Riv. Mat. Univ. Parma (4) 6 (1980), 103-110

Silvio Massa (*)

Generalized multicontractive mappings (**)

1 - Introduction

Let (X, d) be a complete metric space, $cb(X)$ be the family of all nonempty closed bounded subsets of X and H be the Hausdorff metric induced by d.

Let's consider a mapping $f: X \to cb(X)$ which satisfies for every x, y in X the condition

$$(1.1) \qquad H(f(x), f(y)) \leqq a(x, y)d(x, f(x)) + a'(x, y)d(y, f(y)) + b(x, y)d(x, f(y)) + b'(x, y)d(y, f(x)) + c(x, y)d(x, y)$$

with $a, a', b, b', c: X \times X \to R^+$ (1) and $s(x, y) = (a + a' + b + b' + c)(x, y) < 1$.

A previous paper (2) contains some fixed point theorems for single valued (s.v.m.) and multi valued (m.v.m.) mappings satisfying (1.1) with $b(x, y) = b(y, x)$, $\sup_{x,y \in X} a(x, y) < 1$ and with s satisfying a Boyd-Wong condition (3).

In this paper we study the case when b is not symmetric and we prove that if f satisfies a generalized Rakotch condition, then there exist fixed points and the method of successive approximations converges. Moreover the condition imposed on s cannot be weakened in the sense that it cannot be allowed to be a Boyd-Wong condition.

(*) Indirizzo: Istituto Matematico, Università, Via Saldini 50, 20133 Milano, Italy.

(**) Lavoro eseguito nell'ambito del G.N.A.F.A. (C.N.R.). — Ricevuto: 25-IX-1978.

(1) Without any loss of generality we may assume that $a'(x, y) = a(y, x)$ and $b'(x, y) = b(y, x)$.

(2) See [4]. This paper contains also up to date references on the argument.

(3) I.e. $\limsup_{d(x,y) \to d_0^+} s(x, y) < 1 \ \forall d_0 > 0$.

2 - Results for single valued mappings

Let f be a s.v.m. satisfying (1.1) and set

$$M(x, y) = \operatorname{Max}\{d(x, y), d(x, f(x)), d(y, f(y))\} .$$

The following Theorems hold.

Theorem 1. *If*

$$s(x, y) \to 1 \Rightarrow M(x, y) \to 0 \text{ or } \infty \tag{2.1}$$

and there exists x_0 with bounded orbit, then f has a unique fixed point y and $f^n(x_0) \to y$.

Theorem 2. *If*

$$s(x, y) \to 1 \Rightarrow M(x, y) \to \infty \quad \text{or} \quad d(x, y) \to 0 , \tag{2.1)'$$

$$\limsup_{d(x,y)\to 0} (a + b')(x, y) < 1 \tag{2.2}$$

and there exists x_0 with bounded orbit, then f has a unique fixed point y and $f^n(x_0) \to y$.

Theorem 3. *If*

$$s(x, y) \to 1 \Rightarrow d(x, y) \to 0 \tag{2.1)''$$

and (2.2) *holds, then for every x in X $\{f^n(x)\}$ converges to the unique fixed point of f.*

3 - Results for multi valued mappings

Let f be a m.v.m. satisfying (1.1) with $c \equiv 0$ and let $M(x, y)$ be as in **2**. The following Theorems hold.

Theorem 4. *If f satisfies* (2.1) *and there exists x_0 with a bounded sequence $\{x_n\}$ of iterates* ([4]), *then*

(i) *the set A of the fixed points of f is non empty,*

(ii) $f(y) = A \ \forall y \in A$,

(iii) $f(x_n) \xrightarrow[cb(X)]{} A$.

([4]) I.e. a sequence $\{x_n\}$ such that $x_{n+1} \in f(x_n)$.

