f_i \in \mathcal{N}^+$ $(i = 1, 2)$ and $N_0 \setminus M_0 \neq \emptyset$, $M_0 \setminus N_0 \neq \emptyset$. f is a solution of $(*)_2$ if and only if f_1 and f_2 depend only on x_2 and x_1 (or x_1 and x_2) respectively.*

Proof. The «if» part is obvious.

Now let f be a solution of $(*)_2$. If $\{n_k\}$ and $\{m_k\}$ are the sequences that characterize the sets $\{N_k\}$ and $\{M_k\}$, then, by Th. 7, $\{n_k\}$ and $\{m_k\}$ are contained in r and s. If $\{n_k\} \subset r$, then $N_0 \cap M_k \neq \emptyset$ for every k (Lemma 9) implies that $\{m_k\} \subset s$. Analogously if $\{m_k\} \subset r$ then $\{n_k\} \subset s$. Thus f_1 and f_2 depend only on one variable and the theorem follows.

Theorem 11. *Let $f = (f_1, f_2)$, where $f_i \in \mathcal{N}^+$ $(i = 1, 2)$ and $M_0 \subset N_0$ $(M_0 \neq N_0)$. Then f is a solution of $(*)_2$ if and only if there exist two real increasing sequences $\{u_k\}$, $\{v_k\}$, $u_0 = v_0 = 0$, $u_k \to +\infty$, $v_k \to +\infty$, and three positive real numbers a, b, c such that*

$$f_1(x_1, x_2) = hb + jc \quad \text{if } (x_1, x_2) \in L \text{ and } u_j \leqslant x_1 < u_{j+1}, \quad v_h \leqslant x_2 < v_{h+1},$$

$$f_2(x_1, x_2) = ja \quad \text{if } (x_1, x_2) \in L \text{ and } u_j \leqslant x_1 < u_{j+1},$$

$$(\text{or } f_2(x_1, x_2) = ha \quad \text{if } (x_1, x_2) \in L \text{ and } v_h \leqslant x_2 < v_{h+1}).$$

Proof. Let f be a solution of $(*)_2$. By Th. 6, $f_2(N_j) = ja$, $a > 0$, and, by Th. 7, $N_j = (\bar{L}_{n_j} \setminus \bar{L}_{n_j+1}) \cap L$, $\{n_j\} \subset r$ (or in s). Assume that $\{n_j\} \subset r$, then $u_j = n_{j,1}$. As in the proof of Th. 8 we can prove that f_1 in N_k does not depend on x_1, so there is v_1 such that $M_0 = \{(x_1, x_2) : (x_1, x_2) \in N_0 \text{ and } x_2 < v_1\}$. Let b be the value assumed by f_1 on the set $\{(x_1, v_1) : 0 < x_1 < u_1\}$; then there is $v_2 > v_1$ such that $f_1^{-1}(b) \cap N_0 = \{(x_1, x_2) : (x_1, x_2) \in N_0 \text{ and } v_1 \leqslant x_2 < v_2\}$. We claim that f_1 on the set $V_{2,0} = \{(x_1, v_2) : 0 < x_1 < u_1\}$ takes the value $2b$. Let $y \in V_{2,0}$, there is $x \in f_1^{-1}(b) \cap N_0$ such that $x < y$; by $(*)_2$ there is $z \in L$ such that $f(y) = f(z) + f(x)$, i.e. $f(z) = (f_1(y) - b, 0)$; therefore $z \in N_0$ and $f_1(z) = f_1(y) - b$, but $f_1(z) \geqslant b$ so $f_1(y) \geqslant 2b$. $(b, 0) \in f(L)$ implies that $(2b, 0) \in f(L)$, so, f_1 being not decreasing, $f_1(y) \leqslant 2b$, then $f_1(y) = 2b$. Then, by induction, we obtain an increasing sequence $\{v_h\}$ such that if $0 < x_1 < u_1$ and $v_h \leqslant x_2 < v_{h+1}$ then $f_1(x_1, x_2) = hb$.