Theorem 5. *If f satisfies* (2.1)′ *and* (2.2) *and there exists x_0 with a bounded sequence $\{x_n\}$ of iterates, then* (i), (ii) *and* (iii) *hold.*

Theorem 6. *If f satisfies* (2.1)″ *and* (2.2), *then* (i) and (ii) *hold and moreover*

(iii)′ $\forall x_0 \in X \ \forall \{x_n\}$ *such that* $x_{n+1} \in f(x_n)$, $f(x_n) \xrightarrow[cb(X)]{} A$.

4 - Remarks

(1) (2.1), (2.1)′ and (2.1)″ cannot be replaced by a Boyd-Wong condition even if $a = a' = c \equiv 0$. Indeed let (X, d) be the subset of the points $\{x_n\}$ $(n = 1, 2, \ldots)$ of l^∞ of the form $x_n = \sum_{i=1}^{n} e_i + \sum_{i=1}^{\infty} e_{n+i}/(i+1)$ where $\{e_n\} = \{\delta_{i,n}\}_{i=1}^{\infty}$ $(n = 1, 2, \ldots)$. Let $f: x_n \rightarrowtail x_{n+1}$. X is a (bounded) complete metric space, f has no fixed point and, if $n < m$, (1.1) holds with (e.g.) $b(x_n, x_m) = \big(4(m-n)^2+1\big)(m-n+2)/5(m-n)(m-n+1)^2$, $b'(x_n, x_m) = 1/5$.

(2) In Theorems 1, 2, 4 and 5 the assumption that there exists a point with bounded orbit cannot be dropped. Indeed let $X = \{1, 2, \ldots, n, \ldots\}$ with the usual metric. The map $f: n \rightarrowtail n+1$ satisfies (1.1) with $b(n, m) = s(n, m) = (m-n)/(m-n+1)$ for $n < m$ and has no fixed point.

(3) Theorem 1 contains the analogous theorems of [**2**] and [**3**].

(4) Theorem 1 is not contained in Theorem 2. Indeed let $X = [-1, \infty)$ and $f(x) = -1/x$ if $-1 \leqq x < 0$, $f(x) = 0$ if $x \geqq 0$. f satisfies the hypotheses of Theorem 1 but does not satisfy (2.2) (consider $x = -1/n$, $n = 1, 2, \ldots$ and $y = 0$).

(5) Theorem 2 (and its corollary Theorem 3) is not contained in Theorem 1. Indeed consider the compact subset of $\boldsymbol{C}$ of the points $0, 1$, $\exp[i\pi/3]$, $\frac{1}{2}\exp[i\pi/3]$ and $x_n = 1 + 1/n$ $(n = 1, 2, \ldots)$ and let

$$f(1) = 0\,, \quad f(x_n) = \exp[i\pi/3]\,, \quad f(0) = f(\exp[i\pi/3]) = f(\tfrac{1}{2}\exp[i\pi/3]) = \tfrac{1}{2}\exp[i\pi/3]\,.$$

f satisfies the assumptions of Theorem 3, but does not satisfy (2.1) (consider $x = x_n$ and $y = 1$).

(6) In Theorems 4, 5 and 6, if c is not identically zero, in general (ii) and (iii) (or (iii)') fall to be true (5). The problem whether (i) holds is still open (if f is not single valued).

5 - Proofs of the Theorems of 2

For every x in X we set $O(x) = \bigcup_{n=0}^{\infty} f^n(x)$, $\delta(x) = \operatorname{diam}(O(x))$ and $N(x, y) = \operatorname{Max}\{d(x, y),\ d(x, f(x)),\ d(y, f(y)),\ d(x, f(y)),\ d(y, f(x))\}$.

It is easy to prove that (6), for every x in X,

$$\delta(x) = \operatorname*{Sup}_n d(x, f^n(x))$$

and then $\delta(f(x)) \leqq \delta(x)$.

In order to prove Theorems 1 and 2, we observe that, if $\delta(x_0) < \infty$, the non increasing sequence $\{\delta(f^n(x_0))\}$ converges. Let's suppose, by contradiction, that its limit δ is positive. Then for every n there exists $m_n > n$ such that

$$d(f^n(x_0), f^{m_n}(x_0)) \to \delta \qquad \text{for } n \to \infty .$$

We have

$$d(f^n(x_0), f^{m_n}(x_0)) \leqq s(f^{n-1}(x_0), f^{m_n-1}(x_0)) \cdot \delta(f^{n-1}(x_0))$$

and then

$$s(f^{n-1}(x_0), f^{m_n-1}(x_0)) \to 1 . \tag{5.1}$$

Theorem 1. As $\delta(x_0) < \infty$, (5.1) implies $M(f^{n-1}(x_0), f^{m_n-1}(x_0)) \to 0$, therefore

$$d(f^n(x_0), f^{m_n}(x_0)) \leqq d(f^n(x_0), f^{n-1}(x_0)) + d(f^{n-1}(x_0), f^{m_n-1}(x_0)) + d(f^{m_n-1}(x_0), f^{m_n}(x_0)) \leqq 3M(f^{n-1}(x_0), f^{m_n-1}(x_0)) ,$$

absurd.

---

(5) See [4], Remark 1.

(6) See [6], [5_1], [5_2] and Lemma of 6.

Hence $\{f^n(x_0)\}$ is a Cauchy sequence, and if y is its limit we have

$$d(y, f(y)) \leqq d(y, f^{n+1}(x_0)) + d(f^{n+1}(x_0), f(y)) \leqq s(f^n(x_0), y) N(f^n(x_0), y) + o(1)$$

$$\leqq s(f^n(x_0), y) d(y, f(y)) + o(1)$$

and necessarily $y = f(y)$.

Theorem 2. $\delta(x_0) < \infty$ and (5.1) imply $d(f^{n-1}(x_0), f^{m_n-1}(x_0)) \to 0$. We have

$$d(f^n(x_0), f^{m_n}(x_0)) \leqq (a + b')(f^{n-1}(x_0), f^{m_n-1}(x_0)) d(f^{n-1}(x_0), f^n(x_0))$$

$$+ (a' + b)(f^{n-1}(x_0), f^{m_n-1}(x_0)) d(f^{m_n-1}(x_0), f^{m_n}(x_0)) + o(1)$$

and then, from (2.2), $d(f^{n-1}(x_0), f^n(x_0)) \to \delta$, but

$$d(f^{n-1}(x_0), f^n(x_0)) \leqq s(f^{n-2}(x_0), f^{n-1}(x_0)) \cdot \delta(f^{n-2}(x_0))$$

and this implies $d(f^{n-2}(x_0), f^{n-1}(x_0)) \to 0$ which is absurd. Then $\delta = 0$ and $\{f^n(x_0)\}$ is a Cauchy sequence; its limit point is obviously the unique fixed point of f.

Theorem 3. It is sufficient to prove that $\delta(x) < \infty$ $\forall x \in X$.

Let's suppose, by contradiction, that $\delta(x) = \infty$ for some x in X. Then there exist an increasing and divergent sequence of real numbers $\{K_i\}$ and a sequence of integers $\{n_i\}$, $n_i = n_i(K_i)$, such that $d(x, f^{n_i}(x)) > K_i$ and $d(x, f^n(x)) \leqq K_i$ for $n < n_i$. We have

$$d(x, f^{n_i}(x)) \leqq d(x, f(x)) + d(f(x), f^{n_i}(x)) \leqq d(x, f(x)) + s(x, f^{n_i-1}(x)) d(x, f^{n_i}(x)) ,$$

so $f^{n_i-1}(x) \to x$ and (with an obvious meaning of the symbols)

$$d(x, f^{n_i}(x)) \leqq d(x, f(x)) + ad(x, f(x)) + a' d(f^{n_i-1}(x), f^{n_i}(x)) + bd(x, f^{n_i}(x))$$

$$+ b' d(f^{n_i-1}(x), f(x)) + cd(x, f^{n_i-1}(x))$$

$$\leqq (1 + a + b') d(x, f(x)) + (a' + b) d(x, f^{n_i}(x)) + o(1) ,$$

absurd from (2.2).