Now let $x \in N_1$, then $f(x) = (\gamma, a)$. $f(L_x \cap N_1) = f(N_0) + f(x)$, then $f_1(L_x \cap N_1) = \bigcup_{h \in N} \{\gamma + hb\}$. If $y \in N_1$ and $y < x$, then $\gamma = f_1(y) + h_0 b$, i.e. $f_1(y) = \gamma - h_0 b$; thus f_1 takes a minimum value, say $c > 0$, on N_1 and there is v_1' such that $f_1^{-1}(c) \cap N_1 = \{(x_1, x_2) : (x_1, x_2) \in N_1 \text{ and } 0 < x_2 < v_1'\}$. $v_1' = v_1$: if $v_1' > v_1$, we take $x \in f_1^{-1}(b) \cap N_0$ and $y \in f_1^{-1}(c) \cap N_1$ such that $x < y$; then $f(y) - f(x) \in f(L)$ implies the existence of z such that $f(z) = f(y) - f(x) = (c - b, a)$, but c is the minimum of f_1 in N_1, so we have a contradiction. If $v_1' < v_1$, we take $x \in N_0 \cap M_0$ such that $x_2 > v_1'$, then $f(L_x) = f(L)$, but $(c, a) \in f(L)$ and $(c, a) \notin f(L_x)$; a contradiction.

Then, by induction, we have $f_1(x_1, x_2) = hb + c$ if $u_1 \leqslant x_1 < u_2$ and $v_h \leqslant x_2 < v_{h+1}$. $f_1(L)$, by th. 8 in [**3**]$_2$, is the semigroup generated by b and c, then, in the same way, we obtain that $\min f_1(N_2) = 2c$ and, by induction on j and h, the proof follows. By a simple verification, the «if» part follows, indeed

$$f(L) = \bigcup_{\substack{j \geqslant 0 \\ h \geqslant 0}} \{(hb + jc, ja)\}, \qquad f(x) = (h_0 b + j_0 c, j_0 a),$$

$$f(L_x) = \bigcup_{\substack{j \geqslant j_0 \\ h \geqslant h_0}} \{(hb + jc, ja)\} = \bigcup_{\substack{j \geqslant 0 \\ h \geqslant 0}} \{((h + h_0)b + (j + j_0)c, (j + j_0)a)\} = f(L) + f(x).$$

Remark 1.2. The value c in Th. 11 can be of the form $\bar{h}b$.

4 – In order to explain the way of using the previous results, we give a theorem concerning the solutions $f = (f_1, \ldots, f_m)$ of $(*)_m$ when some f_i are in $\mathcal{N}^+$ and some are in $\mathcal{A}^+$.

Before we need the following

Lemma 1.3. *Let $L = (0, +\infty)$ and $f = (f_1, f_2)$, where $f_i \in \mathcal{N}^+$ $(i = 1, 2)$. Then f is a solution of $(*)_2$ if and only if $f_2(x) = kf_1(x)$ for some $k \in \mathbf{R}^+ \setminus \{0\}$.*

Proof. We recall that if $L = (0, +\infty)$, $g \in \mathcal{N}^+$ if and only if $g(x) = nq$ if $x_n \leqslant x < x_{n+1}$, $q \in \mathbf{R}^+ \setminus \{0\}$, where $\{x_n\}$ is an increasing sequence such that $x_0 = 0$ and $x_n \to +\infty$ (see [**1**]). The «if» part of the theorem is obvious.