6 - Proofs of the Theorems of 3

Let y_0 be an arbitrary point in X and $\{y_n\}_{n=1}^{\infty}$ be a sequence of iterates of y_0. The following Lemma holds.

Lemma. *For $n < m$ we have*

$$H\big(f(y_n), f(y_m)\big) \leqq \operatorname*{Max}_{k \leqq m} d\big(y_0, f(y_k)\big) .$$

Proof. Indeed $H\big(f(y_n), f(y_m)\big) \leqq \operatorname*{Max}_{n \leqq i,j \leqq m} d\big(y_i, f(y_j)\big)$ but $i > n$ implies $d\big(y_i, f(y_j)\big) \leqq H\big(f(y_{i-1}), f(y_j)\big)$ and we obtain, for recurrence, $H\big(f(y_n), f(y_m)\big) \leqq \operatorname*{Max}_{n \leqq j \leqq m} d\big(y_n, f(y_j)\big)$ and Lemma follows.

We set $\delta(y_0) = \delta(y_0, y_1, \ldots, y_n, \ldots) = \operatorname*{Sup}_n d\big(y_0, f(y_n)\big)$ and we remark that if $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence of iterates of x_0, $\delta(x_0) < \infty$ $\big($indeed $d\big(x_0, f(x_i)\big) \leqq d(x_0, x_{i+1})\big)$ and $\{\delta(x_n)\}$ is non increasing.

The proofs of Theorems 4, 5 and 6 are now somehow similar to those of Theorems 1, 2 and 3. Set

$$N^0(x, y) = \operatorname{Max}\{d\big(x, f(x)\big), d\big(y, f(y)\big), d\big(x, f(y)\big), d\big(y, f(x)\big)\}$$

and let's suppose, by contradiction, $\lim \delta(x_n) = \delta > 0$. Then for every n there exists $m_n > n$ such that $d\big(x_{n+1}, f(x_{m_n})\big) \to \delta$. As $d\big(x_{n+1}, f(x_{m_n})\big) \leqq H\big(f(x_n), f(x_{m_n})\big) \leqq s(x_n, x_{m_n})\delta(x_n)$ we have

$$s(x_n, x_{m_n}) \to 1 . \tag{6.1}$$

Theorem 4. (6.1) and $\delta(x_0) < \infty$ imply $M(x_n, x_{m_n}) \to 0$, therefore $N^0(x_n, x_{m_n}) \to 0$ and $d\big(x_{n+1}, f(x_{m_n})\big) \leqq N^0(x_n, x_{m_n}) = o(1)$. Hence $\delta = 0$ and $\{f(x_n)\}$ is a Cauchy sequence in $cb(X)$ $\big($indeed Lemma gives $H\big(f(x_n), f(x_{m_n})\big) \leqq \delta(x_n)\big)$. Let A be the limit $\big($in $cb(X)\big)$ of $\{f(x_n)\}$ and let $y \in A$.

$$\begin{aligned} H\big(f(y), A\big) &\leqq H\big(f(y), f(x_n)\big) + H\big(f(x_n), A\big) \\ &\leqq s(y, x_n)N^0(y, x_n) + o(1) \leqq s(y, x_n)H\big(f(y), A\big) + o(1) , \end{aligned}$$

hence $f(y) = A$.

Theorem 5. (6.1) and $\delta(x_0) < \infty$ imply $d(x_n, x_{m_n}) \to 0$ and then

$$\begin{aligned} d\big(x_{n+1}, f(x_{m_n})\big) &\leqq (a + b')(x_n, x_{m_n})d\big(x_n, f(x_n)\big) \\ &\quad + (a' + b)(x_n, x_{m_n})d\big(x_{m_n}, f(x_{m_n})\big) + o(1) . \end{aligned}$$

In view of (2.2) we have $d(x_n, f(x_n)) \to \delta$ which leads to the contradiction $d(x_{n-1}, x_n) \to 0$. Then $\delta = 0$ and $\{f(x_n)\}$ converges (in $cb(X)$) to A. If (by contradiction) $H(f(y), A) > 0$, then (as in proof of Theorem 4)

$$H(f(y), A) \leqq s(y, x_n) H(f(y), A) + o(1).$$

So $s(y, x_n) \to 1$, hence $x_n \to y$ and

$$H(f(y), A) \leqq H(f(y), f(x_n)) + o(1) \leqq a(y, x_n) d(y, f(y)) + a'(y, x_n) d(x_n, f(x_n)) + b(y, x_n) d(y, f(x_n)) + b'(y, x_n) d(x_n, f(y)) + o(1) = (a + b') d(y, f(y)) + o(1) \leqq (a + b') H(f(y), A) + o(1),$$

which is absurd.

Theorem 6. It is sufficient to prove that, for every x_0 in X and for every sequence $\{x_n\}$ of iterates of x_0, $\delta(x_0) < \infty$.

As in the proof of Theorem 3, let's suppose, by contradiction, that there exist $\{K_i\}$ and $\{n_i\}$ such that $K_i \uparrow \infty$ and

$$d(x_0, f(x_{n_i})) > K_i, \quad d(x_0, f(x_n)) \leqq K_i \quad \text{for } n < n_i .$$

We have

$$d(x_0, f(x_{n_i})) \leqq d(x_0, f(x_0)) + H(f(x_0), f(x_{n_i}) \leqq d(x_0, f(x_0)) + s(x_0, x_{n_i}) d(x_0, f(x_{n_i})),$$

therefore $x_{n_i} \to x_0$ and

$$d(x_0, f(x_{n_i})) \leqq (1 + a + b') d(x_0, f(x_0)) + (a' + b) d(x_0, f(x_{n_i})),$$

absurd from (2.2).

References

[1] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458-464.

[2] F. E. Browder, *On the convergence of the successive approximations for nonlinear functional equations*, Indag. Math. **30** (1968), 27-35.

[3] J. Daneš, *Two fixed point theorems in topological and metric spaces*, Bull. Austral. Math. Soc. **14** (1976), 259-265.

[4] G. GAREGNANI and S. MASSA, *Multivalued mappings of contractive type*, Ist. Lombardo Accad. Sci. Lett. Rend. A (to appear).

[5] S. MASSA: [•]$_1$ *Un'osservazione su un teorema di Browder-Roux-Soardi*, Boll. Un. Mat. Ital. (4) **7** (1973), 151-155; [•]$_2$ *Generalized contractions in metric spaces*, Boll. Un. Mat. Ital. (4) **10** (1974), 689-694.

[6] D. ROUX and P. SOARDI, *Alcune generalizzazioni del teorema di Browder-Göhde-Kirk*, Atti Accad. Naz. Lincei Rend. (8) **52** (1972), 682-688.

[7] E. RAKOTCH, *A note on contractive mappings*, Proc. Amer. Math. Soc. **13** (1962), 459-465.

Sunto

Siano (X, d) *uno spazio metrico completo,* $cb(X)$ *l'insieme delle parti di* X *non vuote, chiuse e limitate,* H *la distanza di Hausdorff indotta da* d *su* $cb(X)$. *Sia* $f: X \to cb(X)$ *una multifunzione che soddisfa la condizione* $H(f(x), f(y)) \leqslant s(x, y) \operatorname{Max}\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}$ *con* $s(x, y) < 1$ $\forall x, y \in X$. *Si dimostrano (per funzioni e per multifunzioni) alcuni teoremi di punto fisso, e si assicura la convergenza del metodo delle approssimazioni successive. Inoltre, nel caso delle multifunzioni, si studiano le proprietà dell'insieme dei punti fissi di* f.

* * *