Let f be a solution of $(*)_2$ and let $\{x_n\}$ and $\{y_n\}$ be the sequences of f_1 and f_2 respectively. It is enough to prove that for every n, $x_n = y_n$. $x_1 = y_1$, otherwise Th. 1 and Remark 4 imply that either $f_1 \equiv 0$ or $f_2 \equiv 0$. We assume $x_k = y_k$ for every $k \leqslant n$ and we prove that $x_{n+1} = y_{n+1}$. If $x_n < y_{n+1} < x_{n+1}$, then $(nq_1, nq_2) \in f(L)$ and $(nq_1, (n+1)q_2) \in f(L)$ $(q_1 = f_1(x_1),\ q_2 = f_2(y_1))$, so $(2nq_1, 2nq_2) \in f(L)$, $(2nq_1, (2n+1)q_2) \in f(L)$, $(2nq_1, 2(n+1)q_2) \in f(L)$. $(2nq_1, 2 \cdot (n+1)q_2) \in f(L)$ implies that $y_{2n+2} < x_{2n+1}$; since $(q_1, q_2) \in f(L)$, then $((2n+1)q_1, (2n+1)q_2) \in f(L)$ and $((2n+1)q_1, 2(n+1)q_2) \in f(L)$, thus $x_{2n+1} \leqslant y_{2n+2}$; a contradiction. Then $y_{n+1} \geqslant x_{n+1}$ and, by simmetry, $x_{n+1} \geqslant y_{n+1}$, so $x_{n+1} = y_{n+1}$.

Theorem 1.4. *Let* $L \subset \mathbf{R}^2$ *and* $f = (f_1, \dots, f_j, g_1, \dots, g_n)$, $n + j = m$, *where* $f_i \in \mathscr{A}^+$ $(i = 1, \dots, j)$ *and* $g_k \in \mathscr{N}^+$ $(k = 1, \dots, n)$. *Then* f *is a solution of* $(*)_m$ *if and only if* $f_i(x) = \alpha_i p(x)$, *where* $p \in \mathscr{A}^+$, $\alpha_i \in \mathbf{R}^+ \setminus \{0\}$ *and* p *depends only on* x_1 *(or* x_2*) and* $g_k(x) = \beta_k q(x)$, *where* $q \in \mathscr{N}^+$, $\beta_k \in \mathbf{R}^+ \setminus \{0\}$ *and* q *depends only on* x_2 *(or* x_1*).*

Proof. Let f be a solution of $(*)_m$ and fix f_i and g_k; by Th. 8, g_k depends only on one variable, say x_2, and f_i depends on x_1. Changing k, we have that every g_k depends on x_2 and, analogously, every f_i depends on x_1. The function $(g_1, \dots, g_n)$ is a solution of $(*)_n$, then, by Lemma 13, $g_k(x) = \beta_k g_1(x)$, $\beta_k > 0$.

$(f_1, \dots, f_j)$ is a solution of $(*)_j$, then by corollary 7 and theorem 13 in [**2**], $f_i(x) = \alpha_i f_1(x)$, $\alpha_i > 0$.

Now let $f = (\alpha_1 p, \dots, \alpha_j p, \beta_1 q, \dots, \beta_n q)$, $p \in \mathscr{A}^+$ depending on x_1 and $q \in \mathscr{N}^+$ depending on x_2

$$\begin{aligned} f(L_x) &= \{(\alpha_1 p, \dots, \alpha_j p)(L_x)\} \times \{\beta_1 q, \dots, \beta_n q)(L_x)\} \\ &= \{(\alpha_1 p, \dots, \alpha_j p)(L) + (\alpha_1 p(x), \dots, \alpha_j p(x))\} \times \\ &\qquad \times \{(\beta_1 q, \dots, \beta_n q)(L) + (\beta_1 q(x), \dots, \beta_n q(x))\} \\ &= \{(\alpha_1 p, \dots, \alpha_j p)(L)\} \times \{\beta_1 q, \dots, \beta_n q)(L)\} \\ &\qquad + (\alpha_1 p(x), \dots, \alpha_j p(x), \beta_1 q(x), \dots, \beta_n q(x)) = f(L) + f(x). \end{aligned}$$

(The second equality follows from corollary 7 in [**2**]).

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Sunto

In questa nota vengono caratterizzate alcune classi di soluzioni non continue dell'equazione funzionale $f(L + x) = f(L) + f(x)$, *dove* L *è un cono aperto convesso in* $\boldsymbol{R}^2$, $x \in L$ *e* $f: L \to \boldsymbol{R}^2$. *I risultati trovati sono poi utilizzati per determinare una particolare classe di soluzioni della stessa equazione, quando* f *assume valori in* $\boldsymbol{R}^m$.

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